

ABSTRACT SET THEORY

by

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1. Historical remarks. Outlines of Cantor's theory

Almost 100 years ago the German mathematician Georg Cantor was studying the representation of functions of a real variable by trigonometric series. This problem interested many mathematicians at that time. Trying to extend the uniqueness of representation to functions with infinitely many singular points he was led to the notion of a derived set. This was not only the beginning of his study of point sets but led him later to the creation of transfinite ordinal numbers. This again led him to develop his general set theory. The further development of this, the different variations or modifications of it that have been proposed in more recent years, the discussions and criticisms with regard to this subject, will constitute the contents of my lectures on set theory.

One ought to notice that there have been some anticipations of Cantor's theory. For example B. Bolzano wrote a paper with the title: *Paradoxien des Unendlichen* (1951) (*Paradoxes of the Infinite*), where he mentioned some of the astonishing properties of infinite sets. Already Galilei had noticed the remarkable fact that a part of an infinite set in a certain sense contained as many elements as the whole set. On the other hand it ought to be remarked that about the same time that Cantor exposed his ideas some other people were busy in developing what we today call mathematical logic. These investigations concerned among other things the fundamental notions and theorems of mathematics, so that they should naturally contain set theory as well as other more elementary or ordinary parts of mathematics. A part of the work of another German mathematician, R. Dedekind, was also devoted to studies of a similar kind. In particular, his book "*Was sind und was sollen die Zahlen*" belongs hereto.

In my following first talks I will however confine my subject to just an exposition of the most characteristic ideas in Cantor's work, mostly done in the years 1874-97.

The real reason for a mathematician to develop a general set theory was of course the fact that in mathematics we often have to do not only with single mathematical objects but also with collections of them. Therefore the study of properties of such collections, even infinite ones, must be of very great importance.

There is one fact to which I would like to call attention. Most of mathematics and perhaps above all the classical set theory has been developed in accordance with the philosophical attitude called Platonism. This standpoint means that we consider the mathematical objects as existing before and independent of our actual thinking. Perhaps an illustrating way of expressing it is to say that when we are thinking about mathematical objects we are looking at eternal preexisting objects. It seems clear that the word "existence"

according to Platonism must have an absolute meaning so that everything we talk about shall either exist or not in a definite way. This is the philosophical background for classical mathematics generally and perhaps in particular for classical set theory. Being aware of this, Cantor explicitly cites Plato. Everybody is used to saying that a mathematical fact has been discovered, not that it has been invented. That shows our natural tendency towards Platonism. Whether this philosophical attitude is justified or not, however, I will not discuss now. It will be better to postpone that to a later moment.

When Cantor developed his theory of sets he liked of course to conceive the notion "set" as general as possible. He therefore desired to give a kind of definition of this notion in accordance with this most general conception. A definition in the proper sense this could not be, because a definition in the proper sense means an explanation of a notion by means of more primitive or previously defined notions. However, it is evident that the notion "set" is too fundamental for such an explanation. Cantor says that a set is a collection of arbitrary well-defined and well-distinguished objects. What is achieved, perhaps, by this explanation is the emphasizing that there shall be no restriction whatever with regard to the nature of the considered objects or to the way these objects are collected into a whole. Taking the Platonist standpoint, it is clear that this whole, the collection, must itself again be considered as one of the objects the set theory talks about and therefore can be taken as an object in other collections. This is indeed clear, because there are no restrictions as to the nature of the objects.

Now we are very well acquainted with sets in daily life. These sets are finite, but I shall not now enter into the distinction between finite and infinite sets. The most important mathematical property of the finite sets is the number of their elements. By the way I write

$$m \in M,$$

expressing that m is an element of or belongs to M . Indeed this notation is used everywhere in the literature. If we shall compare two finite sets M and N with regard to number, we may do that in the way of pairing off the elements by distributing as far as possible the elements of M and N into disjoint pairs. Let us for simplicity assume M and N disjoint, that is, without common elements. If it is possible to distribute the elements of M and N into disjoint pairs (m,n) , $m \in M$, $n \in N$, such that all $m \in M$ and all $n \in N$ occur in these pairs, then it is evident that there are just as many elements m in M as elements n in N . If, however, we may build a set of pairs (m,n) such that all m occur, but not all n , then in the case of finite sets M possesses less elements than N . It is clear that it must be possible to compare sets by considering such sets of disjoint pairs in the case of infinite sets as well. This leads to one of the most important notions not only in the classical set theory but also in ordinary mathematics, namely, the notation of one-to-one correspondence or mapping. We say that f is a one-to-one correspondence between the sets M and N , if f is a set of mutually disjoint pairs (m,n) such that each $m \in M$ and each $n \in N$ occur in one of the pairs. In order to be able to take into account the case that M and N have some common elements, it is necessary to replace the simple notion pair $\{a,b\}$, which means the set containing a and b as elements, with the notion ordered pair (a,b) , which can be conceived as $\{\{a,b\}, \{a\}\}$. However I will here, to begin with, use the notion ordered pair, triple etc. as known ideas without worrying about an analysis of them.

Possessing the notion one-to-one correspondence or mapping, we may obtain this generalisation of the number concept:

M and N have the same cardinal number, if a mapping f exists of M on N. This circumstance is written $M \sim N$.

Cantor says that the cardinal number \overline{M} of M is what remains, if we make an abstraction with regard to the individual characters of its elements. This definition is made much clearer by Russell, who says that \overline{M} is the set of all sets N being $\sim M$.

Further, this definition of the relation \leq between cardinals was natural: $\overline{M} \leq \overline{N}$ if M is \sim a subset of N. Further $\overline{M} < \overline{N}$ if M \sim a subset of N, but N not $\sim M$.

Let us again introduce some notations. I shall write $A \subseteq B$ when the set A is contained in B, and $A \subset B$, if A is contained in B, but not inversely B in A. Then we know that for the finite sets as we encounter them in everyday life, there is never a mapping of the set on a proper part of itself. Thus, if M is finite,

$$N \subset M \rightarrow \overline{N} < \overline{M}.$$

Dedekind uses this as a definition of finite sets: A set M is finite, if it is not \sim any proper part N of itself.

On the other hand, if we look at the simplest infinite set we know, namely the number series 0,1,2,..., then it is easily seen that this set admits a mapping on a proper part of itself, for example, the set of positive integers 1,2,... It is said that already Galilei wondered about this, and found it an astonishing property of an infinite set, that a proper part of it could in a certain sense be said to possess just as many elements as the whole set.

Some further notations may suitably be mentioned now. We write

$$M \cup N \text{ resp. } M \cap N$$

as the notation for the union of M and N, resp. the intersection of M and N. Thus $M \cup N$ contains as elements all the elements of M and N and only these, while $M \cap N$ contains as elements just the common elements of M and N. If $M \cap N$ is empty, i.e., M and N are disjoint, I shall often write $M + N$ instead of $M \cup N$. Both operations can be generalized very far. Let T be a set whose elements are again sets A,B,C,... Then I will write ST and DT as denotations for the union of all sets A,B,C,..., resp. the intersection of all A,B,C,...

In natural analogy to the arithmetic of finite sets, addition of cardinals is defined thus:

$\overline{M} + \overline{N} = \overline{M + N}$, if M and N are disjoint, and generally, if A,B,C,..., constituting all elements of T, are disjoint in pairs, then \overline{ST} is said to be the sum of the cardinals of all the elements A,B,C,... of T.

These definitions are justified by the simple theorem:

If $A \sim A'$, $B \sim B'$, $C \sim C'$,, any two of A,B,C,... as well as any two of A',B',C' being disjoint, then $ST \sim ST'$, T' denoting the set of all A',B', C',

The proof of this theorem is of course quite trivial, but as we shall see later, the so-called axiom of choice must be applied.

Multiplication of cardinals is defined in the simplest way by taking again disjoint sets. If M and N are two such sets, we may build the set of all pairs $\{m,n\}$, where m and n run independently through all elements of M and N . This set will be written $M \cdot N$. It is again easy to see that if $M \sim M'$, $N \sim N'$, where M' and N' are again disjoint, then the set $M' \cdot N'$ of all pairs $\{m', n'\}$ will be $\sim M \cdot N$. Therefore we may define an operation on the cardinal numbers called multiplication by putting

$$\overline{M} \cdot \overline{N} = \overline{M \cdot N}.$$

This can then in an obvious way be generalized to the general case, where T is a set of mutually disjoint sets A, B, C, \dots . Letting PT denote the set of all sets which consists of just one element from each of A, B, C, \dots , we say that \overline{PT} is the product of the cardinal numbers $\overline{A}, \overline{B}, \overline{C}, \dots$. Using ordered pairs we may define the Cartesian product $M \times N$ of M and N . This is the set of all ordered pairs (m,n) such that $m \in N, n \in N$. Of course $\overline{M \times N} = \overline{M} \cdot \overline{N}$.

A natural assumption after the discovery that the natural number series is \sim proper parts of itself was that many sets of mathematical objects ought to possess the same cardinal number as the number series, even if they contained the latter as a proper subset. This assumption Cantor proved to be correct. Quite trivial is the remark that the series of integers $>$ a certain negative integer is of the same cardinality as the series of non-negative integers. A little more remarkable is the fact that this is true of the set of all rational integers, negative, positive or zero. The last fact is verified by writing the integers for ex. in this order:

$$0, -1, 1, -2, 2, -3, 3, \dots$$

Or in other words, if we put for $x \geq 0$

$$y = 2x$$

and for $x < 0$

$$y = -2x - 1,$$

then this function y of x furnishes a 1-1- correspondence between all integers on the one hand and the non-negative ones on the other hand.

Let P denote the set of all pairs of non-negative integers, while N is the set of the non negative integers themselves. Then one finds that

$$z = \binom{x+y+1}{2} + \binom{x}{1}$$

yields a one-to-one correspondence between P and N . Indeed to every pair (x, y) corresponds a unique value of z and to each value of z there is only one pair of non-negative integers x, y such that the above equation is fulfilled.

Similarly the set of all ordered n -tuples (x_1, \dots, x_n) all $x_i \in N$ has the same cardinal number as N . All sets possessing this cardinal number are called denumerable.

Turning to the more often considered sets of numbers, Cantor proved that the set of all rational numbers is denumerable. We can take the rationals in the form $\frac{a}{b}$, $b > 0$, a and b coprime integers. Then we arrange the rationals so that $|a| + b$ successively takes the values $1, 2, 3, \dots$ and the $\frac{a}{b}$ for

which $|a| + b$ has the same value we arrange according to their magnitude. Thus we obtain the sequence

$$\frac{0}{1}, \frac{-1}{1}, \frac{+1}{1}, \frac{-2}{1}, \frac{-1}{2}, \frac{+1}{2}, \frac{+2}{1}, \frac{-3}{1}, \frac{-1}{3}, \frac{+1}{3}, \frac{+3}{1}, \frac{-4}{1}, \frac{-3}{2}, \frac{-2}{3}, \frac{-1}{4}, \frac{+1}{4}, \frac{+2}{3}, \frac{+3}{2}, \frac{+4}{1},$$

containing all the rational numbers.

Cantor proved also that even the set of all algebraic numbers is denumerable. This can be done in the simplest way as follows. Every algebraic number is a root in an irreducible equation $a_n x^n + \dots + a_0 = 0$ for some n , the a_0, \dots, a_n being integers with 1 as g.c. div. Now we can arrange the n -tuples a_n, \dots, a_0 in a sequence by taking the successively increasing values of

$$m = |a_n| + \dots + |a_0| + n.$$

Those with the same m we can take according to increasing values of n , and for those with the same value of m and n , which are only finite in number, we arrange the corresponding roots first according to their absolute value and finally those which have the same absolute value we arrange according to increasing amplitude.

One might get the impression that all infinite sets were denumerable. However, Cantor proved that the set of all real numbers, even all reals between 0 and 1, is not denumerable. His proof is performed by the diagonal method, called after him in the literature: Cantor's diagonal method.

We know that every real number ≥ 0 and < 1 can be written as a decimal fraction

$$0. a_1 a_2 \dots$$

and this decimal fraction is unique, if we require that there shall not occur only 9's from a certain place on. Then let us assume that

$$\alpha_1 = 0. a_{11} a_{21} \dots$$

$$\alpha_2 = 0. a_{12} a_{22} \dots$$

.....

were all reals ≥ 0 and < 1 . Let the real number β be $0.b_1 b_2 \dots$, where b_r for each r is the next digit after a_{rr} (0 when a_{rr} is 9) except when all a_{ji} from a certain i on are all 8; then we take the b_i as 7 for example. Then obviously $0 \leq \beta < 1$, while β is \neq every α_i . Thus the set of reals ≥ 0 and < 1 is not denumerable.

This means that in Cantor's theory we have to do with different infinite cardinals. It is now natural to ask, if spaces of higher dimensions would yield greater cardinals. Cantor showed that this is not the case. His result that e.g., a plane could be mapped onto a line or say onto a segment of a straight line astonished the mathematical world at that time. I shall now expose a proof of the fact that the 1. quadrant of a plane, say in Cartesian coordinates the set of all pairs of positive real numbers x, y , can be mapped on the real numbers $z > 0$. The definition of such a mapping is particularly easy when we make use of the development of reals in continued fractions. Any positive real number α can be developed thus:

$$\alpha = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \dots}}$$

where $a_0 \geq 0$, while a_1, a_2, \dots are all ≥ 1 . Now I define the correspondence so that the points (x, y) , where x and y are both irrational, are mapped on the irrational $z > 2$, the points (x, y) , where x is irrational, y rational, are mapped on the irrational z such that $1 < z < 2$, the points (x, y) , where x is rational, y irrational, are mapped on the irrationals $z < 1$, and finally the rational points (x, y) are mapped on the rationals z . This mapping is defined as follows. As often as x and y are both irrational, their continued fractions being

$$x = x_0 + \frac{1}{x_1 + \frac{1}{x_2 + \dots}}, \quad y = y_0 + \frac{1}{y_1 + \frac{1}{y_2 + \dots}}$$

the corresponding z shall be

$$z = x_0 + 2 + \frac{1}{y_0 + 1 + \frac{1}{x_1 + \frac{1}{y_1 + \frac{1}{x_2 + \frac{1}{y_2 + \dots}}}}}$$

If x is irrational, but y rational, the corresponding z shall be

$$z = 1 + \frac{1}{n + \frac{1}{x_0 + 1} + \frac{1}{x_1 + \frac{1}{x_2 + \dots}}},$$

where n is the number given to y in an enumeration of all rationals. If x is rational, y irrational, the corresponding z is, when n is the number of x ,

$$z = \frac{1}{n + \frac{1}{y_0 + 1 + \frac{1}{y_1 + \frac{1}{y_2 + \dots}}}}$$

Finally the (x, y) where x and y are both rational and > 0 , are mapped in an arbitrary way on the rational numbers $z > 0$.

Cantor also proved generally that the set UM of subsets of a set M was of higher cardinality than M ; however, I will talk about this theorem later.