

2. Ordered sets. A theorem of Hausdorff.

One obtains a more complete idea of Cantor's work by studying his theory of ordered sets. As to the notion "ordered set" this is nowadays mostly defined in the following way:

A set M is ordered by a set $P \subseteq M^2$, if and only if the following statements are valid:

- 1) No pair (m, m) , $m \in M$, is $\in P$.
- 2) For any two different elements m and n of M either $(m, n) \in P$ or $(n, m) \in P$ but not both at the same time.
- 3) for all $m, n, p \in M$ we have $(m, n) \in P$ & $(n, p) \in P \rightarrow (m, p) \in P$ (transitivity).

As often as $(m, n) \in P$, we also say m is less than n or m precedes n , written $m < n$.

If M and N are ordered sets and there exists a one-to-one order-preserving correspondence between them, Cantor said that they were of the same order type and wrote $M \approx N$. They are also called similar. Evidently two ordered sets of the same order type possess the same cardinal number; but the inverse need not be the case. Only for finite sets is it so that two ordered sets of the same cardinality are also of same type. Cantor denoted by \bar{M} the order type of M .

That two infinite ordered sets possessing the same cardinal number may have different order types is seen by the simple example of the set of positive integers on the one hand and that of the negative integers on the other. Both sets are denumerable, but obviously not ordered with the same type, because the former has a first member, which the other has not, whereas the latter has a last member, which the former does not possess. Cantor studied to a certain extent the denumerable types, also types of the same cardinality as the continuum, but above all he studied the so-called well-ordered sets. In this short survey of Cantor's theory I shall only mention some of the most remarkable of his results and add a theorem of Hausdorff.

It will be necessary to define addition and multiplication of ordered sets. If A and B are ordered by P_A and P_B while A and B are disjoint, the sum set $A + B$ will be ordered by $P_A + P_B + A \cdot B$. We have of course to distinguish between $A + B$ and $B + A$. This addition may be extended to the case of an ordered set T of ordered sets A, B, C, \dots which are mutually disjoint. Indeed the union (or sum) ST will then be ordered by the sum of the sets P_A, P_B, P_C, \dots and the products $X \cdot Y$ when (X, Y) run through all pairs which are the elements of the ordering set P_T for T .

By the product of two ordered sets A and B we understand $A \cdot B$ ordered lexicographically: that means that a_1, b_1 precedes a_1, b_2 if either a_1 precedes a_2 , or $a_1 = a_2$, but b_1 precedes b_2 . This definition also admits generalization, but that will not be necessary just now.

If a 1-1-correspondence exists between the ordered sets M and N such that the order is reversed by the correspondence, then \bar{N} is said to be the inverse order type of \bar{M} . For example the order type of the set of negative integers is the inverse of the type of the positive integers. Cantor denotes the inverse of the order type α by α^* . Thus ω and ω^* denote the types of the sets of positive and of negative integers.

An interesting class of ordered sets are the dense ones. An ordered set is called dense, if there is always an element between two arbitrary ones. The simplest example is the set of rational numbers in their natural order. This set is also open, that means that there is no first and no last member. Now we have the remarkable theorem:

There is one and only one open and dense denumerable ordertype.

Proof. Let $A = \{a_1, a_2, \dots\}$ and $B = \{b_1, b_2, \dots\}$ be two denumerable sets, both open and dense. First we let a_1 correspond to b_1 . Then a_1 divides the remaining elements of A into those $< a_1$ and those $> a_1$. Let a_{m_1} be the a_i with least index $< a_1$ and a_{m_2} the a_i with least index $> a_1$. Either m_1 or m_2 is 2. Letting b_{n_1} be the b_j with least $j < b_1$, while b_{n_2} is the b_j with least $j > b_1$, then either n_1 or n_2 is 2. We let b_{n_1} correspond to a_{m_1} and b_{n_2} to a_{m_2} . Now every remaining a_i from A is either $< a_{m_1}$ or $> a_{m_1}$ but $< a_1$, or $> a_1$ but $< a_{m_2}$ or $> a_{m_2}$, which gives 4 cases. There are 4 corresponding cases for the remaining b_j . Then if a_{m_3} is the a_i with least i such that $a_i < a_{m_1}$ and b_{n_3} the b_j with least j such that $b_j < b_{m_1}$, we let a_{m_3} correspond to b_{n_3} and so on. It is easily seen how we obtain in this way an order-preserving correspondence between the a_i and the b_j . One has only to remark that if a_m is the a_i with the least i which has not already got any corresponding b_j , then it gets one when in the different intervals between the already chosen a_{m_r} the further a_{m_s} are chosen.

We have further:

In an open and dense denumerable set a subset can be found similar to any given denumerable ordered set. This is seen in a similar way as in the proof of the preceding theorem. Indeed if b_1, b_2, \dots are elements of an arbitrary denumerable ordered set while a_1, a_2, \dots is an open and dense denumerable set, then we may map b_1 on a_1 . Then according as $b_2 < b_1$ or $> b_1$ we map b_2 on an element $a'' < a_1$ or $> a_1$. Then b_3 is either less than both b_1 and b_2 or lies between b_1 and b_2 or is greater than both. Respectively we map b_3 on an element a''' having the same order relation to a_1 and a'' and so on.

Let us use the term scattered set for a set having no dense subset. Then an interesting theorem of Hausdorff says that every ordered set is either scattered or the sum of a set T of such sets, where T is dense.

Proof: It is easy to understand that if an interval a to b in an ordered set is scattered and the interval b to c as well, then the whole interval a to c has the same property. Indeed, if $d < e$ both belong to a dense set S then the set of all $x \in S$ such that $d \leq x \leq e$ constitute a dense subset of S , and an eventual dense subset of the interval a to c must either contain at least 2 elements in the interval a to b or at least 2 in the interval b to c . Therefore the statement that the interval between a and b in an ordered set M is scattered is transitive so that we can divide M into classes A, B, C, \dots such that in each class any two different elements furnish a scattered interval. These classes are therefore successive parts of M , each of them scattered. On the other hand, if there are

two different ones A and B, there must always be a C between, else A and B would amalgamate into one class. Thus a set T of the successive scattered parts of M must be dense.

As to the denumerable ordered sets I should like to mention two facts which will be useful when I talk about Cantor's second number class. If a denumerable ordered set M has no first element, then it is cointial with ω^* , and if it has no last element, it is cofinal with ω . These statements mean that we can in the first instance find an infinite sequence of type ω^* in the set such that there is no earlier element than all these in M, and in the second instance we may find an infinite sequence of type ω such that there is no element in M after all these.

Proof: Let in the first case $a_1 \in M$, a_{n_1} be the a_i with least i such that $a_i < a_1$, further a_{n_2} be the a_i with least i such that $a_i < a_{n_1}$, etc. Clearly $1 < n_1 < n_2 < \dots$. If a_m were $<$ every a_{n_r} , then we should have $m > 1, n_1, n_2, \dots$, which is absurd. Similarly in the second case.

Among the ordered sets, the well-ordered ones, namely those possessing a least element in every non-empty subset, are especially important. That well-ordering is equivalent to the principle of transfinite induction is well known. This principle says that if a statement S is always valid for an element of a well-ordered set M when it is valid for all predecessors, then S is valid for all elements of M. Further I ought to mention that the sum of a well-ordered set T of well-ordered sets A,B,C,... is again a well-ordered set. If T is denumerable and a denumeration is simultaneously given for each element A,B,C,... of T, then the sum is a well-ordered denumerable set. Also the product of two well-ordered sets is again well-ordered.

The order types of the well-ordered sets are called ordinal numbers. These ordinals Cantor has introduced by a creative process which is very characteristic of his way of thinking. I will now give an exposition of this creative process.

He begins with the null set 0 containing no element. Then since this 0 is an object of thought he has obtained one thing which he denotes by 1. (We may think of 1 as the set {0}, see the later axiomatic theory). Now, conceiving 0 and 1 as ordinals he has the right to write $0 < 1$. Then he has this set of two ordinals which represents the ordinal 2. Having obtained $0 < 1 < 2$ he has an ordered set representing the ordinal 3. Now he has $0 < 1 < 2 < 3$ which furnishes a well-ordered set with 4 elements, etc. Now he thinks this process continued infinitely so that he obtains the set of all positive integers $0 < 1 < 2 < \dots$. This well-ordered set, however, represents an infinitely great ordinal ω . Then he has

$$0 < 1 < 2 < \dots < \omega,$$

a set containing all finite integers together with ω . This is a well-ordered set representing a greater ordinal than ω , denoted by $\omega + 1$. Proceeding in this way he obtains after a while

$$0 < 1 < 2 < \dots < \omega < \omega + 1 < \omega + 2 < \dots,$$

a well-ordered set consisting of two infinite series of increasing ordinals. This set represents a still greater ordinal written as $\omega + \omega$.

It is evident that all the infinite sets hitherto introduced are denumerable. But now Cantor collects all ordinals of denumerable well-ordered sets. This set represents an ordinal that is not denumerable. Strictly speaking the axiom of choice is being taken into account here, but Cantor uses that as an evident principle without even being aware of it. According to this principle we have that a denumerable set of denumerable or finite sets has a denumerable union. Now let us assume that the ordinals of finite and denumerable sets constitute a denumerable set. Then this set is cofinal with ω , because there is evidently no greatest ordinal of this kind. Thus we may assume that $\alpha_1 < \alpha_2 < \alpha_3 < \dots$ is a sequence of type ω , such that every denumerable ordinal is \leq some α_r . However we could then find finite or denumerable ordinals β_1, β_2, \dots such that

$$\alpha_2 = \alpha_1 + \beta_1, \alpha_3 = \alpha_2 + \beta_2, \dots ;$$

but now the ordinal

$$\gamma = \alpha_1 + \beta_1 + \beta_2 + \dots$$

must be denumerable. Nevertheless γ is clearly $>$ every α_r , so that we get a contradiction. Therefore the sequence of all finite and denumerable ordinals represents a non-denumerable ordinal. This was by Cantor denoted by Ω .

Cantor used the first letter aleph, written \aleph , of the Hebraic alphabet with indices to denote the cardinal numbers of well-ordered sets. The cardinal of ω , that is the cardinal number of the denumerable sets he called \aleph_0 , the cardinal of Ω he called \aleph_1 . He proved that every subset of Ω is either finite or has the cardinal \aleph_0 or the cardinal \aleph_1 . Indeed if we have a subset of Ω we may enumerate successively the elements of the subset by the elements $0, 1, 2, \dots, \omega, \omega + 1, \dots$ of Ω and then either this enumeration will stop with a finite number n or it will stop with some $\alpha < \Omega$ or it does not stop, so that the subsequence also has the ordinal Ω .

The finite ordinals are also called those of the first number class, the denumerable ones those of the second class. Now Cantor again collects the ordinals of cardinal number \aleph_1 and proves similarly that they constitute a sequence of still greater cardinality \aleph_2 . There is no cardinal between \aleph_1 and \aleph_2 . The ordinals belonging to well-ordered sets whose cardinal number is \aleph_1 are said to be the numbers of the third number class. In this way he continues and obtains an increasing infinite sequence of alephs

$$\aleph_0, \aleph_1, \aleph_2, \dots,$$

each \aleph_{n+1} being the cardinal number of the set of all ordinals represented by well-ordered sets with cardinal number \aleph_n . These latter ordinals are those of class $n + 2$.

But now he collects all ordinals belonging to all the classes with finite number. Then he obtains a set with a cardinal number which is suitably denoted \aleph_ω , being $>$ every \aleph_n , n finite, while there is no cardinal between the \aleph_n and this \aleph_ω . From \aleph_ω he then derives $\aleph_{\omega+1}, \aleph_{\omega+2}$ etc. Quite generally there is an \aleph_α for every ordinal α .

It must be conceded that Cantor's set theory, and in particular his creation of ordinals, is a grandiose mathematical idea. But what was at that time the reaction of the mathematical world to all this? In the first instance the

reaction was rather unfavourable. No wonder, these ideas were too new and too strange. However, very soon the reaction got favourable for two reasons: 1) Cantor's way of thinking was of the same nature as, for example, Cauchy's and Weierstrass's treatment of analysis and the theory of functions, 2) Many of the notions introduced by Cantor were useful in ordinary mathematics. There were, however, also some opponents, above all Kronecker and Poincaré. Kronecker did not only attack Cantor's theory of sets but also most of ordinary analysis. He required decidable notions. Poincaré's main objection was that in set theory so called non-predicative definitions are used which according to him (and also Russell) are logically objectionable. The situation for Cantor's theory became indeed very much changed after 1897. In this year the Italian mathematician Burali-Forti discovered that the theory of transfinite ordinals leads to a contradiction. According to the Platonist point of view the existing ordinals are well-defined and well-distinguished objects such that they, according to Cantor's definition, should constitute a set. This set is well-ordered, therefore it represents an ordinal. However the ordinal represented by a well-ordered set of ordinals is always greater than all ordinals in the set. Thus we obtain an ordinal which is greater than all ordinals, which is absurd.

Another still better known antinomy was discovered a few years later (1903) namely Russell's. Ordinary sets are not elements of themselves. According to platonism the existing sets which are not elements of themselves ought to constitute a set U. We have then the logical equivalence

$$x \in \bar{x} \leftrightarrow x \in U.$$

If, however, we put here U instead of x, which should be allowed because the equivalence should be generally valid, we get

$$U \in \bar{U} \leftrightarrow U \in U$$

which of course is absurd. Also Cantor's theorem that the set UM of all subsets of M is of greater cardinality than M leads to an absurdity when we ask if there is a greatest cardinal or not. Indeed according to this theorem there is no greatest cardinal. But the union of all sets ought on the other hand to have the greatest possible cardinal number.