

From this it again follows that if a well-ordered set M is mapped with preservation of order onto another well-ordered set M' , then this mapping is unique. Indeed if f and g both map M onto M' , then fg^{-1} maps M onto M so that $fg^{-1}(x)$ is x and therefore $f(x) = g(x)$ for all x .

Theorem 12. *If M is mapped by f with preservation of order into an initial part A of itself, then $A = M$ and the mapping is the identical one. We may also say: M cannot be mapped onto an initial section of itself.*

Proof: Let f map M onto A , A initial part of M . Then no element m of M can be $>$ every element x of A , because $f(m)$ should belong to A so that $m > f(m)$, which contradicts the previous theorem. Thus every $m \in M$ is \leq an $x \in A$, whence $m \in A$, that is, $A = M$.

Noticing that an initial part of a well-ordered set M is either M itself or a section of M , we have that if $M \simeq N$ (meaning M and N are similar), then M is neither $\simeq N_1$ nor $N \simeq M_1$, M_1 and N_1 denoting sections of M resp. N .

Theorem 13. *Let M and N be well-ordered sets. Then either $M \simeq N_1$, N_1 a section of N or $M \cong N$ or $M_1 \cong N$, M_1 a section of M .*

Proof: Let I be the set of all initial parts of M that are similar to initial parts of N constituting a set J . Then the union SI is in an obvious way similar to SJ . Now either SI must be $= M$ or $SJ = N$. Else SJ will be the section belonging to an element i of M and SJ the section delivered by $j \in N$. But then $SI + \{i\}$ would be similar to $SJ + \{j\}$ which contradicts the definition of I . Now, if $SI = M$, either $M \cong N$ or $M \cong$ a section N_1 of N according as SJ is N or N_1 , else SI is a section M_1 of M while $SJ = N$ so that $M_1 \cong N$.

5. Ordinals and alephs

It is now natural to say that an ordinal α is $<$ an ordinal β , if α is the order-type of a well-ordered set A , β the type of B , such that A is similar to an initial section of B . It is clear that $\alpha < \beta$ & $\beta < \gamma \rightarrow \alpha < \gamma$ and that $\alpha < \beta$ excludes $\beta < \alpha$. Thus all ordinals are ordered. However, this ordering is also a well-ordering. Let us namely consider an arbitrary set or even class C of well-ordered sets. Let M be one of the sets in C . Its ordinal number μ may be the least of all represented by the considered sets. If not there are other sets in C which are similar to sections of M . These sections are furnished by elements of M and among these there is at least one. The corresponding initial section represents then the least ordinal of all furnished by the sets in C .

Theorem 14. *A terminal part or an interval of a well-ordered set is similar to some initial part of it.*

It is obviously sufficient to prove this for a terminal part. According to the comparability theorem, otherwise the whole set M would have to be sim-

ilar to an interval of itself, but that contradicts the fact that we should have $x \leq f(x)$ for all $x \in M$.

A consequence of this is that we always have $\alpha \leq \alpha + \beta$ and $\beta \leq \alpha + \beta$.

I have earlier defined addition and multiplication of ordered sets. We may define multiplication and exponentiation for well-ordered sets in such a way that well-ordered sets result. First I will repeat the definition of addition: Let T be a well-ordered set of well-ordered sets A, B, C, \dots which we assume mutually disjoint. Then the sum ST is well-ordered thus: Any two elements of the same element X of T retain their order in X . If X preceeds Y in T , then every element of X preceeds every element of Y in ST . It is indeed easy to see that ST is well-ordered in that way. Let namely M be $\subseteq ST$ and $\neq 0$. Then the diverse $X \in T$ which furnish elements of M constitute a non-void subset of T . Since T is well-ordered there is a least element of this subset, N say. Since N is well-ordered there is a least element m in the subset $M \cap N$ of N . Obviously m is the least element of M .

Multiplication I will define as follows. Let us again consider a well-ordered set T of mutually disjoint well-ordered sets $A, B, C, \dots \neq 0$. Let a_0, b_0, c_0, \dots be the least elements of A, B, C, \dots . Then I take a subset P of $A.B.C. \dots$ in the previous sense, namely the set P consisting of all elements of $A.B.C. \dots$ which contain only a finite number of elements different from a_0, b_0, c_0, \dots . This set P is then ordered by the principle of last differences, which means that if a, b, c, \dots and a', b', c', \dots are two elements of the product, then $a, b, c, \dots < a', b', c', \dots$ if $m < m'$ but no later element $m_1 > m'_1$.

Exponentiation is defined by letting all factors in a product be similar well-ordered sets.

Lemma. Let T be a well-ordered set of well-ordered sets A, B, C, \dots such that if X and Y are elements of T and $X < Y$ in T , then $X \subseteq Y$ and the order of the elements of X remain unaltered in Y . Then the union ST is well-ordered and two elements of ST are ordered as in some element X of T .

Proof: If T contains a last (greatest) element M , then the truth of the lemma is immediately clear, because in this case $ST = M$. Therefore we may assume that T does not contain any last element. Let us then consider a subset N of ST , $0 \in N$. There will be elements X of T containing elements belonging to N . Let X_0 be the first of these X . Then $X_0 \cap N$ is a subset $\neq 0$ of the well-ordered set X_0 so that there is a first element in $X_0 \cap N$ which obviously is the first element in N . Thus it is proved that ST is well-ordered. It is evident that two elements of ST will both occur in some element of T and have there the same relation of order.

Now let us consider the product P of the well ordered set T of well ordered factors A, B, C, \dots . The product belonging to an initial section of T may be called a partial product and be denoted by P_X , if the section of T is given by X . It is understood that the elements of P_Y shall, for each $Y \geq X$ in T , contain y_0 only. I shall first prove that if all these partial products are well-ordered, so is P . Indeed as often as $X < Y$, $P_X \subseteq P_Y$ so that the partial products constitute a well-ordered set of well-ordered sets of the kind considered in the lemma. Now if there is no last element in T (no last factor in P) then P is the union of all P_X and is therefore well-ordered according to the lemma. If there is a last factor F then $P = P_F$. F where

P_F is well-ordered according to supposition, and since the product of two well-ordered sets is well-ordered, P is well-ordered. Now let us look at the case that some partial products were not well-ordered. There must then be a least X_0 among all the $X \in T$ for which P_X is not well-ordered. Then P_{X_0} is the union of all P_Y , where Y precedes X_0 in T if X_0 has no predecessor, else, if F is the predecessor, we have $P_{X_0} = P_F F$ where P_F and F are well-ordered. Further all these P_Y are well-ordered. But then again according to the lemma P_{X_0} is well-ordered which is a contradiction. Therefore all partial products are well-ordered, which as we just saw implies that P itself is well-ordered. Thus we have proved:

Theorem 15. The product P of a well-ordered set of well-ordered sets is well-ordered.

I would like to prove that the product $\alpha\beta$ can be conceived as the result of adding β sets each of ordinal number α . Let A have the ordinal α , B the ordinal β . Then $\alpha\beta$ is the ordinal number of the set P of pairs (a, b) ordered according to last differences as explained. Let M_b be the set of all pairs with the last element b and T the set of all these M_b . Then ST , well-ordered as explained above, is just the sum P of all M_b . Each of these has the ordinal α .

It is easy to verify that the associative laws hold for addition and multiplication. Also the distributive law $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$ is seen to be valid. On the other hand, the commutative laws do not hold, nor does the distributive formula $(\alpha + \beta)\gamma = \alpha\gamma + \beta\gamma$. I shall give some examples.

$$1 + \omega = \omega < \omega + 1$$

$$2 \cdot \omega = \omega < \omega 2 \text{ and therefore } (1 + 1)\omega = \omega < 1 \cdot \omega + 1 \cdot \omega.$$

One can also notice that not always

$$(\alpha\beta)^\gamma = \alpha^\gamma\beta^\gamma$$

For example

$$(2 \cdot 2)^\omega < 2^\omega \cdot 2^\omega, \quad (2 \cdot (\omega + 1))^2 > 2^2(\omega + 1)^2$$

On the other hand, if $\lambda = \sum_{\eta < \mu} \beta_\eta$, then $\alpha^\lambda = \sum_{\eta < \mu} \alpha^{\beta_\eta}$ and

$$\alpha^\lambda = \prod_{\eta < \mu} \alpha^{\beta_\eta}, \text{ in particular } \alpha^{\beta + \gamma} = \alpha^\beta \cdot \alpha^\gamma$$

$$\alpha^{\beta\gamma} = (\alpha^\beta)^\gamma$$

We have seen that the ordinal numbers are well-ordered by the relation $<$. It is then natural to ask how the cardinal numbers behave. Because of the comparability of the ordinals it is immediately clear that the cardinal numbers are comparable; indeed, if M and N are any two sets and they are in some way well-ordered, then either M is similar to, and thus equivalent to, some initial part of N or inversely. Thus we have either $\overline{M} \leq \overline{N}$ or $\overline{N} \leq \overline{M}$. Now let T be a set of sets. I assert that the cardinal numbers represented by the elements A, B, C, \dots of T are well-ordered by the relation $<$ as earlier defined. Evidently it suffices to prove that there is a least cardinal represented

by the elements of T , because then the same will be true for every subset of T . Now let M be ϵT . If M is the smallest cardinal represented by any element of T , then our assertion is correct. Otherwise there will be some elements X of T representing smaller cardinals. All these X we may assume well-ordered. Then each of them is similar to an initial section of M given by an element m of M . Among these m there will be a least one m_0 . The section given by m_0 then furnishes the least cardinal number among the mentioned X .

Thus the cardinal numbers are also well-ordered by the relation $<$. More exactly expressed: All cardinals \leq a given cardinal constitute a well-ordered sequence according to their magnitude. The least of the transfinite ones, the cardinal of the denumerable sets, we denote, as Cantor did, by \aleph_0 , the following by \aleph_1 , and so on.

If α is a transfinite ordinal, i.e. $\omega \leq \alpha$, then we have $1 + \alpha = \alpha$, because we may write $\alpha = \omega + \beta$, whence $1 + \alpha = 1 + (\omega + \beta) = (1 + \omega) + \beta = \omega + \beta = \alpha$. More generally we have of course $n + \alpha = \alpha$, n finite. Further it may be noticed, that if α is the ordinal of a set M without last element or in other words α is without immediate predecessor, then for every finite ordinal n we have $n\alpha = \alpha$. We can first prove that $\alpha = \omega\beta$, whence $n\alpha = n(\omega\beta) = (n\omega)\beta = \omega\beta = \alpha$ since $n\omega$ is evidently $= \omega$. That α indeed is a multiple of ω is seen by distributing the elements of M into classes by putting any two elements into the same class which are either neighbors or have only a finite number of elements between them. It is clear that every class is of type ω , and the whole set is the sum of a well-ordered set of these classes, which means that $\alpha = \omega\beta$, β denoting the ordinal of the set of the classes.

Among all ordinals whose cardinal number is \aleph_α there will be a least, usually written ω_α . This ω_α belongs to a very remarkable class of ordinals called principal ordinals. The definition is:

An ordinal α is a principal one, if the equation, $\alpha = \beta + \gamma$ only has the solutions $\beta < \alpha$, $\gamma = \alpha$ and $\alpha = \beta$, $\gamma = 0$. One may also say that the ordinal represented by a well-ordered set M is principal, if M is similar to every terminal part of itself.

Proof that ω_α is principal: Let $\omega_\alpha = \beta + \gamma$, $\gamma > 0$. We know that γ is the ordinal of some initial part of M , if M has the ordinal ω_α . If this initial part of M is not M itself, it is an initial section, so that $\gamma < \omega_\alpha$, and according to the definition of ω_α we have that the cardinal number \aleph_γ of γ must be $< \aleph_\alpha$. Further β is also $< \aleph_\alpha$, because β is the ordinal of some initial section of M . But the sum of two alephs $< \aleph_\alpha$ is again $< \aleph_\alpha$. Thus γ must be $= \omega_\alpha$.

Since it is clear that every transfinite cardinal \aleph may be given by a well-ordered set without last element, indeed the least ordinal with cardinal number \aleph cannot have a predecessor because $1 + \aleph = \aleph$, we obtain from the relation $n\alpha = \alpha$ just mentioned that always

$$n\aleph = \aleph$$

for finite n . Hence for every aleph \aleph_α in particular $\aleph_\alpha + \aleph_\alpha = \aleph_\alpha$. Further if $\aleph_\beta < \aleph_\alpha$, we obtain

$$\aleph_\alpha \leq \aleph_\alpha + \aleph_\beta \leq \aleph_\alpha + \aleph_\alpha = \aleph_\alpha$$

which means that

$$\aleph_\alpha + \aleph_\beta = \aleph_\alpha.$$

Thus the sum of two alephs is the greater one of them. Further, if \aleph_β and \aleph_γ are both $< \aleph_\alpha$, also $\aleph_\beta + \aleph_\gamma < \aleph_\alpha$.

The division of ordinals may be performed thus. Let α be given and $\beta > 0$. We consider the ordinals γ which are such that for some δ

$$\alpha = \beta\gamma + \delta$$

I assert that there is a greatest value of γ here. Indeed the assumption that $\beta\gamma_\lambda$ where $\gamma_1 < \gamma_2 < \dots$, are all $\leq \alpha$ yields $\beta \lim \gamma_\lambda \leq \alpha$, where $\lim \gamma_\lambda$ is the least ordinal $>$ every γ_λ . This is perhaps most easily seen by writing $\gamma_2 = \gamma_1 + \gamma_2'$, $\gamma_3 = \gamma_2 + \gamma_3'$, and generally $\gamma_{\lambda+1} = \gamma_\lambda + \gamma_{\lambda+1}'$. Then $\lim \gamma_\lambda = \sum_{\lambda} \gamma_{\lambda}'$ putting $\gamma_1 = \gamma_1'$, and we have by the distributive law for multiplication

$$\beta \sum_{\lambda} \gamma_{\lambda}' = \sum_{\lambda} \beta \gamma_{\lambda}'.$$

But the several $\beta\gamma_{\lambda}'$ will represent the ordinals of different disjoint intervals of a well-ordered set of ordinal α . Thus $\sum_{\lambda} \beta\gamma_{\lambda}' \leq \alpha$.

If κ is the greatest value of γ , we have

$$\alpha = \beta\kappa + \rho, \rho < \beta.$$

Indeed, if ρ were $= \beta + \rho'$, we should obtain $\alpha = \beta(\kappa + 1) + \rho'$ so that κ would not be the maximal γ .

In the particular case $\beta = \omega$ we get

$$\alpha = \omega\kappa + n, n \text{ finite.}$$

Thus we again get the above result, that if α is the ordinal of a well-ordered set without last element, it is of the form $\omega\kappa$.

It is easily seen that $\beta \lim \gamma_\lambda = \lim \beta\gamma_\lambda$. As a consequence of this there is a maximal power $\beta^{\gamma_1} \leq \alpha$. Then the division of α by β^{γ_1} yields

$$\alpha = \beta^{\gamma_1} \nu_1 + \alpha', \alpha' < \beta^{\gamma_1}, \nu_1 < \beta.$$

Now again there is a maximal power of β , β^{γ_2} say $\leq \alpha'$. Then we obtain

$$\alpha' = \beta^{\gamma_2} \nu_2 + \alpha'', \alpha'' < \beta^{\gamma_2}, \nu_2 < \beta.$$

Since the sequence $\alpha, \alpha', \alpha'', \dots$ is decreasing, there is a least one which must be 0. Then we have

$$\alpha = \sum_{r=1}^m \beta^{\gamma_r} \nu_r, m \text{ finite, all } \nu_r < \beta.$$

Of particular interest is the case $\beta = \omega$. We obtain the result that every ordinal can be written in the form

$$\alpha = \sum_{r=1}^m \omega^{\gamma_r} n_r, \gamma_1 > \gamma_2 > \dots$$

m positive and finite, all n_r positive and finite. It is clear by the method of construction that this form is unique.

It is seen that α cannot be principal without being simply a power of ω . On the other hand every power of ω is easily seen to be principal.

If γ_1 is kept fixed in the above expression while $\gamma_2, \gamma_3, \dots, m$, and the n_r vary, we get all numbers $< \omega^{\gamma_1 + 1}$. If also γ_1 varies but is kept $< \alpha$, α a limit number, we get all ordinals $< \omega^\alpha$. I will show how we can set up a very simple one-to-one correspondence between the elements of a well-ordered set M of ordinal equal to a power of ω on the one hand and the ordered pairs (a,b) which are the elements of M^2 on the other. To every pair

$$\alpha = \sum_{k \leq q} \omega^{\sigma k} m_k, \quad \beta = \sum_{k \leq q} \omega^{\sigma k} n_k$$

we let correspond the number

$$\gamma = \sum_{k \leq q} \omega^{\sigma k} f(m_k, n_k)$$

where $f(m_k, n_k)$ is a one-to-one correspondence between the non-negative integers and their pairs. We set $\gamma = 0$ for $\alpha = \beta = 0$.

If this is applied to ω_α considering the cardinal number \aleph_α we obtain

$$\aleph_\alpha^2 = \aleph_\alpha.$$

Of course we then also get $\aleph_\alpha^n = \aleph_\alpha$ by an easy induction.

Because of the well-ordering theorem we then have that $m^2 = m$ for every transfinite cardinal m . It is now very remarkable that, if inversely it is presupposed that this formula is valid for every transfinite cardinal number m , then every set can be well-ordered. Thus we have

Theorem 16. *The general validity of $m^2 = m$ implies the general principle of choice and inversely.*

If we look at the proof of the earlier theorem stating that m and n are comparable when $m + n = mn$, we notice that if n say is an aleph, then we need not use the axiom of choice in the proof. Further, if simultaneously it is known that n is not $\leq m$, we get $m \leq n$ and then m is an aleph.

Now m being an arbitrary cardinal number, it is always possible to define an aleph which is not $\leq m$. This was first done by F. Hartogs (Math. Ann. 76, 438, 1915). Let M be a set such that $\overline{M} = m$. There are some subsets of M which can be well-ordered. We take into account all well-orderings of all these subsets and distribute these well-ordered subsets into classes of similarity. Every such class is then a set corresponding to an ordinal and these sets constitute again a certain set. To the ordinals represented by the members of this set there exist always greater ordinals e.g. the sum of all the ordinals. Among these greater ordinals there is a least one λ say. Then $\overline{\lambda}$ is not $\leq m$, because this would mean that there exists a subset of M which can be well-ordered with ordinal number λ , whereas λ is greater than every ordinal α for which this was the case. Thus $\overline{\lambda}$ is an aleph which cannot be $\leq m$.

Hence the correctness of our assertion, that if always $m + n = mn$ then every set is well-ordered. However, to be perfectly correct we must assume $m^2 = m$ for any inductive infinite cardinal number.

Now if always $m^2 = m$, we have $(m + n)^2 = m + n$, whence at any rate

$$mn \leq m + n.$$

However we have proved earlier that if m and n are ≥ 2 , then $m + n \leq m \cdot n$. Thus we obtain $mn = m + n$.

6. Some remarks on functions of ordinal numbers

A function $f(x)$ is called monotonic, if $(x < y) \rightarrow (f(x) \leq f(y))$. It is called strictly increasing, if

$$(x < y) \rightarrow (f(x) < f(y)).$$

The function is called seminormal, if it is monotonic and continuous, that is if $f(\lim \alpha_\lambda) = \lim f(\alpha_\lambda)$, λ here indicating a sequence with ordinal number of the second kind, i.e., without immediate predecessor, while $(\lambda_1 < \lambda_2) \rightarrow (\alpha_{\lambda_1} < \alpha_{\lambda_2})$.

The function is called normal, if it is strictly increasing and continuous; ξ is called a critical number for f , if $f(\xi) = \xi$.

Theorem 17. *Every normal function possesses critical numbers and indeed such numbers $>$ any α .*

Proof: Let α be chosen arbitrarily and let us consider the sequence $\alpha, f(\alpha), f^2(\alpha), \dots$. Then if $\alpha_\omega = \lim_{n < \omega} f^n(\alpha)$, we have $f(\alpha_\omega) = f(\lim_{n < \omega} f^n(\alpha)) = \lim_{n < \omega} f^{n+1}(\alpha) = \alpha_\omega$, that is, α_ω is a critical number for f .

Examples.

- 1) The function $1 + x$ is normal. Critical numbers are all $x = \omega + \alpha$, α arbitrary.
- 2) The function $2x$ is normal. Critical numbers are all of the form $\omega\alpha$, α arbitrary.
- 3) The function ω^x is normal. Critical numbers of this function are called ϵ -numbers. The least of them is the limit of the sequence $\omega, \omega^\omega, \omega^{\omega^\omega}, \dots$

I will mention the quite trivial fact that every increasing function f is such that $f(x) \geq x$ for every x .

Theorem 18. *Let $g(x) \geq x$ for all x and α be an arbitrary ordinal; then there is a unique semi-normal function f such that*

$$f(0) = \alpha, f(x+1) = g(f(x)).$$

Proof clear by transfinite induction.

Theorem 19. *If f is a semi-normal function and β is an ordinal which is not a value of f , while f possesses values $< \beta$ and values $> \beta$, then there is among the x such that $f(x) < \beta$ a maximal one x_0 such that $f(x_0) < \beta < f(x_0 + 1)$.*