

If we turn to analysis it must be remarked that the classical form of it cannot be obtained. Indeed it will be necessary to distinguish between real numbers of different orders. A class of real numbers of 1. order which is bounded above possesses an upper bound, but this bound may then be a real number of order 2. Nevertheless a great part of analysis can be developed as usual, namely, the most useful part of it dealing with continuous functions, closed point-sets, etc. The reason for this is that it is often possible to prove theorems of reducibility, namely, theorems saying that a class (or relation) of a certain order coincides with one of lower order. I will not enter into this but only refer the reader to the book: "Das Kontinuum" by H. Weyl, where he has developed such a kind of predicative analysis.

15. Lorenzen's operative mathematics

In more recent years the German mathematician P. Lorenzen has set forth a system of mathematics which in some respects resembles the ramified theory of types, but it has also one important feature in common with the simple theory of types, namely, that the simple infinite sequence and similar notions are characterized by an induction principle which is assumed valid within all layers of objects. Lorenzen talks namely about layers of objects, not of types or orders. To begin with he takes into account some original objects, say numerals, figures built up in a so-called calculus as follows. We have the rules of production

$$1 \\ k \rightarrow k1$$

which means that the object or symbol 1 is originally given and whenever we have a symbol or a string of symbols k we may build the string $k1$ obtained by placing 1 after k . He introduces the notion "system". A system is a finite set of symbols. The systems are obtained by the rules

$$x \\ X \rightarrow X, x$$

The length or cardinal number of a system X is denoted by $|X|$. He gives the rules

$$|x| = 1 \\ |X,x| = |X| + 1$$

for these lengths. Now the explanation of the successive layers of language is as follows.

From certain originally given symbols called atoms, say $u_1 \dots u_n$, he constructs strings of symbols by the schema

$$x \rightarrow xu_1 \\ \dots \dots \dots \\ x \rightarrow xu_n$$

In connection with this we may notice that the problem concerning the principle of choice disappears. Indeed, the enumeration in $S_{\theta+1}$ of the objects constituting the layer S_θ makes possible at once the simultaneous choice of one element from every set in S_θ . On the other hand it is not certain that we can find a formula in S_θ furnishing such a choice for a set of sets in S_θ . Thus we have again a relativity with regard to the existence of choice functions.

Now let us consider real numbers—defined, for example, as initial parts of the ordered set R of rational numbers—and sets of reals all belonging to the layer S_θ , where θ is a limit number. Then it is possible to prove for each set M of real numbers, M as well as the elements of M belonging to S_θ , that if M is bounded below, it possesses a lower bound γ also in S_θ . Indeed γ is the intersection of all elements of M considered as initial parts of R . Since $M \in S_\theta$, we have $M \in S_\theta$, θ some ordinal $< \theta$. In the definition of γ all occurring variables belong to S_θ but there is a universal quantifier extended over S_θ . Thus γ is a real number occurring in $S_{\theta+1}$. However, since θ is a limit number we have $\theta+1 < \theta$. Therefore the lower bound γ always again belongs to S_θ . More special theorems, such as the existence of a convergent subsequence of a bounded sequence of reals, and that every convergent sequence (in the sense of Cauchy) has a real number as limit, are easily proved.

The theory of neighborhoods and coverings is more difficult. In order to be able to develop the usual covering theorems, Lorenzen finds it necessary to take into account sets of real numbers belonging to essentially higher layers than the real numbers themselves. He chooses two limit numbers, $\theta_1 < \theta_2$. The considered real numbers shall all belong to S_{θ_1} , whereas sets of, and relations between, these reals are allowed to belong to S_{θ_2} . The classes and relations which already belong to S_{θ_1} are called primary, those which belong to S_{θ_2} but not S_{θ_1} are called secondary. It may be noticed that by taking into account also the secondary sets we are enabled to say that all the reals in an interval constitute a set, namely, a secondary one. Indeed it is clear that all these numbers belonging to S_{θ_1} constitute a set that occurs in S_{θ_1+1} . Similar remarks can be made for neighborhoods.

Lorenzen now succeeds in proving the Heine-Borel theorem, which here has the wording: To every primary covering, that is a primary set of neighborhoods, one can find a finite covering, that is, a finite set of such neighborhoods.

A further important notion is that of a quasi-primary function: That $y = f(x_1, \dots, x_n)$ is quasi-primary means that, whenever x_1, \dots, x_n are primary real numbers, that is, they belong to S_{θ_1} , y is a primary real. Of course every primary function is quasi-primary, but the inverse is not always true. Thus, for example, $x + y$ is quasi-primary but not primary. Indeed the set of all triples (x, y, z) such that $x + y = z$, where x, y, z run through S_{θ_1} , does not belong to S_{θ_1} , but to S_{θ_1+1} .

For the quasi-primary functions Lorenzen proves theorems analogous to the theorems in ordinary analysis concerning functions of real numbers. Thus he proves that a continuous quasi-primary function on a closed interval

is uniformly continuous. Further he proves that the values of such a function on a closed interval are bounded and that the upper and lower bounds are attained. He also proves that such a function takes every value between two of its values. If a quasi-primary function has a derivative for every (primary) real number, then this derivative is again a quasi-primary function.

He also develops a theory of integration, defining first the Riemann integral, later also Lebesgue's. It might seem that a measure theory must be impossible in this system, because by ordinary concepts the measure should be $= 0$ for denumerable sets, and here all sets are denumerable in a sufficiently high layer. However, the distinction between primary and secondary sets makes a definition of measure possible in such a way that the primary sets all get the measure 0 , but not the secondary sets.

This system has one great advantage in distinction to the previous ones, namely, that the objects we are dealing with are all definitely and explicitly given. It is true of course that the unsolvability or even undecidability of many problems remains as before, but we know what we are talking about. In the previous theories it was at any rate not required that our considerations should be restricted to the definable or constructible objects.

16. Some remarks on intuitionist mathematics

Of great interest is the so-called intuitionism which above all is due to the Dutch mathematician L. E. J. Brouwer. This theory is essentially characterized by the requirement that an assertion of the existence of a mathematical object must contain a means of finding or constructing such an object. Further, the use of such a formal logical principle as "tertium non datur" is only justified, if we have a decision procedure. The intuitionist critique of classical mathematics is similar to the critique of Kronecker who also declared that a great part of ordinary mathematics was only words. It would lead too far, however, if I should give in these lectures a detailed exposition of the intuitionist foundation of mathematics. I must confine my exposition here to a few remarks which I hope will give an idea of the intuitionist way of reasoning.

The conjunction $p \ \& \ q$ retains its usual meaning also in intuitionist logic. The disjunction $p \vee q$ can be asserted if and only if either p can be asserted or q can. The negation $\neg p$ shall mean that the assumption p leads to a contradiction. The implication $p \rightarrow q$ means that we are in possession of a certain construction which will furnish a proof of q as soon as a proof of p is available. The assertion $(x)p(x)$ is justified if we possess a schema showing the property $p(x)$ for an arbitrary x , and $(E(x)p(x))$ can be asserted if we know an x with the property p or at least have a method for constructing such an x .

Since we have no general method to prove either p or $\neg p$, the tertium non datur, $p \vee \neg p$, is not generally valid. It can be proved that $p \rightarrow \neg \neg p$ is generally true, but not the inverse implication. Such differences in the propositional logic cause differences in predicate logic of course. As an interest-