

$$\{a_0, b_0, c_0, \dots\},$$

where $a_0 \in A' - A_1, b_0 \in B' - B_1, \dots$. However this element cannot correspond to any element of ST. Indeed it cannot be mapped on an element of A_0 , for example, because if it could, a_0 would have to be one of the elements of A_1 .

4. The well-ordering theorem

After all this I shall now prove, by use of the choice principle, that every set can be well-ordered. First I shall give another version of the notion "well-ordered", different from the usual one.

We may say that a set M is well-ordered, if there is a function R, having M as domain of the argument values and UM as domain of the function values, such that if $N \supset 0$ is arbitrary and ϵUM , there is a unique $n \in N$ such that $N \subseteq R(n)$. I have to show that this definition is equivalent to the ordinary one. If M is well-ordered in the ordinary sense, then every non-void subset N has a unique first element. Then it is clear that if $R(n), n \in M$, means the set of all $x \in M$ such that $n \leq x$, the other definition is fulfilled by this R. Let us, on the other hand, assume that we have a function R of the said kind. Letting N be {a}, one sees that always $a \in R(a)$. Let N be {a,b}, $a \neq b$. Then either a or b is such that $N \subseteq R(a)$ resp. $R(b)$. If $N \subseteq R(a)$, then we put $a < b$. Since then N is not $\subseteq R(b)$, we have $a \notin R(b)$. Now let $b < c$ in the same sense that is, $c \in R(b), b \notin R(c)$. Then it is easy to see that $a < c$. Indeed we shall have $\{a,b,c\} \subseteq$ either $R(a)$ or $R(b)$ or $R(c)$, but $b \notin R(c), a \notin R(b)$. Hence $\{a,b,c\} \subseteq R(a)$ so that $\{a,c\} \subseteq R(a)$, i.e. $a < c$. Thus the defined relation $<$ is linear ordering. Now let N be an arbitrary subset of M and n be the element of N such that $N \subseteq R(n)$. Then if $m \in N, m \neq n$, we have $m \in R(n)$, which means that $n < m$. Therefore the linear ordering is a well-ordering.

Theorem 10. *Let a function ϕ be given such that $\phi(A)$, for every A such that $0 \subset A \subseteq M$, denotes an element of A. Then UM possesses a subset \mathfrak{M} such that to every $N \subseteq M$ and $\supset 0$ there is one and only one element N_0 of \mathfrak{M} such that $N \subseteq N_0$ and $\phi(N_0) \in N$.*

Proof: I write generally $A' = A - \{\phi(A)\}$. I shall consider the sets $P \subseteq UM$ which, like UM, possess the following properties

- 1) $M \in P$
- 2) $A \in P \rightarrow A' \in P$ for all $A \subseteq M$
- 3) $T \in P \rightarrow DT \in P$.

These sets P constitute a subset \mathfrak{T} of UUM. They are called Θ -chains by Zermelo. I shall show that the intersection $D\mathfrak{T}$ of all elements of \mathfrak{T} is again a Θ -chain, that is, $D\mathfrak{T} \in \mathfrak{T}$. It is seen at once that $D\mathfrak{T}$ possesses the properties 1) and 2). Now let $T \subseteq D\mathfrak{T}$. Then, if $P \in \mathfrak{T}$, we have $T \subseteq P$, and since 3) is valid for P, also $DT \in P$. Since this is true for all P, we have $DT \in D\mathfrak{T}$ as asserted. Thus I have proved that $D\mathfrak{T} \in \mathfrak{T}$.

In the sequel I put $D\mathfrak{C} = \mathfrak{M}$ and I assert that \mathfrak{M} has the property mentioned in the theorem. Obviously \mathfrak{M} is the least \emptyset -chain. Let $O \subset N \subseteq M$, and let N_0 be the intersection of all $Q \in M$ for which $N \subseteq Q$, then $N \subseteq N_0$. Further $\phi(N_0) \in N$, because otherwise $N'_0 = N_0 - \{\phi(N_0)\}$ would still contain N and be $\in \mathfrak{M}$, which is a contradiction, since this would mean that N_0 is contained in $N_0 - \{\phi(N_0)\}$.

Thus we have proved the first half of the theorem. The proof of the latter half is considerably more laborious. It will be suitable first to prove the following:

Lemma. Let $A \in \mathfrak{M}$ have the property that for every $X \in \mathfrak{M}$ either $X \subset A$ or $X = A$ or $A \subset X$.

Then A' possesses the same property.

By the way, we may notice that such an A exists, M having this property.

Proof: If $X \in \mathfrak{M}$ is such that $A = X$ or $A \subset X$, then $A' \subset X$. Therefore, we only need to consider the case $X \subset A$. The question is whether some $\mathfrak{U} \in \mathfrak{M}$ could exist such that $\mathfrak{U} \subset A$ but \mathfrak{U} not $\subseteq A'$, or in other words, $\phi(A)$ still $\in \mathfrak{U}$. I will denote by \mathfrak{M}^* the subset of \mathfrak{M} which remains after having removed all these \mathfrak{U} from \mathfrak{M} . I shall show that \mathfrak{M}^* is a \emptyset -chain.

1) $M \in \mathfrak{M}^*$ because $M \in \mathfrak{M}$ and M is not possibly a \mathfrak{U} . Indeed each \mathfrak{U} is $\subset A$.

2) Let $B \in \mathfrak{M}^*$. If $A \subset B$, then B' is not $\subset A$ so that B' is not a \mathfrak{U} . On the other hand $B' \in \mathfrak{M}$, since $B \in \mathfrak{M}$. Then $B' \in \mathfrak{M}^*$ in this case.

If $A = B$, then $B' = A'$ so that $\phi(A) \bar{\in} B'$, whence again B' is not a \mathfrak{U} so that $B' \in \mathfrak{M}^*$. Finally, let $B \subset A$. Then $\phi(A)$ must be $\bar{\in} B$; otherwise B would be a \mathfrak{U} against the supposition $B \in \mathfrak{M}^*$. But then a fortiori $\phi(A) \bar{\in} B'$, so that B' is not a \mathfrak{U} . Therefore $B' \in \mathfrak{M}^*$.

3) Let $T \subseteq \mathfrak{M}^*$. Should DT be a \mathfrak{U} , we would have

$$(DT \subset A) \ \& \ (\phi(A) \in DT).$$

Then $\phi(A)$ is \in every element C of T . Since every C is not a \mathfrak{U} , we must have $C \not\subset A$ for every $C \in T$ and thus, because of the supposed property of A , $A \subseteq C$ for all $C \in T$, whence $A \subseteq DT$, so that DT is no \mathfrak{U} . Hence $DT \in \mathfrak{M}^*$.

However, since \mathfrak{M} is the minimal \emptyset -chain and \mathfrak{M}^* is a \emptyset -chain $\subseteq \mathfrak{M}$, we have $\mathfrak{M}^* = \mathfrak{M}$, which means that the elements \mathfrak{U} do not exist. This proves our lemma.

Now let \mathfrak{M}_1 be the subset of \mathfrak{M} consisting of all $A \in \mathfrak{M}$ such that for every $X \in \mathfrak{M}$ we have either $X \subset A$ or $X = A$ or $A \subset X$. I shall show that \mathfrak{M}_1 is a \emptyset -chain, so that it coincides with \mathfrak{M} .

1) M is $\in \mathfrak{M}_1$. This is evident since every $X \in \mathfrak{M}$ is $\subseteq M$.

2) If $A \in \mathfrak{M}_1$, then $A' \in \mathfrak{M}_1$. That is just the lemma proved above.

3) Let T be $\subseteq \mathfrak{M}_1$. Then for every $N \in T$ and every $X \in \mathfrak{M}$ we have either $N \subseteq X$ or $X \subseteq N$. Let X be an arbitrary element of \mathfrak{M} . Then either there is an element N of T such that $N \subseteq X$, and then $DT \subseteq X$, or we have for all $N \in T$ that $X \subseteq N$, whence $X \subseteq DT$. Thus $DT \in \mathfrak{M}_1$.

Hence it follows that \mathfrak{M}_1 is a \emptyset -chain and therefore $= \mathfrak{M}$. This means that if A and B are $\in \mathfrak{M}$, we always have one of the three cases $A \subset B$, $A = B$, $B \subset A$. Further it ought to be noticed that if $B \subset A$, then $B \subseteq A'$, else we should have $A' \subset B$, which obviously is impossible when $B \subset A$.

All this makes it now possible to prove the latter half of our well-ordering theorem; namely that if $N \neq \emptyset$ is $\subseteq M$ there is only one $N_0 \in \mathfrak{M}$ such that $\phi(N_0) \in N$ and $N \subseteq N_0$. We have seen that there is such an N_0 . Every element P of \mathfrak{M} such that $P \subset N_0$ is $\subseteq N'_0$, so that $\phi(N_0) \notin P$, whence N is not $\subseteq P$. Every other element P of \mathfrak{M} is such that $N_0 \subset P$, whence $N_0 \subseteq P'$, whence again $\phi(P) \notin N_0$ so that also $\phi(P) \in N$. Thus N_0 is the only element of \mathfrak{M} with the two properties $N \subseteq N_0$ and $\phi(N_0) \in N$.

We can now define a function R from M to \mathfrak{M} thus: As often as $N \in \mathfrak{M}$ & $\phi(N) = m$, we write $N = R(m)$. It follows in particular from the theorem just proved that for every $m \in M$ a unique $N \in \mathfrak{M}$ exists such that $\{m\} \subseteq N$ while $m = \phi(N)$ so that $N = R(m)$. Thus R and ϕ are inverse functions.

It is easy to see that ϕ maps \mathfrak{M} onto M . Indeed, if $N_1 \subset N_2$, then $N_1 \subseteq N'_2$ so that $\phi(N_2) \notin N_1$ whereas $\phi(N_1) \in N_1$. Hence $\phi(N_1) \neq \phi(N_2)$ so that ϕ furnishes a one-to-one correspondence between \mathfrak{M} and M . Therefore there exists an inverse function mapping M onto \mathfrak{M} , that is the function R .

Before entering into a more thorough treatment of the well-ordered sets and the ordinals I would like to remind you of some notations I shall use. An initial part A of an ordered set \emptyset shall mean a subset A of \emptyset such that if $x \in A$ and $y < x$, then always also $y \in A$, or in logical symbols $(x)(y)((x \in A) \& (y < x) \rightarrow y \in A)$. Similarly a terminal part C of \emptyset is to be understood. An interval B shall be used in the meaning $B \subseteq \emptyset$ and $(x)(y)(z) (x \in B \& y \in B \& (x < z) \& (z < y) \rightarrow z \in B)$. These parts A, B, C may be closed or open, for example an initial part A may have a last element, then it is said to be closed, or not, then it is open. An interval B may be open or closed or open to the left, closed to the right or inversely. It ought to be noticed that the union of a set of initial parts is again an initial part.

If $\sigma \in \emptyset$, the set of all $x < \sigma$ constitute an initial part. This I shall call the initial section corresponding to σ . It ought to be noticed that if \emptyset is well-ordered, every initial part which is not \emptyset itself is an initial section.

Theorem 11. *Let a well-ordered set M be mapped into itself by a function f which preserves the order, that is $a < b \rightarrow f(a) < f(b)$ for all a and $b \in M$. Then for all $m \in M$ we have $m \leq f(m)$.*

Proof: Let us assume that the theorem is not true. That would mean that the subset N of M of all those x for which $x > f(x)$ was not void. Let m denote the least element of N . Then we should have

$$m > f(m) = m',$$

and because $m' \in N$,

$$m' \leq f(m').$$

However, since f is order-preserving and $m > m'$, we should have $f(m) > f(m')$, that is $m' > f(m')$.

It follows that if M is mapped by a function f onto M with preservation of order, then $f(x) = x$ for all x . Indeed, according to the theorem, we have $f(x) \leq x$ and $f^{-1}(x) \leq x$, that is, $x = f(x)$.

From this it again follows that if a well-ordered set M is mapped with preservation of order onto another well-ordered set M' , then this mapping is unique. Indeed if f and g both map M onto M' , then fg^{-1} maps M onto M so that $fg^{-1}(x)$ is x and therefore $f(x) = g(x)$ for all x .

Theorem 12. *If M is mapped by f with preservation of order into an initial part A of itself, then $A = M$ and the mapping is the identical one. We may also say: M cannot be mapped onto an initial section of itself.*

Proof: Let f map M onto A , A initial part of M . Then no element m of M can be $>$ every element x of A , because $f(m)$ should belong to A so that $m > f(m)$, which contradicts the previous theorem. Thus every $m \in M$ is \leq an $x \in A$, whence $m \in A$, that is, $A = M$.

Noticing that an initial part of a well-ordered set M is either M itself or a section of M , we have that if $M \simeq N$ (meaning M and N are similar), then M is neither $\simeq N_1$ nor $N \simeq M_1$, M_1 and N_1 denoting sections of M resp. N .

Theorem 13. *Let M and N be well-ordered sets. Then either $M \simeq N_1$, N_1 a section of N or $M \cong N$ or $M_1 \cong N$, M_1 a section of M .*

Proof: Let I be the set of all initial parts of M that are similar to initial parts of N constituting a set J . Then the union SI is in an obvious way similar to SJ . Now either SI must be $= M$ or $SJ = N$. Else SJ will be the section belonging to an element i of M and SJ the section delivered by $j \in N$. But then $SI + \{i\}$ would be similar to $SJ + \{j\}$ which contradicts the definition of I . Now, if $SI = M$, either $M \cong N$ or $M \cong$ a section N_1 of N according as SJ is N or N_1 , else SI is a section M_1 of M while $SJ = N$ so that $M_1 \cong N$.

5. Ordinals and alephs

It is now natural to say that an ordinal α is $<$ an ordinal β , if α is the order-type of a well-ordered set A , β the type of B , such that A is similar to an initial section of B . It is clear that $\alpha < \beta$ & $\beta < \gamma \rightarrow \alpha < \gamma$ and that $\alpha < \beta$ excludes $\beta < \alpha$. Thus all ordinals are ordered. However, this ordering is also a well-ordering. Let us namely consider an arbitrary set or even class C of well-ordered sets. Let M be one of the sets in C . Its ordinal number μ may be the least of all represented by the considered sets. If not there are other sets in C which are similar to sections of M . These sections are furnished by elements of M and among these there is at least one. The corresponding initial section represents then the least ordinal of all furnished by the sets in C .

Theorem 14. *A terminal part or an interval of a well-ordered set is similar to some initial part of it.*

It is obviously sufficient to prove this for a terminal part. According to the comparability theorem, otherwise the whole set M would have to be sim-