

Chapter 7. Universal constructions in category-theoretic terms.

The language of category theory has enabled us to give general definitions of “free object”, “product”, “coproduct”, “equalizer” and various other universal constructions. It is clear that these different constructions have many properties in common. Let us now look for ways to unify them, so that we will be able to prove results about them by general arguments, rather than piecemeal.

7.1. Universality in terms of initial and terminal objects. In all the above constructions, we deal with mathematical entities with certain “extra” structure, and seek one entity E with such structure that is “universal”. This suggests that we make the class of entities with such extra structure into a category, and examine the universal property of E there.

For instance, the free group on three generators is universal among systems (G, a, b, c) where G is a group, and $a, b, c \in |G|$. If we define a category whose objects are these systems (G, a, b, c) , and where a morphism $(G, a, b, c) \rightarrow (G', a', b', c')$ means a group homomorphism $f: G \rightarrow G'$ such that $f(a) = a'$, $f(b) = b'$, $f(c) = c'$, we see that the universal property of the free group (F, x, y, z) says that it has a unique morphism into every object of the category – in other words, that it is an initial object.

Similarly, given a group G , the *abelianization* of G is universal among pairs (A, f) where A is an abelian group, and f a group homomorphism $G \rightarrow A$. If we define a morphism from one such pair (A, f) to another such pair (B, g) to mean a group homomorphism $m: A \rightarrow B$ such that $mf = g$, we see that the definition of the abelianization of G says that it is initial in *this* category.

Finally, a group, a ring, a lattice, etc., with a presentation $\langle X \mid R \rangle$ clearly means an initial object in the category whose objects are groups, etc., with specified X -tuples of elements satisfying the system of equations R , and whose morphisms are homomorphisms respecting these distinguished X -tuples of elements.

The above were examples of what we named “left universal” properties in §3.8. Let us look at one “right universal” property, that of a *product* of two objects A and B in a category \mathbf{C} . We see that the relevant auxiliary category should have for objects all 3-tuples (X, a, b) , where $X \in \text{Ob}(\mathbf{C})$, $a \in \mathbf{C}(X, A)$, $b \in \mathbf{C}(X, B)$, and for morphisms $(X, a, b) \rightarrow (Y, a', b')$ all morphisms $X \rightarrow Y$ in \mathbf{C} making commuting triangles with the maps into A and B . A direct product of A and B in \mathbf{C} is seen to be a *terminal* object (P, p_1, p_2) in this category.

You can likewise easily translate the universal properties of *pushouts*, *pullbacks* and *coproducts* in arbitrary categories to those of initial or terminal objects in appropriately defined auxiliary categories.

So all the universal properties we have considered reduce to those of being an initial or a terminal object in an appropriate category. This view of universal constructions is emphasized by Lang [31, p.57 et seq.], who gives these two types of objects the more poetic designations “universally repelling” and “universally attracting”. Since a terminal object in \mathbf{C} is an initial object in \mathbf{C}^{op} , all these universal properties ultimately reduce to that of initial objects!

Lemma 6.8.2 tells us that initial (and hence terminal) objects are *unique* up to unique isomorphism. This gives us, in one fell swoop, uniqueness up to canonical isomorphism for free groups, abelianizations, products, coproducts, pushouts, pullbacks, objects presented by generators and relations, and all the other universal constructions we have considered. The canonical

isomorphisms that these constructions are “unique up to” correspond to the unique morphisms between any two initial objects of a category. I.e., given two realizations of one of our universal constructions, these isomorphisms will be the unique morphisms from each to the other that preserve the extra structure.

We will look at questions of *existence* of initial objects in §7.10.

7.2. Representable functors, and Yoneda’s Lemma. The above approach to universal constructions is impressive for its simplicity; but we would also like to relate these universal objects to the original categories in question: Though the free group on an S -tuple of generators is initial in the category of groups given with S -tuples of elements, and the kernel of a group homomorphism $f: G \rightarrow H$ is terminal in the category of groups L given with homomorphisms $L \rightarrow G$ having trivial composite with f , we also want to understand these constructions in relation to the category **Group**.

Note that the objects of the various auxiliary categories we have used can be written as pairs (X, a) , where X is an object of the original category \mathbf{C} , and a is some additional structure on X . If we write $F(X)$ for the set of *all possible values* of this additional structure (e.g., in the case that leads to the free group on a set S , the set of all S -tuples of elements of X), we find that F is in general a functor, covariant or contravariant, from \mathbf{C} to **Set**. The condition characterizing a *left* universal pair (R, u) is that for every $X \in \text{Ob}(\mathbf{C})$ and $x \in F(X)$, there should be a unique morphism $f: R \rightarrow X$ such that $F(f)(u) = x$. This condition – which we see requires a covariant F so that the latter equation will make sense – is equivalent to saying that for each object X , the set of morphisms $f \in \mathbf{C}(R, X)$ is sent bijectively to the set of elements of $F(X)$ by the map $f \mapsto F(f)(u)$. The bijectivity of this correspondence for each X leads to an isomorphism between the functor $\mathbf{C}(R, -)$, i.e., $h_R: \mathbf{C} \rightarrow \mathbf{Set}$, and the given functor $F: \mathbf{C} \rightarrow \mathbf{Set}$. Thus, the universal property of R can be formulated as a statement of this isomorphism:

Theorem 7.2.1. *Let \mathbf{C} be a category, and $F: \mathbf{C} \rightarrow \mathbf{Set}$ a functor. Then the following data are equivalent:*

- (i) *An object $R \in \text{Ob}(\mathbf{C})$ and an element $u \in F(R)$ having the universal property that for all $X \in \text{Ob}(\mathbf{C})$ and all $x \in F(X)$, there exists a unique $f \in \mathbf{C}(R, X)$ such that $F(f)(u) = x$.*
- (ii) *An initial object (R, u) in the category whose objects are all ordered pairs (X, x) with $X \in \text{Ob}(\mathbf{C})$ and $x \in F(X)$, and whose morphisms are morphisms among the first components of these pairs which respect the second components.*
- (iii) *An object $R \in \text{Ob}(\mathbf{C})$ and an isomorphism of functors $i: h_R \cong F$ in $\mathbf{Set}^{\mathbf{C}}$.*

Namely, given (R, u) as in (i) or (ii), one obtains an isomorphism i as in (iii) by letting $i(X)$ take $f \in h_R(X)$ to $F(f)(u) \in F(X)$, while in the reverse direction, one obtains u from i as $i(R)(\text{id}_R)$.

Sketch of Proof. The equivalence of the structures described in (i) and (ii) is immediate.

Concerning our description of how to pass from these structures to that of (iii), it is a straightforward verification that for any $u \in F(R)$, the map i described there gives a *morphism of functors* $h_R \rightarrow F$. That this is an isomorphism is then the content of the universal property of (i). In the opposite direction, given an isomorphism i as in (iii), if u is defined as indicated, then the universal property of (i) is just a restatement of the bijectivity of the maps $i(X): h_R(X) \rightarrow F(X)$.

Finally, it is easy to check that if one goes as above from universal element to isomorphism of functors and back, one recovers the original element, and if one goes from isomorphism to

universal element and back, one recovers the original isomorphism. \square

Exercise 7.2:1. Write out the “straightforward verifications” referred to in the second sentence of the above proof, and those implied in the phrases “is then the content of” and “is just a restatement of” in the next two sentences.

Dualizing (i.e., applying Theorem 7.2.1 to \mathbf{C}^{op} and stating the resulting assertion in terms of \mathbf{C}), we get

Theorem 7.2.2. *Let \mathbf{C} be a category, and F a contravariant functor from \mathbf{C} to \mathbf{Set} (i.e., a functor $\mathbf{C}^{\text{op}} \rightarrow \mathbf{Set}$). Then the following data are equivalent:*

- (i) *An object $R \in \text{Ob}(\mathbf{C})$ and an element $u \in F(R)$ with the universal property that for any $X \in \text{Ob}(\mathbf{C})$ and $x \in F(X)$, there exists a unique $f \in \mathbf{C}(X, R)$ such that $F(f)(u) = x$.*
- (ii) *A terminal object (R, u) in the category whose objects are all ordered pairs (X, x) with $X \in \text{Ob}(\mathbf{C})$ and $x \in F(X)$, and whose morphisms are morphisms among the first components of these pairs which respect the second components.*
- (iii) *An object $R \in \text{Ob}(\mathbf{C})$ and an isomorphism of contravariant functors $i: h^R \cong F$ in $\mathbf{Set}^{\mathbf{C}^{\text{op}}}$.*

Namely, given (R, u) as in (i) or (ii), one obtains an isomorphism i as in (iii) by letting $i(X)$ take $f \in h^R(X)$ to $F(f)(u) \in F(X)$, while in the reverse direction, one obtains u from i as $i(R)(\text{id}_R)$. \square

Note that in Theorem 7.2.1(ii), the last phrase, “which respect second components”, meant that for a morphism $f: X \rightarrow Y$ to be considered a morphism $(X, x) \rightarrow (Y, y)$, we required $F(f)(x) = y$, while in Theorem 7.2.2(ii), the corresponding condition is $F(f)(y) = x$.

We remark that the auxiliary categories used in point (ii) of the above two theorems are comma categories, $(1 \downarrow F)$ (Exercise 6.8:26(iii)).

The properties described above have names:

Definition 7.2.3. *Let \mathbf{C} be a category.*

A covariant functor $F: \mathbf{C} \rightarrow \mathbf{Set}$ is said to be representable if it is isomorphic to a covariant hom-functor h_R for some $R \in \text{Ob}(\mathbf{C})$.

A contravariant functor $F: \mathbf{C}^{\text{op}} \rightarrow \mathbf{Set}$ is likewise said to be representable if it is isomorphic to a contravariant hom-functor h^R for some $R \in \text{Ob}(\mathbf{C})$.

In each case, R is called the representing object for F , and if i is the given isomorphism of functors, then $i(R)(\text{id}_R)$ is called the associated universal element of $F(R)$.

So from this point of view, universal problems of the sort considered above in a category \mathbf{C} are questions of the *representability* of certain set-valued functors on \mathbf{C} . Let us examine a few set-valued functors, and see which of them are representable.

If U is the underlying-set functor on **Group**, a representing object for U should be a group with a universal element of its underlying set. The object with this property is the free group on one generator. More generally, if a category has free objects with respect to a concretization U , then U will be represented by the free object on one generator, while the free object on a general set I can be characterized as representing the functor U^I (Definition 6.8.5).

The functor associating to every group the set of its elements of exponent 2 is represented by the group \mathbb{Z}_2 . More generally, the group with presentation by generators and relations $\langle X \mid R \rangle$ represents the functor associating to every group G the set of X -tuples of members of G which

satisfy the relations R .

Is the functor associating to every commutative ring K the set $|K[t]|$ of all polynomials over K in one indeterminate t representable? A representing object would be a ring R with a universal polynomial $u(t) \in R[t]$. The universal property would say that given any polynomial $p(t)$ over any ring K , there should exist a unique homomorphism $R \rightarrow K$ which, applied coefficient-wise to polynomials, carries $u(t)$ to $p(t)$. But clearly there is a problem here: The polynomial u will have some degree n , and if we choose a polynomial p of degree $> n$, it cannot be obtained from u in this way. So the set-of-polynomials functor is not representable.

However, there is a concept close to that of polynomial but not subject to the restriction that only finitely many of the coefficients be nonzero, that of a *formal power series* $a_0 + a_1t + a_2t^2 + \dots$. If K is a ring, then the ring of formal power series over K is denoted $K[[t]]$; its underlying set $|K[[t]]| = \{a_0 + a_1t + a_2t^2 + \dots\}$ can be identified with the set of all sequences (a_0, a_1, \dots) of elements of K , i.e., with $|K|^\omega = U^\omega(K)$. We know that the functor U^ω is represented by the free commutative ring on an ω -tuple of generators, that is, the polynomial ring $\mathbb{Z}[A_0, A_1, \dots]$. And indeed, the formal power series ring over this polynomial ring contains the element $A_0 + A_1t + A_2t^2 + \dots$, which clearly has the property of a universal power series.

Exercise 7.2:2. (i) Show that the functor associating to every monoid S the set of its invertible elements is representable, but that the functor associating to S the set of its right-invertible elements is not.

(ii) What about the functor associating to every monoid S the set of pairs (x, y) such that $xy = e$ and $yx = e$? The set of pairs (x, y) merely satisfying $xy = e$? The set of 3-tuples (x, y, z) such that $xy = xz = e$?

(iii) Determine which, if any, of the functors mentioned in (i) and (ii) are isomorphic to one another.

Exercise 7.2:3. Let P denote the contravariant power-set functor, associating to every set X the set $\mathbf{P}(X)$ of its subsets, and E the contravariant functor associating to every set X the set $\mathbf{E}(X)$ of equivalence relations on X . Determine whether each of these is representable.

Exercise 7.2:4. Let A, B be objects of a category \mathbf{C} . Describe a set-valued functor F on \mathbf{C} such that a *product* of A and B , if it exists in \mathbf{C} , means a representing object for F , and likewise a functor G such that a *coproduct* of A and B in \mathbf{C} means a representing object for G . (One of these will be covariant and the other contravariant.)

Exercise 7.2:5. Let (\mathbf{C}, U) be a concrete category. Show that the following conditions are equivalent. (a) The concretization functor U is *representable*. (b) \mathbf{C} has a free object on one generator. Moreover, show that if \mathbf{C} has coproducts, then these are also equivalent to (b') \mathbf{C} has free objects on all sets.

Students who know some Lie group theory might try

Exercise 7.2:6. Let \mathbf{LieGp} denote the category of Lie groups and continuous group homomorphisms. Let $T: \mathbf{LieGp} \rightarrow \mathbf{Set}$ denote the functor associating to a Lie group L the set of tangent vectors to L at the neutral element. Which of the following covariant functors $\mathbf{LieGp} \rightarrow \mathbf{Set}$ are representable? (a) the functor T , (b) the functor $T^2: L \mapsto T(L) \times T(L)$, (c) the functor $L \mapsto \{(x, y) \in T(L) \times T(L) \mid [x, y] = x\}$.

Exercise 7.2:7. Given a set X , let $\mathbf{GpStruct}(X)$ denote the set of all group-structures on X (consisting of a composition operation μ , an inverse operation ι , and a neutral element e). A group can be considered as a set X given with an element $s \in \mathbf{GpStruct}(X)$, and the category \mathbf{Group} has an initial object. This looks as though it should mean the underlying set of this group is a representing object for $\mathbf{GpStruct}$; but something is clearly wrong, since a map from

this set into a set X does not determine a group structure on X . Resolve this paradox.

The equivalence, in each of Theorems 7.2.1 and 7.2.2, of parts (ii) and (iii) shows that the concept of representable functor can be characterized in terms of initial and terminal objects. The reverse is also true:

Exercise 7.2:8. Let \mathbf{C} be any category. Display a covariant functor F and a contravariant functor G from \mathbf{C} to \mathbf{Set} such that an initial, respectively a terminal object of \mathbf{C} is equivalent to a representing object for F , respectively G .

The implication (i) \Rightarrow (iii) in Theorem 7.2.1 shows that an isomorphism between the hom-functor h_R associated with an object R , and an arbitrary functor F , is equivalent to a specification of an element of $F(R)$ with the universal property given in (i). In fact, every morphism, invertible or not, from a hom-functor h_R to a functor F corresponds to a choice of some element of $F(R)$. Though utterly simple to prove, this is an important tool. We give both this result and its contravariant dual in

Theorem 7.2.4 (Yoneda's Lemma). *Let \mathbf{C} be a category, and R an object of \mathbf{C} .*

If $F: \mathbf{C} \rightarrow \mathbf{Set}$ is a covariant functor, then morphisms $f: h_R \rightarrow F$ are in one-to-one correspondence with elements of $F(R)$, under the map $f \mapsto f(R)(\text{id}_R)$.

Likewise, if $F: \mathbf{C}^{\text{op}} \rightarrow \mathbf{Set}$ is a contravariant functor, morphisms $f: h^R \rightarrow F$ are in one-to-one correspondence with elements of $F(R)$, again under the map $f \mapsto f(R)(\text{id}_R)$.

Proof. In the covariant case, we must describe how to get from an element $x \in F(R)$ an appropriate morphism $f_x: h_R \rightarrow F$. We define f_x to carry $a \in h_R(X) = \mathbf{C}(R, X)$ to $F(a)(x) \in F(X)$. The verification that this is a morphism of functors, and that this construction is inverse to the indicated map from morphisms of functors to elements of $F(R)$, is immediate.

The contravariant case follows by duality (or by the dualized argument). \square

Again –

Exercise 7.2:9. Show the verifications omitted in the proof of the above result.

The following line of thought yields some intuition on Yoneda's Lemma. Recall that if G is a group, then a G -set, i.e., a functor from the category G_{cat} to \mathbf{Set} , can be looked at as a (possibly non-faithful) representation of G by permutations. In the same way, for any category \mathbf{C} , a functor $F: \mathbf{C} \rightarrow \mathbf{Set}$ can be thought of as a (possibly non-faithful) representation of \mathbf{C} by sets and set maps. Like a G -set, such a representation F can be regarded as a mathematical ‘‘object’’; in this case the ‘‘elements’’ of that object are the members of the sets $F(X)$ ($X \in \text{Ob}(\mathbf{C})$). This was the point of view of our development of Cayley's Theorem for small categories. In proving that result, we constructed such an object by introducing one generator in each set $F(X)$, and no relations; in the discussion that followed we observed that if one introduced only a generator in the set $F(X)$ for a particular $X \in \text{Ob}(\mathbf{C})$, and again no relations, the resulting ‘‘freely generated’’ object would be essentially the hom-functor which we named h_X . Yoneda's Lemma is the statement of the universal property of this ‘‘free’’ construction – that a morphism from this ‘‘representation of \mathbf{C} by sets’’ to any other ‘‘representation of \mathbf{C} by sets’’ is uniquely determined by specifying where the one generator, the identity element $\text{id}_X \in h_X(X)$, is to be sent. We make this formulation explicit in

Corollary 7.2.5. *Let \mathbf{C} be a category and R an object of \mathbf{C} .*

In the (large) category whose objects are pairs (F, x) where F is a functor $\mathbf{C} \rightarrow \mathbf{Set}$ and x an element of $F(R)$, the pair (h_R, id_R) is the initial object. Equivalently, the object $h_R \in \text{Ob}(\mathbf{Set}^{\mathbf{C}})$ is a representing object for the “evaluation at R ” functor $\mathbf{Set}^{\mathbf{C}} \rightarrow \mathbf{Set}$, the universal element being $\text{id}_R \in h_R(R)$.

Likewise, in the category whose objects are pairs (F, x) where F is a functor $\mathbf{C}^{\text{op}} \rightarrow \mathbf{Set}$ and x an element of $F(R)$, the pair (h^R, id_R) is the initial object; equivalently, the object $h^R \in \text{Ob}(\mathbf{Set}^{\mathbf{C}^{\text{op}}})$ represents the (again covariant!) “evaluation at R ” functor $\mathbf{Set}^{\mathbf{C}^{\text{op}}} \rightarrow \mathbf{Set}$. \square

This points to a general principle worth keeping in mind: when dealing with a morphism from a hom-functor to an arbitrary set-valued functor, look at its value on the identity map!

What if we apply Yoneda’s Lemma (covariant or contravariant) to the case where the arbitrary functor F is another hom-functor h_S , respectively h^S ? We get

Corollary 7.2.6. *Let \mathbf{C} be a category.*

Then for any two objects $R, S \in \text{Ob}(\mathbf{C})$, the morphisms from h_R to h_S as functors $\mathbf{C} \rightarrow \mathbf{Set}$ are in one-to-one correspondence with morphisms $S \rightarrow R$. Thus, the mapping $R \mapsto h_R$ gives a contravariant full embedding of \mathbf{C} in $\mathbf{Set}^{\mathbf{C}}$, the “Yoneda embedding”.

Likewise, morphisms from h^R to h^S as functors $\mathbf{C}^{\text{op}} \rightarrow \mathbf{Set}$ correspond to morphisms $R \rightarrow S$, giving a covariant full “Yoneda embedding” of \mathbf{C} in $\mathbf{Set}^{\mathbf{C}^{\text{op}}}$.

These two embeddings may both be obtained from the bivariate hom-functor $\mathbf{C}^{\text{op}} \times \mathbf{C} \rightarrow \mathbf{Set}$ by distinguishing one or the other argument, i.e., regarding this bifunctor in one case as a functor $\mathbf{C}^{\text{op}} \rightarrow \mathbf{Set}^{\mathbf{C}}$, and in the other as a functor $\mathbf{C} \rightarrow \mathbf{Set}^{\mathbf{C}^{\text{op}}}$.

Sketch of Proof. By Lemma 6.10.1 the bivariate hom functor does indeed yield functors $\mathbf{C}^{\text{op}} \rightarrow \mathbf{Set}^{\mathbf{C}}$ and $\mathbf{C} \rightarrow \mathbf{Set}^{\mathbf{C}^{\text{op}}}$ on distinguishing one or the other argument, and we see that the object R is sent to h_R , respectively h^R . Given a morphism $f: S \rightarrow R$ in \mathbf{C} , one verifies that the induced morphism of functors $h_f: h_R \rightarrow h_S$ takes id_R to $f \in h_S(R)$. Yoneda’s Lemma with $F = h_S$ tells us that the map $f \mapsto h_f$ is one-to-one and onto, so our functor $\mathbf{C}^{\text{op}} \rightarrow \mathbf{Set}^{\mathbf{C}}$ is full and faithful. The contravariant case follows by duality. \square

Exercise 7.2:10. Verify the above characterization of the morphism of functors induced by a morphism $f: S \rightarrow R$.

Exercise 7.2:11. Show how to answer most of the parts of Exercise 6.9:5, and also Exercise 6.9:7(i), using Yoneda’s Lemma.

Remark 7.2.7. It may seem paradoxical that we get the *contravariant* Yoneda embedding using *covariant* hom-functors, and the *covariant* Yoneda embedding using *contravariant* hom-functors, but there is a simple explanation. When we write the hom bifunctor $\mathbf{C}^{\text{op}} \times \mathbf{C} \rightarrow \mathbf{Set}$ as a functor to a functor category, $\mathbf{C} \rightarrow \mathbf{Set}^{\mathbf{C}^{\text{op}}}$ or $\mathbf{C}^{\text{op}} \rightarrow \mathbf{Set}^{\mathbf{C}}$, by distinguishing one variable, the variance in that variable determines the variance of the resulting Yoneda embedding, while the variance in the other variable determines the variance of the hom-functors that the embedding takes on as its values. Whichever way we slice it, we get covariance in one, and contravariance in the other.

What is the value of the Yoneda embedding? First, note that categories of the form $\mathbf{Set}^{\mathbf{C}}$ have very good properties; e.g., they have small limits and colimits. Hence Yoneda embeddings

embed arbitrary categories into “good” categories. Moreover, if one wishes to extend a category \mathbf{C} by adjoining additional objects with particular properties, one can often do this by identifying \mathbf{C} with the category of representable contravariant functors on \mathbf{C} , or the opposite of the category of representable covariant functors, and then taking for the additional objects certain other functors that are not quite representable.

In §6.5 we saw that systems of universal constructions could frequently be linked together, by natural morphisms among the constructed objects, to give functors. From the above corollary, we see that this should happen in situations where the functors that these universal objects are constructed to *represent* are linked by a corresponding system of morphisms of functors, in other words (by Lemma 6.10.1) where they form the components of a *bifunctor*. There is a slight complication in formulating this precisely, because the given representable functors are not themselves the hom-functors h_R or h^R , but only isomorphic to these, and the choice of representing objects R is likewise determined only up to isomorphism. To prepare ourselves for this complication, let us prove a lemma showing that a system of objects separately isomorphic to the values of a functor in fact form the values of an isomorphic functor.

Lemma 7.2.8. *Let $F: \mathbf{C} \rightarrow \mathbf{D}$ be a functor, and for each $X \in \text{Ob}(\mathbf{C})$, let $i(X)$ be an isomorphism of $F(X)$ with another object $G(X) \in \text{Ob}(\mathbf{D})$.*

Then there is a unique way to assign to each morphism of \mathbf{C} , $f \in \mathbf{C}(X, Y)$ a morphism $G(f) \in \mathbf{D}(G(X), G(Y))$ so that the objects $G(X)$ and morphisms $G(f)$ constitute a functor $G: \mathbf{C} \rightarrow \mathbf{D}$, and i constitutes an isomorphism of functors, $F \cong G$.

Proof. If G is to be a functor and i a morphism of functors, then for each $f \in \mathbf{C}(X, Y)$ we must have $G(f) i(X) = i(Y) F(f)$. Since $i(X)$ is an isomorphism, we can rewrite this as $G(f) = i(Y) F(f) i(X)^{-1}$. It is straightforward to verify that G , so defined on morphisms, is indeed a functor. This definition of $G(f)$ insures that i is a morphism of functors $F \rightarrow G$, and it clearly has an inverse, defined by $i^{-1}(X) = i(X)^{-1}$. \square

Exercise 7.2:12. Write out the verification that G , constructed as above, is a functor.

We can now get our desired result about tying representing objects together into a functor. In thinking about results such as the next lemma, I find it useful to keep in mind the case where $\mathbf{C} = \mathbf{Set}$, $\mathbf{D} = \mathbf{Group}$, and A is the bifunctor associating to every set X and group G the set $|G|^X$ of X -tuples of elements of G .

Lemma 7.2.9. *Suppose that \mathbf{C} and \mathbf{D} are categories, and that for each $X \in \text{Ob}(\mathbf{C})$ we are given a functor $A(X, -): \mathbf{D} \rightarrow \mathbf{Set}$ and an object $F(X) \in \text{Ob}(\mathbf{D})$ representing this functor, via an isomorphism $i(X): A(X, -) \cong h_{F(X)}$.*

Then if the given functors $A(X, -)$ are in fact the values of a bifunctor $A: \mathbf{C}^{\text{op}} \times \mathbf{D} \rightarrow \mathbf{Set}$ at the objects of \mathbf{C} , then the objects $F(X)$ of \mathbf{D} can be made the values of a functor $F: \mathbf{C} \rightarrow \mathbf{D}$ in a unique way so that the isomorphisms $i(X)$ comprise an isomorphism of bifunctors

$$(7.2.10) \quad i: A(-, -) \cong \mathbf{D}(F(-), -).$$

Conversely, if the objects $F(X)$ are the values at the objects X of a functor $F: \mathbf{C} \rightarrow \mathbf{D}$, we can make the family of functors $A(X, -)$ into a bifunctor $A: \mathbf{C}^{\text{op}} \times \mathbf{D} \rightarrow \mathbf{Set}$ in a unique way so that the isomorphisms $i(X)$ again give an isomorphism (7.2.10) of bifunctors.

Proof. On the one hand, if $A: \mathbf{C}^{\text{op}} \times \mathbf{D} \rightarrow \mathbf{Set}$ is a bifunctor, the induced system of functors $A(X, -): \mathbf{D} \rightarrow \mathbf{Set}$ will together constitute a single functor which we may call $B: \mathbf{C}^{\text{op}} \rightarrow \mathbf{Set}^{\mathbf{D}}$ (Lemma 6.10.1). For each $X \in \text{Ob}(\mathbf{C})$ we have an isomorphism $i(X)$ of $B(X)$ with a hom-functor $h_{F(X)}$, so by the preceding lemma we get an isomorphic functor $C: \mathbf{C}^{\text{op}} \rightarrow \mathbf{Set}^{\mathbf{D}}$, such that $C(X) = h_{F(X)}$, and the isomorphism $i: B \cong C$ is made up of the $i(X)$'s. Now by Corollary 7.2.6, the covariant hom-functors h_Y ($Y \in \text{Ob}(\mathbf{D})$) form a full subcategory of $\mathbf{Set}^{\mathbf{D}}$ isomorphic to \mathbf{D} via the Yoneda embedding $Y \mapsto h_Y$. Hence the functor $C: \mathbf{C}^{\text{op}} \rightarrow \mathbf{Set}^{\mathbf{D}}$ is induced by precomposing this embedding $\mathbf{D}^{\text{op}} \rightarrow \mathbf{Set}^{\mathbf{D}}$ with a unique functor $\mathbf{C}^{\text{op}} \rightarrow \mathbf{D}^{\text{op}}$, which is equivalent to a functor $F: \mathbf{C} \rightarrow \mathbf{D}$, and this F is the functor of the statement of the lemma.

Inversely, if F is given as a functor, let us consider each functor $A(X, -)$ as an object $B(X)$ of $\mathbf{Set}^{\mathbf{D}}$. Then for each X we have an isomorphism $i(X): B(X) \cong h_{F(X)}$, and applying the preceding lemma to the isomorphisms $i(X)^{-1}$, we conclude that the objects $B(X)$ are the values of a functor $B: \mathbf{C}^{\text{op}} \rightarrow \mathbf{Set}^{\mathbf{D}}$, which we may regard as a bifunctor $A: \mathbf{C}^{\text{op}} \times \mathbf{D} \rightarrow \mathbf{Set}$, and again the values of i become an isomorphism of bifunctors. \square

The above lemma concerns systems of objects representing covariant hom-functors; let us state the corresponding result for contravariant hom-functors. A priori, this means replacing \mathbf{D} by \mathbf{D}^{op} . But it is then natural to replace the ‘‘parametrizing’’ category \mathbf{C}^{op} by \mathbf{C} so as to keep the parametrization of the constructed objects of \mathbf{D} covariant. And having done that much, why not interchange the names of \mathbf{C} and \mathbf{D} so as to get a set-up parallel to that of the preceding case? Doing so, we get

Lemma 7.2.11. *Suppose that \mathbf{C} and \mathbf{D} are categories, and that for each $Y \in \text{Ob}(\mathbf{D})$ we are given a functor $A(-, Y): \mathbf{C}^{\text{op}} \rightarrow \mathbf{Set}$ and an object $U(Y) \in \text{Ob}(\mathbf{C})$ representing this contravariant functor, via an isomorphism $j(Y): A(-, Y) \cong h^{U(Y)}$.*

Then if the given functors $A(-, Y)$ are the values of a bifunctor $A: \mathbf{C}^{\text{op}} \times \mathbf{D} \rightarrow \mathbf{Set}$ at the objects of \mathbf{D} , the family of objects $U(Y)$ of \mathbf{C} can be made the values of a functor $U: \mathbf{D} \rightarrow \mathbf{C}$ in a unique way so that the isomorphisms $j(Y)$ constitute an isomorphism of bifunctors

$$(7.2.12) \quad j: A(-, -) \cong \mathbf{C}(-, U(-)).$$

Conversely, if the objects $U(Y)$ are the images of the objects Y under a functor $U: \mathbf{D} \rightarrow \mathbf{C}$, we can make the family of functors $A(-, Y)$ into a bifunctor $A: \mathbf{C}^{\text{op}} \times \mathbf{D} \rightarrow \mathbf{Set}$ in a unique way so that the isomorphisms $j(Y)$ together give an isomorphism (7.2.12) of bifunctors. \square

7.3. Adjoint functors. Let us look at some examples of the situation of the two preceding lemmas – families of objects that we characterized individually as the representing objects for certain naturally occurring functors, but that turned out, themselves, to fit together into a functor. By those lemmas, this means that the system of functors that these objects represented fit together into a bifunctor. We shall see that in each of these cases, this structure of bifunctor was actually present in the original situation, providing an explanation of why our constructions yielded functors.

The free group on each set X is the object of **Group** representing the functor $G \mapsto |G|^X = \mathbf{Set}(X, U(G))$. So the free group functor arises by representing the family of functors **Group** \rightarrow **Set** obtained by inserting all sets as the first argument of the bifunctor

$$\mathbf{Set}(-, U(-)): \mathbf{Set}^{\text{op}} \times \mathbf{Group} \rightarrow \mathbf{Set}.$$

The analogous description obviously applies in any category \mathbf{C} having free objects with respect to

a concretization $U: \mathbf{C} \rightarrow \mathbf{Set}$.

If G is a group, the *abelianization* of G is the object of \mathbf{Ab} representing the functor $\mathbf{Ab} \rightarrow \mathbf{Set}$ given by $A \mapsto \mathbf{Group}(G, A)$. The symbol $\mathbf{Group}(G, A)$ makes sense because \mathbf{Ab} is a subcategory of \mathbf{Group} , but to put this example in the context of the general pattern, let us write V for the inclusion functor of \mathbf{Ab} in \mathbf{Group} . We then see that the abelianization functor arises by representing the family of set-valued functors obtained by inserting values in the first argument of the bifunctor

$$\mathbf{Group}(-, V(-)): \mathbf{Group}^{\text{op}} \times \mathbf{Ab} \rightarrow \mathbf{Set}.$$

In the same way, if W denotes the forgetful functor $\mathbf{Group} \rightarrow \mathbf{Monoid}$, then the functor taking a monoid to its universal enveloping group arises by representing the family of set-valued functors obtained by inserting values in the first argument of the bifunctor

$$\mathbf{Monoid}(-, W(-)): \mathbf{Monoid}^{\text{op}} \times \mathbf{Group} \rightarrow \mathbf{Set}.$$

The above were left universal examples, that is, constructions $F: \mathbf{C} \rightarrow \mathbf{D}$ such that each object $F(X)$ represented a covariant functor $\mathbf{D} \rightarrow \mathbf{Set}$. We see that in each such case, the bifunctor from which these covariant functors were extracted had the form

$$(7.3.1) \quad \mathbf{C}(-, U(-)): \mathbf{C}^{\text{op}} \times \mathbf{D} \rightarrow \mathbf{Set},$$

for some functor $U: \mathbf{D} \rightarrow \mathbf{C}$. Taking (7.3.1) to be the A in Lemma 7.2.9, we see that the universal properties of the objects $F(X)$ in terms of U can be formulated in each of these cases as

$$\mathbf{C}(-, U(-)) \cong \mathbf{D}(F(-), -)$$

– a strikingly symmetrical condition!

Let us consider one right universal example. Given a monoid S , we considered above the construction of the universal group G with a homomorphism of S into G_{md} ; but there is also a universal group G with a homomorphism of G_{md} into S , namely the group $G = S_{\text{inv}}$ of invertible elements (“units”) of S . If we write $F: \mathbf{Group} \rightarrow \mathbf{Monoid}$ for the forgetful functor $G \mapsto G_{\text{md}}$, and call the above group-of-units functor $U: \mathbf{Monoid} \rightarrow \mathbf{Group}$, we see that $U(S)$ represents the contravariant functor associating to each group G the set $\mathbf{Monoid}(F(G), S)$. If we write \mathbf{C} and \mathbf{D} for \mathbf{Group} and \mathbf{Monoid} , then on taking $\mathbf{D}(F(-), -)$ for the bifunctor A in the last formulation of Lemma 7.2.11, we get an isomorphism characterizing this right universal construction U :

$$\mathbf{D}(F(-), -) \cong \mathbf{C}(-, U(-)).$$

This is exactly the same as the isomorphism characterizing our examples of left universal constructions – but written in reverse order, and looked at as characterizing U in terms of F , rather than F in terms of U ! The fact that these two situations are characterized by the same isomorphism means that a functor F gives objects representing the covariant functors $\mathbf{C}(X, U(-))$ if and only if U gives objects representing the contravariant functors $\mathbf{D}(F(-), Y)$.

Let us test this conclusion, by turning our characterization of the free group construction upside down. Since the free group $F(X)$ on a set X is left universal among groups G with set maps of X into their underlying sets $U(G)$, the *underlying set* $U(G)$ of a group G should be right-universal among all sets X with group homomorphisms from the free group $F(X)$ into G . And indeed, though it may seem bizarre to treat the free-group construction as given and the

underlying-set construction as something to be characterized, the universal property certainly holds: For any group G , $U(G)$ is a set with a homomorphism $u: F(U(G)) \rightarrow G$, such that given any homomorphism f from a free group $F(X)$ on any set into G , there is a unique set map $h: X \rightarrow U(G)$ (which you should be able to describe) such that $f = uF(h)$. This property of underlying sets is sometimes even useful. For instance, in showing that every group can be presented by generators and relations, one wishes to write an arbitrary group G as a homomorphic image of a free group on some set X . The above property says that there is a universal choice of such X , namely the underlying set $U(G)$ of G .

Before setting out to tie together all our ways of describing these universal constructions, let us prove a lemma that will allow us to relate isomorphisms of bifunctors as above to systems of maps $X \rightarrow U(F(X))$ and $F(U(Y)) \rightarrow Y$. (This observation is an instance of the general principle noted following Corollary 7.2.5.)

Lemma 7.3.2. *Let \mathbf{C} and \mathbf{D} be categories and $U: \mathbf{D} \rightarrow \mathbf{C}$, $F: \mathbf{C} \rightarrow \mathbf{D}$ functors, and consider the two bifunctors $\mathbf{C}^{\text{op}} \times \mathbf{D} \rightarrow \mathbf{Set}$,*

$$\mathbf{C}(-, U(-)), \quad \mathbf{D}(F(-), -).$$

Then a morphism of bifunctors

$$(7.3.3) \quad a: \mathbf{C}(-, U(-)) \rightarrow \mathbf{D}(F(-), -)$$

is determined by its values on identity morphisms $\text{id}_{U(D)} \in \mathbf{C}(U(D), U(D))$ ($D \in \text{Ob}(\mathbf{D})$). In fact, given a as above, if we write $\alpha(D) = a(U(D), D)(\text{id}_{U(D)}) \in \mathbf{D}(F(U(D)), D)$, then this family of morphisms comprises a morphism of functors,

$$(7.3.4) \quad \alpha: FU \rightarrow \text{Id}_{\mathbf{D}}$$

and this construction yields a bijection between morphisms (7.3.3) and morphisms (7.3.4). Given a morphism (7.3.4), the corresponding morphism (7.3.3) can be described as acting on $f \in \mathbf{C}(C, U(D))$ by first applying F to get $F(f): F(C) \rightarrow FU(D)$, then composing this with $\alpha(D): FU(D) \rightarrow D$, getting $a(f) = \alpha(D)F(f): F(C) \rightarrow D$.

Likewise, a morphism of bifunctors in the opposite direction to (7.3.3),

$$(7.3.5) \quad b: \mathbf{D}(F(-), -) \rightarrow \mathbf{C}(-, U(-))$$

is determined by its values on identity morphisms, in this case morphisms $\text{id}_{F(C)} \in \mathbf{D}(F(C), F(C))$ ($C \in \text{Ob}(\mathbf{C})$), and writing $\beta(C) = b(C, F(C))(\text{id}_{F(C)}) \in \mathbf{C}(C, U(F(C)))$, we get a bijection between morphisms (7.3.5) and morphisms

$$(7.3.6) \quad \beta: \text{Id}_{\mathbf{C}} \rightarrow UF.$$

Given β , the corresponding morphism b can be described as taking $f \in \mathbf{D}(F(C), D)$ to $U(f)\beta(C) \in \mathbf{C}(C, U(D))$.

Sketch of Proof. Consider a morphism a as in (7.3.3). For each $D \in \text{Ob}(\mathbf{D})$ this gives a morphism of functors $\mathbf{C}(-, U(D)) \rightarrow \mathbf{D}(F(-), D)$. Since the first of these functors is $h^{U(D)}$, the Yoneda Lemma says this morphism is determined by its value on the identity morphism of $U(D)$. It is straightforward to verify that the condition that these morphisms of functors $\mathbf{C}(-, U(D)) \rightarrow \mathbf{D}(F(-), D)$ should comprise a single morphism of bifunctors (7.3.3) is equivalent to the condition that the values of these morphisms on identities should comprise a morphism of functors (7.3.4).

The reader can easily check that the description of how to recover (7.3.3) from (7.3.4) also leads to a morphism of functors, and that this construction is inverse to the first.

The second paragraph follows by duality. \square

Exercise 7.3:1. Write out the ‘‘straightforward’’ verification and the ‘‘easy check’’ referred to in the first paragraph of the proof of the lemma.

To get a feel for the above construction, you might start with the morphism of bifunctors a that associates to every set map from a set X to the underlying set $U(G)$ of a group G the induced group homomorphism from the free group $F(X)$ into G . Determine the morphism of functors α that the above construction yields, and check explicitly that the ‘‘inverse’’ construction described does indeed recover a . In this example, one finds that a is invertible; calling its inverse b , you might similarly work out for this b the constructions of the second assertion of the lemma.

With the help of Lemmas 7.2.9, 7.2.11 and 7.3.2, we can now give several descriptions of the type of universal construction discussed at the beginning of this section.

Theorem 7.3.7. *Let \mathbf{C} and \mathbf{D} be categories. Then the following data are equivalent:*

(i) *A pair of functors $U: \mathbf{D} \rightarrow \mathbf{C}$, $F: \mathbf{C} \rightarrow \mathbf{D}$, and an isomorphism*

$$i: \mathbf{C}(-, U(-)) \cong \mathbf{D}(F(-), -)$$

of functors $\mathbf{C}^{\text{op}} \times \mathbf{D} \rightarrow \mathbf{Set}$.

(ii) *A functor $U: \mathbf{D} \rightarrow \mathbf{C}$, and for every $C \in \text{Ob}(\mathbf{C})$, an object $R_C \in \text{Ob}(\mathbf{D})$ and an element $u_C \in \mathbf{C}(C, U(R_C))$ which are universal among such object-element pairs, i.e., which represent the covariant functor $\mathbf{C}(C, U(-)): \mathbf{D} \rightarrow \mathbf{Set}$ (cf. Theorem 7.2.1 and Definition 7.2.3).*

(ii*) *A functor $F: \mathbf{C} \rightarrow \mathbf{D}$, and for every $D \in \text{Ob}(\mathbf{D})$, an object $R_D \in \text{Ob}(\mathbf{C})$ and an element $v_D \in \mathbf{D}(F(R_D), D)$ which are universal among such object-element pairs, i.e., which represent the contravariant functor $\mathbf{D}(F(-), D): \mathbf{C}^{\text{op}} \rightarrow \mathbf{Set}$.*

(iii) *A pair of functors $U: \mathbf{D} \rightarrow \mathbf{C}$, $F: \mathbf{C} \rightarrow \mathbf{D}$, and a pair of morphisms of functors*

$$\eta: \text{Id}_{\mathbf{C}} \rightarrow UF, \quad \varepsilon: FU \rightarrow \text{Id}_{\mathbf{D}},$$

such that the two composites

$$U \xrightarrow{\eta \circ U} UFU \xrightarrow{U \circ \varepsilon} U, \quad F \xrightarrow{F \circ \eta} FUF \xrightarrow{\varepsilon \circ F} F,$$

are the identity morphisms of U and F respectively. (For the ‘‘ \circ ’’ notation see Lemma 6.10.2.)

Sketch of Proof. The equivalence of (i) with (ii) and with (ii*) is given by Lemma 7.2.9 with $A(-, -) = \mathbf{C}(-, U(-))$, and Lemma 7.2.11 with $A(-, -) = \mathbf{D}(F(-), -)$, respectively. By Lemma 7.3.2, an isomorphism of bifunctors as in (i) must correspond to a pair of morphisms of functors $\eta: \text{Id}_{\mathbf{C}} \rightarrow UF$, $\varepsilon: FU \rightarrow \text{Id}_{\mathbf{D}}$ which induce mutually inverse morphisms of bifunctors. I claim that the conditions needed for these induced morphisms to be mutually inverse are those shown diagrammatically in (iii).

In the verification of this statement (made an exercise below), one assumes α and β given as in Lemma 7.3.2, uses the formulas for a and b in terms of these to express the composites ab and ba , and must prove that these composites are the identity morphisms. By Yoneda’s Lemma, it suffices to check these equalities on appropriate identity morphisms. (With what objects of \mathbf{C} and \mathbf{D} in the slots of $\mathbf{D}(F(-), -)$, respectively $\mathbf{C}(-, U(-))$?) This approach quickly gives the

desired statements. However, if one prefers to see directly that these statements are equivalent to ab and ba fixing *all* morphisms $f \in \mathbf{D}(F(C), D)$, respectively $g \in \mathbf{C}(C, U(D))$, then one may combine the equations saying that the latter conclusions hold with the commutativity of the diagram expressing the functoriality of a , respectively b , applied to the morphism f , respectively g . \square

- Exercise 7.3:2.** (i) Write out the verification sketched in the last paragraph of the above proof.
 (ii) Show that η will be composed of the “universal morphisms” u_C of point (ii) of the theorem, and ε will be composed of the universal morphisms v_D of point (ii*).*
 (iii) Take one universal construction, e.g., that of free groups, write down the equalities expressed diagrammatically in part (iii) of the above theorem for this construction in terms of maps of set- and group-elements, and explain why they hold in *this case*.

Definition 7.3.8. Given categories \mathbf{C} and \mathbf{D} and functors $U: \mathbf{D} \rightarrow \mathbf{C}$, $F: \mathbf{C} \rightarrow \mathbf{D}$, an isomorphism

$$i: \mathbf{C}(-, U(-)) \cong \mathbf{D}(F(-), -)$$

of bifunctors $\mathbf{C}^{\text{op}} \times \mathbf{D} \rightarrow \mathbf{Set}$, or equivalently, a pair of morphisms of functors ε, η satisfying the condition of point (iii) of the above theorem, is called an adjunction between U and F .

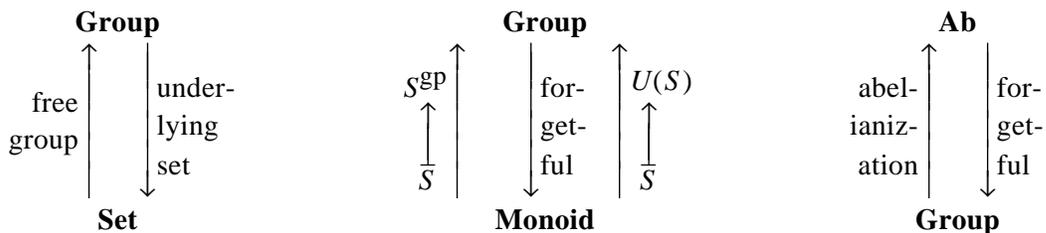
In this situation, U is called the “right adjoint” of F , and F the “left adjoint” of U (referring to their occurrence in the right and left slots of the hom-bifunctors in the above isomorphism). The morphisms of functors η and ε are called, respectively, the unit and counit of the adjunction.

Historical note: The term “adjoint” was borrowed from analysis, where the adjoint of a bounded operator between Hilbert spaces, $A: X \rightarrow Y$, is the operator $B: Y \rightarrow X$ characterized by the condition on inner products $(x, By) = (Ax, y)$.

The student who finds condition (iii) of Theorem 7.3.7 hard to grasp will be happy to know that we will not make much use of it in the next few chapters. (I have trouble with it myself.) But we will use the morphisms η and ε named in that condition, so you should get a clear idea of how these act. (What we will seldom use is the fact that the indicated compositional condition on a pair of morphisms η, ε is equivalent to their being the unit and counit of an adjunction. Nevertheless, I recommend working Exercise 7.3:2 this once.)

The terms “unit” and “counit” will be easier to explain when we consider the concepts of *monad* and *comonad* in Chapter 10 (not yet written).

We can now characterize as right or left adjoints many of the universal constructions we are familiar with. The three diagrams below show the cases we used above to motivate the concept. In each of these, a pair of successive vertical arrows between two categories represents a pair of mutually adjoint functors, the right adjoint being shown on the right and the left adjoint on the left.



The middle diagram is interesting in that the forgetful functor there (in the notation of §3.11, $G \mapsto G_{\text{md}}$) has both a left and a right adjoint. In the first diagram, we can, as mentioned, replace **Group** with any category \mathbf{C} having free objects with respect to a concretization U . A still wider generalization is noted in the next exercise.

Exercise 7.3:3. If you did not do Exercise 7.2:5, prove that if \mathbf{C} is a category with small coproducts and $U: \mathbf{C} \rightarrow \mathbf{Set}$ a functor, then U has a left adjoint if and only if it is *representable*.

(Exercise 7.2:5 was essentially the case of this result where U was faithful, so that it could be called a “concretization” and its left adjoint a “free object” construction; but faithfulness played no part in the proof. In Chapter 9 we shall extend the concept of “representable functor” from set-valued functors to algebra-valued functors, and generalize the above result to the resulting much wider context.)

Exercise 7.3:4. Show that the left (or right) adjoint of a functor, if one exists, is unique up to canonical isomorphism, and conversely, that if A and B are isomorphic functors, then any functor which can be made a left (or right) adjoint of A can also be made a left (or right) adjoint of B .

Exercise 7.3:5. Show that if $A: \mathbf{C} \rightarrow \mathbf{D}$, $B: \mathbf{D} \rightarrow \mathbf{C}$ give an equivalence of categories, then B is both a right and a left adjoint to A .

The next exercise is a familiar example in disguise.

Exercise 7.3:6. Let \mathbf{C} be the category with $\text{Ob}(\mathbf{C}) = \text{Ob}(\mathbf{Group})$, but with morphisms defined so that for groups G and H , $\mathbf{C}(G, H) = \mathbf{Set}(|G|, |H|)$. Thus **Group** is a subcategory of \mathbf{C} , with the same object set but smaller morphism sets. Does the inclusion functor $\mathbf{Group} \rightarrow \mathbf{C}$ have a left and/or a right adjoint?

There are many other constructions whose universal properties translate into adjointness statements: The forgetful functor $\mathbf{Ring}^1 \rightarrow \mathbf{Monoid}$ that remembers only the multiplicative structure has as left adjoint the *monoid ring* construction. The forgetful functor $\mathbf{Ring}^1 \rightarrow \mathbf{Ab}$ that remembers only the additive structure has for left adjoint the *tensor ring* construction. (These two constructions were discussed briefly toward the end of §3.12.) The inclusion of the category of compact Hausdorff spaces in that of arbitrary topological spaces has for left adjoint the Stone-Ćech compactification functor (§3.17). The functor associating to every commutative ring its Boolean ring of idempotent elements has as left adjoint the construction asked for in Exercise 3.14:3(iv). The forgetful functors going from **Lattice** to \vee -**Semilattice** and \wedge -**Semilattice**, and from these in turn to **POSet**, have left adjoints which you were asked to construct in Exercise 5.1:8.

The student familiar with Lie algebras (§8.7 below) will note that the functor associating to an associative algebra A the Lie algebra A_{Lie} with the same underlying vector space as A , and with the commutator operation of A for Lie bracket, has for left adjoint the *universal enveloping algebra* construction. (The Poincaré-Birkhoff-Witt Theorem gives a normal form for this universal object; I hope to treat such normal form results in a much later chapter. Cf. [40])

Suppose \mathbf{C} is a category having *products* and *coproducts* of all pairs of objects. We know that each of these constructions will give a functor $\mathbf{C} \times \mathbf{C} \rightarrow \mathbf{C}$. Can these functors be characterized as adjoints of some functors $\mathbf{C} \rightarrow \mathbf{C} \times \mathbf{C}$? Similarly, can the *tensor product* functor $\mathbf{Ab} \times \mathbf{Ab} \rightarrow \mathbf{Ab}$ be characterized as an adjoint of some functor $\mathbf{Ab} \rightarrow \mathbf{Ab} \times \mathbf{Ab}$?

The universal property of the product functor $\mathbf{C} \times \mathbf{C} \rightarrow \mathbf{C}$ is a right universal one, so if it arises as an adjoint, it should be a right adjoint to some functor $A: \mathbf{C} \rightarrow \mathbf{C} \times \mathbf{C}$. No such functor was evident in our definition of products. However, we can search for such an A by posing the universal problem whose solution would be a *left* adjoint to the product functor: Given

$X \in \text{Ob}(\mathbf{C})$, does there exist $(Y, Z) \in \text{Ob}(\mathbf{C} \times \mathbf{C})$ with a universal example of a morphism $X \rightarrow Y \times Z$? Since a morphism $X \rightarrow Y \times Z$ corresponds to a morphism $X \rightarrow Y$ and a morphism $X \rightarrow Z$, this asks whether there exists a pair (Y, Z) of objects of \mathbf{C} universal for having a morphism from X to each member of this pair. In fact, the pair (X, X) is easily seen to have the desired universal property. This leads us to define the “diagonal functor” $\Delta: \mathbf{C} \rightarrow \mathbf{C} \times \mathbf{C}$ taking each object X to (X, X) , and each morphism f to (f, f) . It is now easy to check that the universal property of the direct product construction is that of a right adjoint to Δ . Moreover, similar reasoning shows that the universal property of the coproduct is that of a left adjoint of Δ . So in a category \mathbf{C} having both products and coproducts, we have the diagram of adjoint functors

$$\begin{array}{ccc} & \mathbf{C} & \\ \uparrow & \downarrow \Delta & \uparrow \\ \mathbf{C} & & \mathbf{C} \\ \uparrow & & \uparrow \\ \mathbf{C} & & \mathbf{C} \end{array}$$

$\mathbf{C} \times \mathbf{C}$

We recall that if \mathbf{C} is \mathbf{Ab} , the constructions of pairwise products and coproducts (“direct products and direct sums”) coincide. So in that case we get a “cyclic” diagram of adjoints.

Exercise 7.3:7. Does the direct product construction on \mathbf{Set} have a *right* adjoint? Does the coproduct construction have a *left* adjoint?

The next exercise is one of my favorites:

Exercise 7.3:8. Recall that $\mathbf{2}$ denotes the category with two objects, 0 and 1 , and exactly one nonidentity morphism, $0 \rightarrow 1$, so that for any category \mathbf{C} , an object of $\mathbf{C}^{\mathbf{2}}$ corresponds to a choice of two objects $A_0, A_1 \in \text{Ob}(\mathbf{C})$ and a morphism $f: A_0 \rightarrow A_1$.

Let $p_0: \mathbf{Group}^{\mathbf{2}} \rightarrow \mathbf{Group}$ denote the functor taking each object (A_0, A_1, f) to its first component A_0 , and likewise every morphism $(a_0, a_1): (A_0, A_1, f) \rightarrow (B_0, B_1, g)$ of $\mathbf{Group}^{\mathbf{2}}$ to its first component a_0 .

Investigate whether p_0 has a left adjoint, and whether it has a right adjoint. If a left adjoint is found, investigate whether this in turn has a left adjoint (clearly it has a right adjoint – namely p_0); likewise if p_0 has a right adjoint, investigate whether this in turn has a right adjoint; and so on, as long as further adjoints on either side can be found.

Exercise 7.3:9. Let G be a group, and $G\text{-Set}$ the category of all G -sets.

You can probably think of one or more very easily described functors from \mathbf{Set} to $G\text{-Set}$, or vice versa. Choose one of them, and apply the idea of the preceding exercise; i.e., look for a left adjoint and/or a right adjoint, and for further adjoints of these, as long as you can find any.

When you are finished, does the chain of functors you have gotten contain all the “easily described functors” between these two categories that you were able to think of? If not, take one that was missed, and do the same with it.

Exercise 7.3:10. Translate the idea indicated in observation (a) following Exercise 3.8:1 into questions of the existence of adjoints to certain functors between categories $G_1\text{-Set}$ and $G_2\text{-Set}$, determine whether these adjoints do in fact exist, and if they do, describe them as well as you can.

Let us now consider the case of the tensor product construction, $\otimes: \mathbf{Ab} \times \mathbf{Ab} \rightarrow \mathbf{Ab}$. It is the solution to a left universal problem, and we can characterize this problem as arising, as described in Lemma 7.2.9, from the bifunctor $\text{Bil}: (\mathbf{Ab} \times \mathbf{Ab})^{\text{op}} \times \mathbf{Ab} \rightarrow \mathbf{Set}$, where for abelian groups A, B, C we let $\text{Bil}((A, B), C)$ denote the set of bilinear maps $(A, B) \rightarrow C$. From the preceding

examples, we might expect $\text{Bil}((A, B), C)$ to be expressible in the form $(\mathbf{Ab} \times \mathbf{Ab})((A, B), U(C))$ for some functor $U: \mathbf{Ab} \rightarrow \mathbf{Ab} \times \mathbf{Ab}$.

But, in fact, it cannot be so expressed; in other words, the tensor product construction $\mathbf{Ab} \times \mathbf{Ab} \rightarrow \mathbf{Ab}$, though it is a left universal construction, is *not* a left adjoint. The details (and a different sense in which the tensor product *is* a left adjoint functor construction) are something you can work out:

Exercise 7.3:11. (i) Show that the functor $\otimes: \mathbf{Ab} \times \mathbf{Ab} \rightarrow \mathbf{Ab}$ has no left or right adjoint.

(ii) On the other hand, show that for any fixed abelian group A , the functor $A \otimes -: \mathbf{Ab} \rightarrow \mathbf{Ab}$ is left adjoint to the functor $\text{Hom}(A, -): \mathbf{Ab} \rightarrow \mathbf{Ab}$. (I am writing $\text{Hom}(A, B)$ for the *abelian group* of homomorphisms from A to B , in contrast to $\mathbf{Ab}(A, B)$ the *set* of such homomorphisms – an arbitrary and ad hoc notational choice.)

(iii) Investigate whether the functor $A \otimes -: \mathbf{Ab} \rightarrow \mathbf{Ab}$ has a left adjoint, and whether $\text{Hom}(A, -): \mathbf{Ab} \rightarrow \mathbf{Ab}$ has a right adjoint. If such adjoints do not *always* exist, do they exist for *some* choices of A ?

If you are familiar enough with ring theory, generalize the above problems to modules over a fixed commutative ring k , or to bimodules over pairs of noncommutative rings.

Exercise 7.3:12. For a fixed set A , does the functor $\mathbf{Set} \rightarrow \mathbf{Set}$ given by $S \mapsto S \times A$ have a left or right adjoint?

A situation which is similar to that of the tensor product, in that the question of whether a construction is an adjoint depends on what we take as the variable, is considered in

Exercise 7.3:13. In this exercise “ring” will mean commutative ring with 1; recall that we denote the category of such rings $\mathbf{CommRing}^1$.

If R is a ring and X any set, $R[X]$ will denote the polynomial ring over R in an X -tuple of indeterminates.

(i) Show that for X a nonempty set, the functor $P_X: \mathbf{CommRing}^1 \rightarrow \mathbf{CommRing}^1$ taking each ring R to $R[X]$ has neither a right nor a left adjoint, and similarly that for R a ring, the functor $Q_R: \mathbf{Set} \rightarrow \mathbf{CommRing}^1$ taking each set X to $R[X]$ has neither a right nor a left adjoint.

(ii) On the other hand, show that the functor $\mathbf{CommRing}^1 \times \mathbf{Set} \rightarrow \mathbf{CommRing}^1$ taking a pair (R, X) to $R[X]$ is an adjoint (on the appropriate side) of an easily described functor.

(iii) For any ring R , let $\mathbf{CommRing}_R^1$ denote the category of commutative R -algebras (rings S given with homomorphisms $R \rightarrow S$), and R -algebra homomorphisms (ring homomorphisms making commuting triangles with R . In the notation of Exercise 6.8:26(ii), this is the comma category $(R \downarrow \mathbf{CommRing}^1)$.)

Similarly, for any set X , let $\mathbf{CommRing}_X^1$ denote the category of rings S given with set maps $X \rightarrow |S|$, and again having for morphisms the ring homomorphisms making commuting triangles. (This is the comma category $(X \downarrow U)$, where U is the underlying set functor of $\mathbf{CommRing}^1$. Note that to keep the symbols $\mathbf{CommRing}_R^1$ and $\mathbf{CommRing}_X^1$ unambiguous, we must remember to use distinct symbols for rings and sets.)

Show that for any R , the functor $\mathbf{Set} \rightarrow \mathbf{CommRing}_R^1$ taking X to $R[X]$ can be characterized as an adjoint, and that for any X , the functor $\mathbf{CommRing}^1 \rightarrow \mathbf{CommRing}_X^1$ taking R to $R[X]$ can also be characterized as an adjoint.

(iv) Investigate similar questions for the formal power series construction, $R[[X]]$; in particular, whether the analog of (i) is true.

Here is still another way to make the tensor product construction into an adjoint functor:

Exercise 7.3:14. (i) Let **Bil** be the category whose objects are all 4-tuples (A, B, β, C) where A, B, C are abelian groups, and $\beta: (A, B) \rightarrow C$ is a bilinear map, and with morphisms defined in the natural way. (Say what this natural way is!) Show that the forgetful functor $\mathbf{Bil} \rightarrow \mathbf{Ab} \times \mathbf{Ab}$, taking each such 4-tuple to its first two components, has a left adjoint, which is “essentially” the tensor-product construction.

(ii) Show that an analogous trick can be used to convert any isomorphism of bifunctors as in the Lemma 7.2.9 into an adjunction. (Between what categories?) Do the same for the situation of Lemma 7.2.11.

Exercise 7.3:15. Describe all pairs of adjoint functors at least one member of which is a *constant* functor, i.e., a functor taking all objects of its domain category to a single object X of its codomain category, and all morphisms of its domain category to id_X .

What happens when we compose two functors arising from adjunctions?

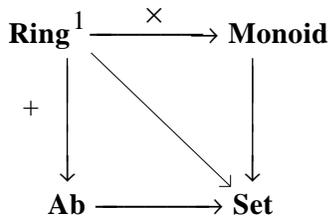
Note that the *abelianization* of the *free* group on a set X is a *free abelian* group on X . That is, when we compose these two functors, each of which is a left adjoint, we get another functor with that property. The general statement is simple, and is delightfully easy to prove.

Theorem 7.3.9. Suppose $\mathbf{E} \begin{matrix} \xrightarrow{U} \\ \xleftarrow{F} \end{matrix} \mathbf{D} \begin{matrix} \xrightarrow{V} \\ \xleftarrow{G} \end{matrix} \mathbf{C}$ are pairs of adjoint functors, with U and V the right adjoints, F and G the left adjoints. Then $\mathbf{E} \begin{matrix} \xrightarrow{VU} \\ \xleftarrow{FG} \end{matrix} \mathbf{C}$ are also adjoint, with VU the right adjoint and FG the left adjoint.

Proof. $\mathbf{C}(-, VU(-)) \cong \mathbf{D}(G(-), U(-)) \cong \mathbf{E}(FG(-), -)$. \square

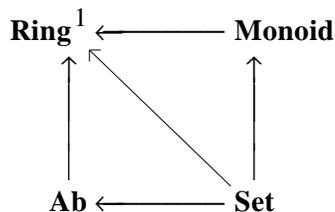
Exercise 7.3:16. Suppose U, V, F and G are as above, η and ε are the unit and counit of the adjoint pair U, F , and η' and ε' are the unit and counit of the adjoint pair V, G . Describe the unit and counit of the adjoint pair VU, FG .

For further examples of the above theorem, consider two ways we can factor the forgetful functor from \mathbf{Ring}^1 to \mathbf{Set} . We can first pass from a ring to its multiplicative monoid, then go to the underlying set thereof, or we can first pass from the ring to its additive group, and then to the underlying set:



Taking left adjoints, we get the two decompositions of the *free ring* construction noted in §3.12: as the free-monoid functor followed by the monoid-ring functor, and as the free abelian group functor

followed by the tensor algebra functor:



7.4. Number-theoretic interlude: the p -adic numbers, and related constructions. While you digest the concept of adjunction (fundamentally simple, yet daunting in its multiple facets), let us look at some constructions of a different sort, which we did not meet any examples of in the ‘‘Cook’s tour’’ of Chapter 3. In this section we will develop a particular case important in number theory; the general category-theoretic concept will be defined in the next section. A much broader generalization, which also embraces several constructions we *have* studied, will be developed in the section after that.

Suppose we are interested in solving the equation

$$(7.4.1) \quad x^2 = -1$$

in the integers, \mathbb{Z} . Of course, we know it has no solution in the real numbers, let alone the integers, but we will ignore that dreary fact for the moment.

We may observe that the above equation does have a solution in the finite ring \mathbb{Z}_5 , in fact, two solutions, 2 and 3. Up to sign, these are the same, so let us look for a solution of (7.4.1) in \mathbb{Z} satisfying

$$x \equiv 2 \pmod{5}.$$

An integer x which is $\equiv 2 \pmod{5}$ has the form $5y+2$, so we may rewrite (7.4.1) as

$$(5y+2)^2 = -1$$

and expand, to see what information we can learn about y . We get $25y^2+20y = -5$. Hence $20y \equiv -5 \pmod{25}$, and dividing by 5, we get $4y \equiv -1 \pmod{5}$. This has the unique solution

$$y \equiv 1 \pmod{5},$$

which, substituted back, determines x modulo 25:

$$x = 5y+2 \equiv 5 \cdot 1 + 2 = 7 \pmod{25}.$$

We continue in the same fashion: At the next stage, putting $x = 25z+7$ we have $(25z+7)^2 = -1$. You should verify that this implies

$$z \equiv 2 \pmod{5},$$

which leads to

$$x \equiv 57 \pmod{125}.$$

Can we go on indefinitely? This is answered in

Exercise 7.4:1. (i) Show that given $i > 0$, and $c \in \mathbb{Z}$ such that $c^2 \equiv -1 \pmod{5^i}$, there exists $c' \in \mathbb{Z}$ such that $c'^2 \equiv -1 \pmod{5^{i+1}}$, and $c' \equiv c \pmod{5^i}$.

(ii) Show that any integer is uniquely determined by its residues modulo $5, 5^2, 5^3, \dots, 5^i, \dots$.

Part (ii) of the above exercise shows that if there *were* an integer satisfying (7.4.1), the sequence of residues arising by repeated application of the step of part (i) would determine it. But now let us return to our senses, and remember that (7.4.1) has no real solution, and ask what, if anything, we *have* found.

Clearly, we have shown that there exists a sequence of residues, $x_1 \in |\mathbb{Z}_5|$, $x_2 \in |\mathbb{Z}_{5^2}|$, \dots , $x_i \in |\mathbb{Z}_{5^i}|$, \dots , each of which satisfies (7.4.1) in the ring in which it lives, and which are “consistent”, in the sense that each x_{i+1} is a “lifting” of x_i , under the series of natural ring homomorphisms

$$\dots \rightarrow \mathbb{Z}_{5^{i+1}} \rightarrow \mathbb{Z}_{5^i} \rightarrow \dots \rightarrow \mathbb{Z}_{5^2} \rightarrow \mathbb{Z}_5.$$

Let us name the i th homomorphism in the above sequence $f_i: \mathbb{Z}_{5^{i+1}} \rightarrow \mathbb{Z}_{5^i}$; thus, f_i takes the residue of any integer n modulo 5^{i+1} to the residue of n modulo 5^i . Now note that the set of all strings

$$(7.4.2) \quad (\dots, x_i, \dots, x_2, x_1) \text{ such that } x_i \in |\mathbb{Z}_{5^i}| \text{ and } f_i(x_{i+1}) = x_i \text{ (} i = 1, 2, \dots \text{)}$$

forms a ring under componentwise operations. What we have shown is that *this ring* contains a square root of -1 . Since, as we have noted, an integer n is determined by its residues modulo the powers of 5 , the ring \mathbb{Z} is *embedded* in this ring, though of course the square root, in this ring, of $-1 \in |\mathbb{Z}|$ does not lie in the embedded copy of the ring \mathbb{Z} . (“If the fool would persist in his folly, he would become wise,” William Blake [52].)

The ring of sequences (7.4.2) is called the *ring of 5-adic integers*. The corresponding object constructed for any prime p , using the system of maps

$$(7.4.3) \quad \dots \rightarrow \mathbb{Z}_p^{i+1} \rightarrow \mathbb{Z}_p^i \rightarrow \dots \rightarrow \mathbb{Z}_p^2 \rightarrow \mathbb{Z}_p,$$

is called the ring of *p-adic integers*. These rings are of fundamental importance in modern number theory, and come up in many other areas as well. The notation for them is not uniform; the symbol we will use here is $\hat{\mathbb{Z}}_{(p)}$. (The (p) in parenthesis denotes the *ideal* of the ring \mathbb{Z} generated by the element p . What is meant by putting it as a subscript of \mathbb{Z} and adding a hat will be seen a little later. Many number-theorists simply write \mathbb{Z}_p for the p -adic integers, denoting the field of p elements by $\mathbb{Z}/p\mathbb{Z}$ or \mathbb{F}_p ; cf. [24, p.272], [31, p.162, Example].)

The construction of \mathbb{Z}_p is in some ways analogous to the construction of the real numbers from the rationals. Real numbers are entities that can be approximated by rational numbers under the *distance metric*; p -adic integers are entities that can be approximated by integers via *congruences* modulo arbitrarily high powers of p . This analogy is made stronger in

Exercise 7.4:2. Let p be a fixed prime number. If n is any integer, let $v_p(n)$ denote the greatest integer e such that p^e divides n , or the symbol $+\infty$ if $n = 0$. The *p-adic metric* on \mathbb{Z} is defined by $d_p(m, n) = p^{-v_p(m-n)}$. Thus, it makes m and n “close” if they are congruent modulo a high power of p .

(i) Verify that d_p is a metric on \mathbb{Z} , and that the ring operations are continuous in this metric. Deduce that the *completion* of \mathbb{Z} with respect to this metric (the set of Cauchy sequences modulo the usual equivalence relation) can be made a ring containing \mathbb{Z} .

- (ii) Show that this completion is isomorphic to $\hat{\mathbb{Z}}_{(p)}$.
- (iii) Show that every element x of this completion has a unique “left-facing base- p expression” $x = \sum_{0 \leq i < \infty} c_i p^i$, where each $c_i \in \{0, 1, \dots, p-1\}$. In particular, show that any such infinite sum is convergent in the p -adic metric. What is the expression for -1 in this form?

We showed above that one could find a solution to the equation $x^2 = -1$ in $\hat{\mathbb{Z}}_{(5)}$. Let us note some simpler equations one can solve:

- Exercise 7.4.3.** (i) Show that every integer n not divisible by p is invertible in $\hat{\mathbb{Z}}_{(p)}$.
- (ii) Are the “base- p expressions” (in the sense of the preceding exercise) for the elements n^{-1} eventually periodic?

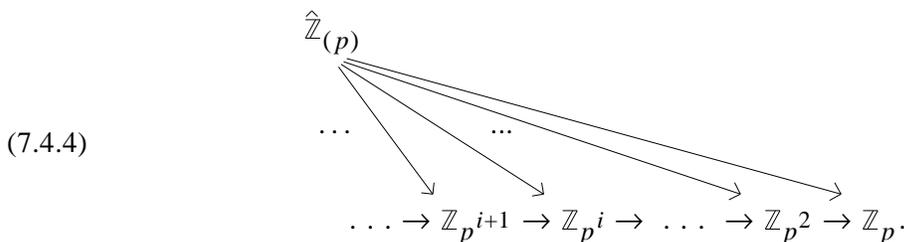
It follows from point (i) of the above exercise that we can embed into the p -adic integers not only \mathbb{Z} , but the subring of \mathbb{Q} consisting of all fractions with denominators not divisible by p . Now when one adjoins to a commutative ring R inverses of all elements not lying in some prime ideal P , the resulting ring (which, if R is an integral domain, is a subring of the field of fractions of R) is denoted R_P , so what we have embedded in the p -adic integers is the ring $\mathbb{Z}_{(p)}$. In $\mathbb{Z}_{(p)}$, every nonzero element is clearly an invertible element times a power of p , from which it follows that the nonzero ideals are precisely the ideals (p^i) . It is easy to verify that the factor-ring $\mathbb{Z}_{(p)}/(p^i)$ is isomorphic to \mathbb{Z}_p^i ; hence the system of finite rings and homomorphisms (7.4.3) can be described as consisting of *all* the proper factor-rings of $\mathbb{Z}_{(p)}$, together with the canonical maps among them. Hence the p -adic integers can be thought of as elements which can be approximated by members of $\mathbb{Z}_{(p)}$ modulo all *nonzero ideals* of that ring. Ring-theorists call the ring of such elements the *completion* of $\mathbb{Z}_{(p)}$ with respect to the system of its nonzero ideals, hence the symbol $\hat{\mathbb{Z}}_{(p)}$.

We will not go into a general study of what algebraic equations have solutions in the ring of p -adic integers. A result applicable to a large class of rings including the p -adics is *Hensel’s Lemma*; see [24, Theorem 8.5.6] or [22, §III.4.3] for the statement.

Let us characterize abstractly the relation between the diagram (7.4.3) and the ring of p -adic integers which we have constructed from it. Since a p -adic integer is by definition a sequence $(\dots, x_i, \dots, x_2, x_1)$ with each $x_i \in \mathbb{Z}_p^i$, the ring of p -adic integers has *projection* homomorphisms p_i onto each ring \mathbb{Z}_p^i . (Apologies for the double use of the letter “ p ”!) Since the components x_i of each element satisfy the compatibility conditions $f_i(x_{i+1}) = x_i$, these projection maps satisfy

$$f_i p_{i+1} = p_i,$$

i.e., they make a commuting diagram



I claim that $\hat{\mathbb{Z}}_{(p)}$ is right universal for these properties. Indeed, given any ring R with homomorphisms $r_i: R \rightarrow \mathbb{Z}_p^i$ which are “compatible”, i.e., satisfy $f_i r_{i+1} = r_i$, we see that for

any $a \in R$, the system of images $(\dots, r_i(a), \dots, r_2(a), r_1(a))$ defines an element $r(a) \in \hat{\mathbb{Z}}_{(p)}$. The resulting map $r: R \rightarrow \hat{\mathbb{Z}}_{(p)}$ will be a homomorphism such that $r_i = p_i r$ for each i , and will be uniquely determined by these equations.

This universal property is expressed by saying that $\hat{\mathbb{Z}}_{(p)}$ is the *inverse limit* of the system (7.4.3); one writes

$$\hat{\mathbb{Z}}_{(p)} = \varprojlim_i \mathbb{Z}_p^i.$$

We will give the formal definition of this concept in the next section.

A very similar example of an inverse limit is that of the system

$$(7.4.5) \quad \dots \rightarrow k[x]/(x^{i+1}) \rightarrow k[x]/(x^i) \rightarrow \dots \rightarrow k[x]/(x^2) \rightarrow k[x]/(x),$$

where $k[x]$ is the ring of polynomials in x over a field k , and (x^i) the ideal of all multiples of x^i . A member of $k[x]/(x^i)$ can be thought of as a polynomial in x specified modulo terms of degree $\geq i$. If we take a sequence of such partially specified polynomials, each extending the next, these determine a *formal power series* in x . So the inverse limit of the above system is the formal power series ring $k[[x]]$. This ring is well known as a place where one can solve various sorts of equations. Some of these results are instances of Hensel's Lemma, referred to above; others, such as the existence of formal-power-series solutions to differential equations, fall outside the scope of that lemma.

We constructed the p -adic integers using the canonical *surjections* $\mathbb{Z}_p^{i+1} \rightarrow \mathbb{Z}_p^i$. Now there are also canonical *embeddings* $\mathbb{Z}_p^i \rightarrow \mathbb{Z}_p^{i+1}$, sending the residue of n modulo p^i to the residue of pn modulo p^{i+1} . These respect addition but not multiplication, i.e., they are homomorphisms of abelian groups but not of rings. If we write out this system of groups and embeddings,

$$(7.4.6) \quad \mathbb{Z}_p \rightarrow \mathbb{Z}_p^2 \rightarrow \dots \rightarrow \mathbb{Z}_p^i \rightarrow \mathbb{Z}_p^{i+1} \rightarrow \dots$$

it is natural to think of each group as a subgroup of the next, and to try to take their "union" G . But they are not literally subgroups of one another, so we need to think further about what we want this G to be.

Clearly, for every element x of each group in the above system, we want there to be an element of G representing the image of x . Furthermore, if an element x of one of the above groups is mapped to an element y of another by some composite of the maps shown in (7.4.6), then these two elements should have the same image in G . Hence to get our G , let us form a disjoint union of the underlying sets of the given groups, and divide out by the equivalence relation that equates two elements if the image of one under a composite of the given maps is the other. It is straightforward to verify that this *is* an equivalence relation on the disjoint union, and that because the maps in the above diagram are group homomorphisms, the quotient by this relation inherits a group structure. If we call the maps in (7.4.6) $e_i: \mathbb{Z}_p^i \rightarrow \mathbb{Z}_p^{i+1}$, and the maps to the group we have constructed $q_i: \mathbb{Z}_p^i \rightarrow G$, then the identifications we have made have the effect that for each i ,

$$q_{i+1}e_i = q_i,$$

i.e., that the diagram

Definition 7.5.1. Let P be a partially ordered set.

P is said to be directed (or upward directed) if it is nonempty, and for any two elements x, y of P , there exists an element z majorizing both x and y .

P is said to be inversely directed (or downward directed) if it is nonempty and for any two elements x, y of P , there exists an element z which is \leq both x and y ; equivalently, if P^{op} is directed.

(The word “filtered” is sometimes used instead of “directed” in these definitions.)

(If you did Exercise 5.2:9, you will find that these conditions are certain of the “interpolation” properties of that exercise.)

We can now give the general definitions of direct and inverse limits. The formulations we give below assume that the morphisms of our given systems go in the “upward” direction with respect to the ordering on the indexing set. It happens that in our initial example of $\hat{\mathbb{Z}}_{(p)}$, the standard ordering on the positive integers is such that the morphisms went the *opposite* way; in our construction of \mathbb{Z}_p^∞ they went the “right” way; while in the case of germs of analytic functions, if one orders neighborhoods of z by inclusion, the morphisms again go the “wrong” way (namely, from the set of functions on a larger neighborhood to the set of functions on a smaller neighborhood). This can be corrected formally by using, when necessary, the opposite partial ordering on the index set. Informally, in discussing direct and inverse limits one often just specifies the system of *objects and maps*, and understands that to apply the formal definition, one should partially order the set indexing the objects so as to make maps among them go “upward”.

Definition 7.5.2. Let \mathbf{C} be a category, and suppose we are given a family of objects $X_i \in \text{Ob}(\mathbf{C})$ ($i \in I$), a partial ordering on the index set I , and a system (f_{ij}) of morphisms, $f_{ij} \in \mathbf{C}(X_i, X_j)$ ($i < j, i, j \in I$) such that for $i < j < k$, one has $f_{jk}f_{ij} = f_{ik}$. (In brief, suppose we are given a partially ordered set I , and a functor $F: I_{\text{cat}} \rightarrow \mathbf{C}$.)

If I is inversely directed, then $(X_i, f_{ij})_I$ is called an inversely directed system of objects and maps in \mathbf{C} . An inverse limit of this system means an object L given with morphisms $p_i: L \rightarrow X_i$ which are compatible, in the sense that for all $i < j \in I$, $p_j = f_{ij}p_i$, and which is universal for this property, in the sense that given any object W and morphisms $w_i: W \rightarrow X_i$ such that for all $i < j \in I$, $w_j = f_{ij}w_i$, there exists a unique morphism $w: W \rightarrow L$ such that $w_i = p_i w$ for all $i \in I$.

Likewise, if I is directed, then $(X_i, f_{ij})_I$ is called a directed system in \mathbf{C} ; and a direct limit of this system means an object L given with morphisms $q_i: X_i \rightarrow L$ such that for all $i < j \in I$, $q_i = q_j f_{ij}$, and which is universal in the sense that given any object Y and morphisms $y_i: X_i \rightarrow Y$ such that for all $i < j \in I$, $y_i = y_j f_{ij}$, there exists a unique morphism $y: L \rightarrow Y$ such that $y_i = y q_i$ for all $i \in I$.

(Synonyms sometimes used for inverse and direct limit are projective and inductive limit, respectively.)

Loosely, one often writes the inverse limit object $\varprojlim_i X_i$, and the direct limit object $\varinjlim_i X_i$. More precisely, letting F denote the functor $I_{\text{cat}} \rightarrow \mathbf{C}$ corresponding to the inversely directed or directed system (X_i, f_{ij}) , one writes these objects as $\varprojlim F$ and $\varinjlim F$ respectively.

The morphisms $p_j: \varprojlim_i X_i \rightarrow X_j$ are called the projection maps associated with this inverse limit, and the $q_j: X_j \rightarrow \varinjlim_i X_i$ the coprojection maps associated with the direct limit.

In the next-to-last paragraph of the above definition, by the “functor ... corresponding to the ...

system (X_i, f_{ij}) we understand the functor which takes on the value X_i at the object i , the value f_{ij} at the morphism (i, j) ($i < j$ in I), and the value id_{X_i} at the morphism (i, i) . In the case where the indexing partially ordered set consists of the positive or negative integers, note that the full system of morphisms is determined by the morphisms $f_{i, i+1}$ (which can be arbitrary), hence in such cases one generally specifies only these morphisms in describing the system.

Direct and inverse limits in **Set** may be constructed by the techniques we illustrated earlier:

Lemma 7.5.3. *Every inversely directed system (X_i, f_{ij}) of sets and set maps has an inverse limit, given by*

$$(7.5.4) \quad \varprojlim X_i = \{(x_i) \in \prod_I X_i \mid x_j = f_{ij}(x_i) \text{ for } i < j \in I\}, \text{ with}$$

the p_j given by projection maps, $\varprojlim X_i \subseteq \prod X_i \rightarrow X_j$.

Likewise, every directed system (X_i, f_{ij}) of sets and set maps has a direct limit, gotten by forming the disjoint union of the X_i and dividing out by the equivalence relation under which $x \in X_i$ and $x' \in X_{i'}$ are equivalent if and only if they have the same image in some X_j ($j > i, i'$). \square

One may ask what the point is, in our definitions of direct and inverse limit, of requiring that the partially ordered set I be directed or inversely directed. One could set up the definitions without that restriction, and in most familiar categories one can, in fact, construct objects which satisfy the resulting condition. But the behavior of these constructions tends to be quite different from those we have discussed *unless* these directedness assumptions are made. (For instance, the explicit description in Lemma 7.5.3 of the equivalence relation in the construction of a direct limit of sets is no longer correct.) In any case, such a generalized definition would be subsumed by a still more general definition to be made in the next section! So the value of the definition in the form given above is that it singles out a situation in which the limit objects can be studied by certain techniques.

Exercise 7.5:1. (i) If $(X_i, f_{ij})_I$ is a directed system in a category **C**, and J a *cofinal* subset of I , show that $\varinjlim_J X_j \cong \varinjlim_I X_i$; precisely, that J will also be a directed partially ordered set, and that any object with the universal property of the direct limit of the given system can be made into a direct limit of the subsystem in a natural way, and vice versa.

(ii) Show that the isomorphism of (i) is an instance of a morphism (in one direction or the other) between $\varinjlim_J X_j$ and $\varinjlim_I X_i$ which can be defined whenever $J \subseteq I$ are both directed and both limits exist, whether or not J is cofinal.

(iii) State the result corresponding to (i) for inverse limits. (For this we need a term for a subset of a partially ordered set which has the property of being cofinal under the opposite ordering; let us use “downward cofinal”. When speaking of inverse systems, one sometimes just says “cofinal”, with the understanding that this is meant in the only sense that is relevant to such systems.)

(iv) What can you deduce from (i) and (iii) about direct limits over directed partially ordered sets having a greatest element, and inverse limits over inversely directed partially ordered sets having a least element?

(v) Given any directed partially ordered set I and any *noncofinal* directed subset J of I , show that there exists a directed system of sets, (X_i, f_{ij}) , indexed by I , such that $\varinjlim_I X_i \not\cong \varinjlim_J X_j$.

Exercise 7.5:2. (i) Suppose $(X_i, f_{ij})_I$ is a directed system in a category \mathbf{C} , and $f: J \rightarrow I$ a surjective isotone map, such that J , like I , is directed. Show that $\varinjlim_{j \in J} X_{f(j)} \cong \varinjlim_{i \in I} X_i$.

(ii) Deduce that if \mathbf{D} is a subcategory of \mathbf{C} , and L is an object of \mathbf{C} that can be written as a direct limit of objects and morphisms in \mathbf{D} , then L can be written as such a direct limit taken over a directed partially ordered set of the form $\mathbf{P}_{\text{fin}}(S)$, where S is a set, and $\mathbf{P}_{\text{fin}}(S)$ denotes the partially ordered set of all finite subsets of S , ordered by inclusion.

The next few exercises concern direct and inverse limits of *sets*. We shall see in the next chapter that direct and inverse limits of algebras have as their underlying sets the direct or inverse limits of the objects' underlying sets (assuming, in the case of *direct* limits, that the algebras have only finitary operations); hence the results obtained for sets in the exercises below will be applicable to algebras.

The construction of the p -adic integers was based on a system of *surjective* homomorphisms. The first point of the next exercise looks at inverse systems with the opposite property, and the second considers the dual situation for direct limits.

Exercise 7.5:3. (i) Let (S_i, f_{ij}) be an inversely directed system in \mathbf{Set} such that all the morphisms f_{ij} are one-to-one, and let us choose any element $i_0 \in I$. Show that $\varprojlim_i S_i$ can be identified with the intersection, in S_{i_0} , of the sets $f_{ii_0}(S_i)$ ($i < i_0$).

(ii) Let (S_i, f_{ij}) be a directed system in \mathbf{Set} such that all the morphisms f_{ij} are onto, and let us choose any element $i_0 \in I$. Show that $\varinjlim_i S_i$ can be identified with the quotient set of S_{i_0} by the union of the equivalence relations induced by the maps $f_{i_0 i}: S_{i_0} \rightarrow S_i$ ($i > i_0$).

Exercise 7.5:4. (i) Show that the inverse limit of any inverse system of *finite nonempty* sets is nonempty.

(Suggestions: Either build the description of an element of the inverse limit up “from below”, by looking at partial assignments satisfying appropriate extendibility conditions, and apply Zorn's Lemma to get a maximal such assignment, or else “narrow down on an element from above”, by looking at “subsystems” of the given inverse system, i.e., systems of nonempty subsets of the given sets carried into one another by the given mappings, and using Zorn's Lemma to get a minimal such subsystem. You might find it instructive to work out both of these proofs.)

(ii) Show that (i) can fail if the condition “finite” is removed, even for inverse limits over the totally ordered set of negative integers.

(iii) If you have some familiarity with general topology, see whether you can generalize statement (i) to a result on topological spaces, with “compact Hausdorff” replacing “finite”.

As an application of part (i) of the above exercise, suppose we are given a subdivision of the plane into regions, possibly infinitely many, and are studying the problem of coloring these regions with n colors so that no two adjacent regions are the same color. Let the set of all our regions be denoted R , the adjacency relation $A \subseteq R \times R$ (i.e., $(r_1, r_2) \in A$ if and only if r_1 and r_2 are adjacent regions), and the set of colors C . For any subset $S \subseteq R$, let X_S denote the set of all colorings of S (maps $S \rightarrow C$) under which no two adjacent regions have the same color; let us call these “permissible colorings of S ”. If $S \subseteq T$, then the restriction to S of a permissible coloring of T is a permissible coloring of S ; thus we have a restriction map $X_T \rightarrow X_S$. Now –

Exercise 7.5:5. (i) Show that in the above situation, the sets X_S , as S ranges over the *finite* subsets of R , form an inversely directed system, and that X_R may be identified with the inverse limit of this system in \mathbf{Set} .

(ii) Deduce using Exercise 7.5:4(i) that if each finite family $S \subseteq R$ can be colored, then the

whole picture R can be colored. (Note: the assumption that every finite family S can be colored does *not* say that *every* permissible coloring of a finite family S can be extended to a permissible coloring of every larger finite family T !)

- Exercise 7.5:6.** (i) Show that if (X_i, f_{ij}) is a directed system of sets, and each f_{ij} is one-to-one, then the canonical maps $q_j: X_j \rightarrow \varinjlim X_i$ are all one-to-one.
- (ii) Let (X_i, f_{ij}) be an inversely directed system of sets such that each f_{ij} is surjective. Show that if I is *countable*, then the canonical maps $p_j: \varprojlim X_i \rightarrow X_j$ are surjective. (Suggestion: First prove this in the case where I is the set of negative integers. Then show that any countable inversely directed partially ordered set either has a least element, or has a downward-cofinal subset order-isomorphic to the negative integers, and apply Exercise 7.5:1(iii).)
- (iii) Does this result remain true for uncountable I ? In particular, what if I is the opposite of an uncountable cardinal?

Exercise 7.5:7. Show that every group is a direct limit of finitely presented groups.

(This result is not specific to groups. We shall be able to extend it to more general algebras when we have developed the necessary language in the next chapter.)

The remaining exercises in this section develop some particular examples and applications of direct and inverse limits, including some further results concerning the p -adic integers. In these exercises you may assume the result which, as noted earlier, will be proved in the next chapter, that a direct or inverse limit of algebras whose operations are finitary can be constructed by forming the corresponding limit of underlying sets and giving this an induced algebra structure. None of these exercises, or the remarks connecting them, is needed for the subsequent sections of these notes.

One can sometimes achieve interesting constructions by taking direct limits of systems in which all objects are the same; this is illustrated in the next three exercises. The first shows a sophisticated way to get a familiar construction; in the next two, direct limits are used to get curious examples.

Exercise 7.5:8. Consider the directed system (X_i, f_{ij}) in **Ab**, where I is the set of positive integers, partially ordered by divisibility (i considered less than or equal to j if and only if i divides j), each object X_i is the additive group \mathbb{Z} , and for $j = ni$, $f_{ij}: \mathbb{Z} \rightarrow \mathbb{Z}$ is given by multiplication by n .

- (i) Show that $\varinjlim X_i$ may be identified with the additive group of the rational numbers.
- (ii) Show that if you perform the same construction starting with an arbitrary abelian group A in place of \mathbb{Z} , the result is a \mathbb{Q} -vector-space which can be characterized by a universal property relative to A .
- (iii) Can you describe the ring multiplication of \mathbb{Q} in terms of the description of its underlying abelian group in (i)?

Exercise 7.5:9. For this exercise, assume known the facts that every subgroup of a free group is free, and in particular, that in the free group on two generators x, y , the subgroup generated by the two commutators $x^{-1}y^{-1}xy$ and $x^{-2}y^{-1}x^2y$ is free on those two elements.

Let F denote the free group on x and y , and f the endomorphism of F taking x to $x^{-1}y^{-1}xy$ and y to $x^{-2}y^{-1}x^2y$. Let G denote the direct limit of the system $F \rightarrow F \rightarrow F \rightarrow \dots$, where all the arrows shown are the above morphism f .

Show that G is a nontrivial group such that every finitely generated subgroup of G is free, but that G is equal to its own commutator, $G = [G, G]$; i.e., that the abelianization of G is the trivial group. Deduce that though G is “locally free”, it is not free.

Exercise 7.5:10. Let k be a field. Let R denote the direct limit of the system of k -algebras $k[x] \rightarrow k[x] \rightarrow k[x] \rightarrow \dots$, where each arrow is the homomorphism sending x to x^2 . Show that R is an integral domain in which every finitely generated ideal is principal, but not every

ideal is finitely generated. (Thus, for each ideal, the minimum cardinality of a generating set is either 0, 1 or infinite.)

For the student familiar with the Galois theory of finite-dimensional field extensions, the next exercise shows how the Galois groups of infinite-dimensional extensions can be characterized in terms of the finite-dimensional case.

Exercise 7.5:11. Suppose E/K is a normal algebraic field extension, possibly of infinite degree. Let I be the set of subfields of E normal and of *finite* degree over K . If $F_2 \subseteq F_1$ in I , let $f_{F_1, F_2}: \text{Aut}_K F_1 \rightarrow \text{Aut}_K F_2$ denote the map which acts by *restricting* automorphisms of F_1 to the subfield F_2 .

- (i) Show that the definition of f_{F_1, F_2} makes sense, and gives a group homomorphism.
- (ii) Show that if we order I by reverse inclusion of fields, then the groups $\text{Aut}_K F$ ($F \in I$) and homomorphisms f_{F_1, F_2} ($F_1 \leq F_2$) form an inversely directed system of groups.
- (iii) Show that $\text{Aut}_K E$ is the inverse limit of this system in **Group**.
- (iv) Can you find a normal algebraic field extension whose automorphism group is isomorphic to the additive group of the p -adic integers?

Exercise 7.5:12. (i) (Open question.) Suppose a group G is the inverse limit of a system of finite groups. If G is a torsion group (i.e., if all elements of G are of finite order), must G have finite exponent (i.e., must there exist an integer n such that $x^n = e$ is an identity of G)?

Though the above question is very difficult, the next two parts are reasonable exercises, and may help render that question more tractable:

- (ii) Show that (i) is equivalent to the corresponding question in which we assume that G is the inverse limit of a system of finite groups indexed by the negative integers (under the natural ordering), with all connecting morphisms surjective.
- (iii) Translate (i) (possibly with the help of (ii)) into a question on finite groups which you could pose to a person not familiar with the concept of inverse limit. (The more natural-sounding, the better.)

Back to the p -adic integers, now. Part (i) of the next exercise seemed to me too simple to be true when I saw it described (in a footnote in a Ph.D. thesis) as “well-known”. But it is, in fact, not hard to verify

Exercise 7.5:13. (i) Show that $\mathbb{Z}[[x]]/(x-p) \cong \hat{\mathbb{Z}}_{(p)}$, where $\mathbb{Z}[[x]]$, we recall, denotes the ring of formal power series over \mathbb{Z} in one indeterminate x , and $(x-p)$ denotes the ideal of that ring generated by $x-p$.

- (ii) Examine other constructions of factor-rings of formal power series rings. For instance, can you describe $\mathbb{Z}[[x]]/(x-p^2)$? $\mathbb{Z}[[x]]/(x^2-p)$? $\mathbb{Z}[[x]]/(px^2-1)$? $R[[x]]/(f(x))$ for a general commutative ring R and a polynomial or power series $f(x)$, perhaps subject to some additional conditions? $R[[x, y]]/I$ for some fairly general class of ideals I ?

(If you consider $\mathbb{Z}[[x]]/(x-n)$ for n not a prime power, you might first look at Exercise 7.5.15 below.)

Exercise 7.5:14. (i) Show that the function v_p of Exercise 7.4:2 satisfies $v_p(xy) = v_p(x) + v_p(y)$ and $v_p(x+y) \geq \min(v_p(x), v_p(y))$ ($x, y \in \mathbb{Z}$).

- (ii) Deduce that $\hat{\mathbb{Z}}_{(p)}$ is an integral domain.
- (iii) Show that v_p can be extended in a unique manner to a $\mathbb{Z} \cup \{+\infty\}$ -valued function on \mathbb{Q} satisfying the properties noted in (i).
- (iv) Show that the completion of \mathbb{Q} with respect to the metric d_p induced by the above extended function v_p is the field of fractions of $\hat{\mathbb{Z}}_{(p)}$.

(v) Show that elements of this field have expansions $x = \sum_i c_i p^i$, where again $c_i \in \{0, 1, \dots, p-1\}$, and where i now ranges over all integer values (not necessarily positive), but subject to the condition that the set of i such that c_i is nonzero is bounded below.

This field is called the field of *p-adic rationals*, and denoted $\hat{\mathbb{Q}}_{(p)}$ (or \mathbb{Q}_p).

Is the “adic” construction limited to primes p , or can one construct, say, a ring of “10-adic integers”, $\hat{\mathbb{Z}}_{(10)}$? One encounters a trivial difficulty in that there are two ways of interpreting this symbol. But we shall see below that they lead to the same ring; so there is a well-defined object to which we can give this name. However, its properties will not be as nice as those of the p -adic integers for prime p .

Exercise 7.5:15. Let $\mathbb{Z}_{(10)}$ denote the ring of all rational numbers which can be written with denominators relatively prime to 10.

(i) Determine all nonzero ideals $I \subseteq \mathbb{Z}_{(10)}$ and the structures of the factor-rings $\mathbb{Z}_{(10)}/I$. Sketch the diagram of the inverse system of these factor-rings and the canonical maps among them.

(ii) Show that the inverse system $\dots \rightarrow \mathbb{Z}_{10^i} \rightarrow \dots \rightarrow \mathbb{Z}_{100} \rightarrow \mathbb{Z}_{10}$ constitutes a downward *cofinal* subsystem of the above inverse system.

Hence by Exercise 7.5:1 the inverse limits of these two systems are isomorphic, and we shall denote their common value $\hat{\mathbb{Z}}_{(10)}$. It is clear from the form of the second inverse system that elements of $\hat{\mathbb{Z}}_{(10)}$ can be described by “infinite decimal expressions to the left of the decimal point”.

(iii) Show that the relation $[2] \cdot [5] = [0]$ in \mathbb{Z}_{10} can be lifted to get a pair of nonzero elements which have product 0 in \mathbb{Z}_{100} , that these can be lifted to such elements in \mathbb{Z}_{1000} , and so on, and deduce that $\hat{\mathbb{Z}}_{(10)}$ is not an integral domain.

(iv) Prove, in fact, that $\hat{\mathbb{Z}}_{(10)} \cong \hat{\mathbb{Z}}_{(2)} \times \hat{\mathbb{Z}}_{(5)}$.

A construction often used in number theory is characterized in

Exercise 7.5:16. Show that the inverse limit of the system of all factor-rings of \mathbb{Z} by nonzero ideals is isomorphic to $\prod_p \hat{\mathbb{Z}}_{(p)}$, where the direct product is taken over all primes p . (This ring is denoted $\hat{\mathbb{Z}}$.)

A feature we have not yet mentioned, but which is important in the study of inverse limits, is topological structure. Recall that the inverse limit of a system of sets and set maps (X_i, f_{ij}) was constructed as a subset of $\prod X_i$. Let us now regard each X_i as a discrete topological space, and give $\prod X_i$ the product topology. In general, a product of discrete spaces is not discrete; however, a product of compact spaces is compact, so if our discrete spaces X_i are *finite*, their product will be compact. It is not hard to show that the subset $\varprojlim X_i \subseteq \prod X_i$ will be closed in the product topology, and hence, if the X_i are finite, will be compact in the induced topology.

Exercise 7.5:17. (i) Verify the assertion that $\varprojlim X_i \subseteq \prod X_i$ is always closed in the product topology, and is therefore compact if all X_i are finite.

(ii) Show that Exercise 7.5:4(i) (and hence Exercise 7.5:5(ii)) can be deduced using the compactness of $\varprojlim X_i$.

(iii) Show that the compact topology described above agrees in the case of $\hat{\mathbb{Z}}_{(p)}$ with the topology arising from the metric d_p of Exercise 7.4:2.

In fact, results like Exercise 7.5:5(ii), saying that a family of conditions can be satisfied simultaneously if all finite subfamilies of these conditions can be so satisfied, are called by logicians “compactness” results, because the proofs can generally be formulated in terms of the compactness of some topological space.

I can now say that the usual formulation of the open question of Exercise 7.5:12(i) is, “If a compact topological group is torsion, must it have finite exponent?” (Note that a topological group is by definition required to have a Hausdorff topology.) The equivalence of this with the question of that exercise follows from a deep result, that any compact group is an inverse limit of surjective maps of compact Lie groups (see [101, Theorem IV.4.6, p.175]), combined with the observation that if any of these Lie groups had positive dimension, we would get elements of infinite order. Thus, any compact torsion group is an inverse limit of 0-dimensional compact Lie groups, i.e., finite discrete groups, under the product topology.

An inverse limit of finite structures is called *profinite* (based on the synonym “projective limit” for “inverse limit”). I hope to eventually add to these notes a chapter treating profinite algebras (meanwhile, for some interesting results see [48]), and objects with related conditions, such as profinite-dimensionality. Let us look briefly at the latter condition in

Exercise 7.5:18. Let V be a vector space over a field k .

- (i) Show that the dual space V^* is the inverse limit, over all finite-dimensional subspaces $V_0 \subseteq V$, of the spaces V_0^* .
- (ii) Can you get the result of (i) as an instance of a general result describing *duals of direct limits* of vector spaces?
- (iii) If you did Exercise 5.5:5(ii)-(iii), show that the topology described there is that of the inverse limit of the finite-dimensional discrete spaces V_0^* referred to above. Show moreover that the only linear functionals $V^* \rightarrow k$ continuous in this topology are those induced by the elements of V .

The remainder of this section constitutes a digression for curiosity’s sake.

Ordinary real numbers expressed in base p have expansions going endlessly to the right, and finitely many steps to the left of the decimal point, while p -adic rationals (Exercise 7.5:14) have expansions going endlessly to the left, and finitely many steps to the right. Is it possible to define an arithmetic of elements with formal base- p expansions going endlessly in both directions?

Exercise 7.5:19. Let p be a prime. For every integer n , we have a subgroup $p^n\mathbb{Z} \subseteq \mathbb{R}$, hence we can form the quotient group $\mathbb{R}/p^n\mathbb{Z}$. Observe that these groups are each isomorphic to the circle group \mathbb{R}/\mathbb{Z} , and form an inverse system $\dots \rightarrow \mathbb{R}/p^2\mathbb{Z} \rightarrow \mathbb{R}/p\mathbb{Z} \rightarrow \mathbb{R}/\mathbb{Z} \rightarrow \dots$, where the connecting maps take the residue of a real number modulo \mathbb{Z}/p^{i+1} to its residue modulo \mathbb{Z}/p^i . Let G be the *inverse limit* of this system of groups.

- (i) Show how to express elements of G as formal doubly infinite series $\sum_{i \in \mathbb{Z}} c_i p^i$, where $c_i \in \{0, 1, \dots, p-1\}$, ($i = \dots, -1, 0, 1, \dots$). Show that such a representation is unique except for the cases where for all sufficiently small i , c_i either becomes constant with value 0 or constant with value $p-1$.
- (ii) Show that $\hat{\mathbb{Q}}_{(p)}$ and \mathbb{R} both embed as dense subgroups of G .

Groups of the above sort appear in the theory of locally compact abelian groups, where they are called “solenoids”, from a term in electronics meaning “a hollow tightly wound coil of wire”. For students familiar with Pontryagin duality, the solenoid G constructed above will be seen to be the dual of the *discrete* additive group of $\mathbb{Z}[p^{-1}]$ (the ring of rational numbers of the form np^{-i}).

The above group G may also be obtained as a completion: For p a prime, let us define a function v_p on the real numbers, by letting $v_p(x)$ be the supremum of all integers n such that $x \in p^n\mathbb{Z}$. This will be $+\infty$ if $x = 0$, a nonnegative integer if x is a nonzero integer, a negative integer if x is a noninteger rational number of the form m/p^i , and $-\infty$ if none of these cases hold. (This does not agree with the definition of $v_p(x)$ we gave in Exercise 7.5:14 for rational x ,

though it does for x in the subring $\mathbb{Z}[p^{-1}]$.) Now for any two real numbers x, y , define $d_{p,|}(x, y) = \inf_{z \in \mathbb{R}} (p^{-v_p(x-z)} + |z-y|)$. Observe that although $p^{-v_p(x-z)}$ takes on the value $+\infty$ for most z , there exist values of z for which it is finite, so the infimum shown will be finite for all x and y .

- Exercise 7.5:20.** (i) Show that $d_{p,|}$ is a metric on the real line \mathbb{R} , and is bounded above.
 (ii) Show how to obtain from a doubly infinite series $\sum_{i \in \mathbb{Z}} c_i p^i$ a Cauchy sequence in \mathbb{R} under this metric, and show that all elements of the completion of \mathbb{R} in the metric $d_{p,|}$ can be represented by such series.
 (iii) Deduce that this completion is isomorphic to the solenoid G of the preceding exercise.

- Exercise 7.5:21.** (i) Show that the topology on G arising from the above metric agrees with that obtained by regarding G as an inverse limit of compact groups $\mathbb{R}/p^n\mathbb{Z}$. Deduce that the additive group operations of \mathbb{R} extend continuously to this completion.
 (ii) Let r be a real number, and $\bar{r}: \mathbb{R} \rightarrow \mathbb{R}$ the operation of multiplication by r . Show that \bar{r} is continuous in the metric $d_{p,|}$ if and only if $r \in \mathbb{Z}[p^{-1}]$. Deduce that multiplication as a map $\mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is not bicontinuous in this metric. Hence the ring structure on \mathbb{R} does not extend to the solenoid.
 (iii) Can addition of elements of the solenoid be performed by the same operations on digits that one uses to add ordinary real numbers in base p ? What goes wrong if we try to apply the ordinary procedure for *multiplying* numbers in base p ?
 (iv) If n is a positive integer not a power of p , show that the elements “ n^{-1} ” of \mathbb{R} and of $\hat{\mathbb{Q}}_{(p)}$ have distinct images under the embeddings of Exercise 7.5:19(ii). Deduce that the additive group of the solenoid has nonzero elements of finite order. Can you characterize such elements in terms of their “base p ” expansions?
 (v) Show that the solenoid described above is isomorphic to the group $\mathbf{Ab}(\mathbb{Z}[p^{-1}], \mathbb{R}/\mathbb{Z})$. (This is equivalent to the assertion of in the paragraph following Exercise 7.5:19(ii)).

7.6. Limits and colimits. Direct and inverse limits are similar in their universal properties to several other constructions we have seen. Let us recall these.

Given two objects X_1, X_2 of a category \mathbf{C} , a *product* of X_1 and X_2 in \mathbf{C} is an object P given with morphisms p_1 and p_2 into X_1 and X_2 , and universal for this property.

Given a pair of parallel morphisms $X_1 \rightrightarrows X_2$ in \mathbf{C} , an *equalizer* of this system is an object K given with a morphism k into X_1 having equal composites with those two morphisms, and again universal. To improve the parallelism with similar constructions, let us rename the morphism k as k_1 , and let $k_2: K \rightarrow X_2$ denote the common value of the composites of k_1 with the two morphisms $X_1 \rightrightarrows X_2$. Then we can describe K as having a morphism into *each* of X_1, X_2 , such that the composite of $k_1: K \rightarrow X_1$ with each of the two given morphisms $X_1 \rightarrow X_2$ is the morphism $k_2: K \rightarrow X_2$, and such that (K, k_1, k_2) is universal for these properties. We see that this is exactly like the universal property of an inverse limit, except that the indexing category $\cdot \rightrightarrows \cdot$ is not of the form $I_{\mathbf{cat}}$ for a partially ordered set I .

In the same way, a *pullback* of a pair of morphisms $f_1: X_1 \rightarrow X_3, f_2: X_2 \rightarrow X_3$ can be redefined as an object P given with morphisms p_1, p_2, p_3 into X_1, X_2, X_3 respectively, satisfying $f_1 p_1 = p_3$ and $f_2 p_2 = p_3$, and universal for this property.

Let us look at a case we haven't discussed yet. If G is a group and S a G -set, then the *fixed-point set* of the action of G on S means $\{x \in |S| \mid (\forall g \in |G|) gx = x\}$. If we denote the action of each $g \in |G|$ on S by $g_S: |S| \rightarrow |S|$, then the fixed-point set is universal among sets A with maps $i: A \rightarrow |S|$ such that for all $g \in |G|, i = g_S i$. Given an object X of any category

\mathbf{C} , and an action of a group G on X , we can look for an object with the same universal property, and, if it exists, call it the “fixed object” of the action.

We have seen constructions dual to those of product, equalizer and pullback. A construction dual to that of fixed object should take an object X of \mathbf{C} with an action of G on it to an object B of \mathbf{C} with a map $j: X \rightarrow B$ unchanged under composition on the right with the actions of elements of G , and universal for this property. Examples of this concept are examined in

Exercise 7.6:1. Let G be a group.

- (i) If X is a set on which G acts by permutations, and x an element of X , one defines the *orbit* of x under G to be the set $Gx = \{gx \mid g \in |G|\}$. Let B be the set of such orbits Gx , called the *orbit space* of X . Show that this set B , together with the map $X \rightarrow B$ taking x to Gx , has the universal property discussed above.
- (ii) Show that if G acts by automorphisms on (say) a ring R , then there is an object S in the category of rings with this same universal property, but that its underlying set will not in general be the orbit space of the action of G on the underlying set of R .
- (iii) If G acts by automorphisms on an object X of **POSet**, again show the existence of an object B with the above universal property. Show moreover that if G is finite, the underlying set of B will be the orbit space of the underlying set of X , and the universal map $X \rightarrow B$ will be *strictly* isotone; but that if G is infinite, neither statement need be true.
- (iv) Do the assertions of (iii) about the case where G is finite remain true if we replace **POSet** by **Lattice**?

As noted above, the universal properties we have been examining have statements formally identical with those of direct and inverse limits, except that the partially ordered set I of that definition is replaced by other categories \mathbf{D} (for example the two-object category $\cdot \rightrightarrows \cdot$ or the one-object category $G_{\mathbf{cat}}$). As names for the more general concepts embracing such cases, one uses modified versions of the terms “direct limit” and “inverse limit”.

Definition 7.6.1. Let \mathbf{C} and \mathbf{D} be categories, and $F: \mathbf{D} \rightarrow \mathbf{C}$ a functor.

Then a limit of F , written $\varprojlim F$ or $\varprojlim_{\mathbf{D}} F(X)$, means an object $L \in \text{Ob}(\mathbf{C})$ given with morphisms $p(X): L \rightarrow F(X)$ for all $X \in \text{Ob}(\mathbf{D})$, such that for $f \in \mathbf{D}(X, Y)$ one has $p(Y) = F(f)p(X)$, and universal for this property, in the sense that given any object $M \in \text{Ob}(\mathbf{C})$ and family of morphisms $m(X): M \rightarrow F(X)$ ($X \in \text{Ob}(\mathbf{D})$) which similarly make commuting triangles with the morphisms $F(f)$, there exists a unique morphism $h: M \rightarrow L$ such that for all X , $m(X) = p(X)h$.

Likewise, a colimit of F , written $\varinjlim F$ or $\varinjlim_{\mathbf{D}} F(X)$, means an object $L \in \text{Ob}(\mathbf{C})$ given with morphisms $q(X): F(X) \rightarrow L$ for all $X \in \text{Ob}(\mathbf{D})$ such that for $f \in \mathbf{D}(X, Y)$ one has $q(Y) = q(X)F(f)$, and universal for this property, in the sense that given $M \in \text{Ob}(\mathbf{C})$ and morphisms $m(X): F(X) \rightarrow M$ ($X \in \text{Ob}(\mathbf{D})$) making commuting triangles with the morphisms $F(f)$, there exists a unique morphism $h: L \rightarrow M$ such that for all X , $m(X) = hq(X)$.

The morphisms $p(X)$ in the definition of a limit may be called the projection morphisms, and the $q(X)$ in the definition of colimit may be called the coprojection morphisms.

One says that a category \mathbf{C} “has small limits” if all functors from small categories \mathbf{D} into \mathbf{C} have limits; likewise \mathbf{C} “has small colimits” if all functors from small categories into \mathbf{C} have colimits.

Remarks on terminology. Since the above concepts generalize not only direct and inverse limits, but also a large number of other pairs of constructions, they might just as well have been given names suggestive of one of the other pairs. I think that the reason “limit” and “colimit”

were chosen is that each of the other relevant universal constructions involves a more or less fixed diagram, while the diagrams involved in direct and inverse limits are varied. Hence in developing the latter concepts, people were forced to formulate a more general definition, and just a little more generality gave the concepts noted above.

But though the choice is historically explainable, I think it is unfortunate. As we can see from the examples of products and coproducts, or of kernels and cokernels, the objects given by limit and colimit constructions over diagram categories other than directed partially ordered sets are not “approximated arbitrarily closely” by the objects from which they are constructed, as the term “limit” would suggest. The particular cases that best exemplify the general concepts are not, I think, inverse and direct limits, but pullbacks and pushouts, so it would be preferable if the limit and colimit of $F: \mathbf{D} \rightarrow \mathbf{C}$ were renamed the *pullback* and the *pushout* of F (regarded as a system of objects and maps in \mathbf{C}). But it seems too late to turn the tide of usage.

Note also the initially confusing fact that *limits* generalize *inverse limits*, while *colimits* generalize *direct limits*. The explanation is that the words “direct” and “inverse” refer to forward and backward orientation with respect to a partial ordering, while the relation between the terms “limit” and “colimit” is based on looking at which is left and which is right universal, by analogy with “products and coproducts” and “kernels and cokernels”. There is no reason why two such principles of naming should agree as to which concept gets the “plain” and which the “modified” name, and in this case, they do not.

There is another pair of words for the same constructions: Freyd has named them “roots” and “coroots”, probably because if one pictures a system of objects and morphisms as a graph, the addition of the universal object makes it a *rooted* graph, with the universal object at the root. However there is no evident connection with roots of equations etc., and this terminology has not caught on.

Following the associations of the word “limit”, Mac Lane [17] calls a category \mathbf{C} *complete* if it has small limits, *cocomplete* if it has small colimits.

Exercise 7.6.2. If S is a monoid, then as for groups, an S -set is equivalent to a functor $F: S_{\mathbf{cat}} \rightarrow \mathbf{Set}$. Show how to construct the limit (easy) and the colimit (not so easy) of such a functor.

A useful observation is

Lemma 7.6.2. *Let \mathbf{D} be a category and X_0 an object of \mathbf{D} such that there are morphisms from X_0 to every object of \mathbf{D} . Let $F: \mathbf{D} \rightarrow \mathbf{C}$ be a functor having a limit L . Then the projection morphism $p(X_0): L \rightarrow F(X_0)$ is a monomorphism. In particular, all equalizer maps are monomorphisms.*

Likewise, if \mathbf{D} is a category having an object X_0 such that there are morphisms from every object of \mathbf{D} to X_0 , and $F: \mathbf{D} \rightarrow \mathbf{C}$ is a functor having a colimit L , then the coprojection morphism $q(X_0): F(X_0) \rightarrow L$ is an epimorphism. In particular, coequalizer maps are epimorphisms.

Proof. Assume the first situation. The universal property of L implies that a morphism $h: M \rightarrow L$ in \mathbf{C} is uniquely determined by the system of morphisms $p(X)h: M \rightarrow F(X)$ ($X \in \text{Ob}(\mathbf{D})$). But for any $X \in \text{Ob}(\mathbf{D})$, we can find a morphism $f: X_0 \rightarrow X$ in \mathbf{D} , and we then have $p(X) = F(f)p(X_0)$. Thus any $h: M \rightarrow L$ in \mathbf{C} is uniquely determined by the single morphism $p(X_0)h$. This is equivalent to saying $p(X_0)$ is a monomorphism. The result for

colimits follows by duality. \square

We have seen that the constructions of pairwise product and coproduct, when they exist for all pairs of objects of a category \mathbf{C} , give right and left adjoints to the “diagonal” functor $\Delta: \mathbf{C} \rightarrow \mathbf{C} \times \mathbf{C}$. These statements generalize to limits and colimits.

Proposition 7.6.3. *Let \mathbf{C} and \mathbf{D} be categories. Let $\Delta: \mathbf{C} \rightarrow \mathbf{C}^{\mathbf{D}}$ denote the “diagonal” functor, taking every object $X \in \text{Ob}(\mathbf{C})$ to the “constant” functor $\Delta(X) \in \text{Ob}(\mathbf{C}^{\mathbf{D}})$ with value X at all objects of \mathbf{D} and value id_X at all morphisms of \mathbf{D} , and likewise taking each morphism $f \in \mathbf{C}(X, Y)$ to the morphism of functors $\Delta(f): \Delta(X) \rightarrow \Delta(Y)$ with value f at all objects of \mathbf{D} .*

Then a limit of a functor $F: \mathbf{D} \rightarrow \mathbf{C}$ is the same as an object L representing the contravariant functor $\mathbf{C}^{\mathbf{D}}(\Delta(-), F): \mathbf{C}^{\text{op}} \rightarrow \mathbf{Set}$. In particular, if for a given \mathbf{D} all functors $\mathbf{D} \rightarrow \mathbf{C}$ have limits, then the construction $\varprojlim: \mathbf{C}^{\mathbf{D}} \rightarrow \mathbf{C}$ is a right adjoint to the diagonal functor $\Delta: \mathbf{C} \rightarrow \mathbf{C}^{\mathbf{D}}$.

Likewise, a colimit of $F: \mathbf{D} \rightarrow \mathbf{C}$ is an object L representing the covariant functor $\mathbf{C}^{\mathbf{D}}(F, \Delta(-)): \mathbf{C} \rightarrow \mathbf{Set}$. Thus, when all functors $\mathbf{D} \rightarrow \mathbf{C}$ have colimits, the construction $\varinjlim: \mathbf{C}^{\mathbf{D}} \rightarrow \mathbf{C}$ is a left adjoint to the diagonal functor $\Delta: \mathbf{C} \rightarrow \mathbf{C}^{\mathbf{D}}$. \square

These adjointness relationships are shown below.

$$\begin{array}{ccc} & \mathbf{C} & \\ \varinjlim \uparrow & \downarrow \Delta & \uparrow \varprojlim \\ & \mathbf{C}^{\mathbf{D}} & \end{array}$$

Note that if, as above, \mathbf{C} has colimits of all functors $F \in \mathbf{C}^{\mathbf{D}}$, then our observation that $\varinjlim: \mathbf{C}^{\mathbf{D}} \rightarrow \mathbf{C}$ is left adjoint to Δ tells us, in particular, that it is a *functor*. Thus, given a morphism

$$f: F \rightarrow G$$

in $\mathbf{C}^{\mathbf{D}}$, we get an induced morphism

$$\varinjlim_{\mathbf{D}} f: \varinjlim_{\mathbf{D}} F \rightarrow \varinjlim_{\mathbf{D}} G$$

in \mathbf{C} . This will be characterized by the equations

$$(7.6.4) \quad (\varinjlim_{\mathbf{D}} f) q_F(X) = q_G(X) f(X) \quad (X \in \text{Ob}(\mathbf{D}))$$

where $q_F(X): F(x) \rightarrow \varinjlim_{\mathbf{D}} F$ and $q_G(X): G(x) \rightarrow \varinjlim_{\mathbf{D}} G$ are the coprojection maps for these objects and colimits.

Similarly, if functors in $\mathbf{C}^{\mathbf{D}}$ have limits, then $\varprojlim_{\mathbf{D}}: \mathbf{C}^{\mathbf{D}} \rightarrow \mathbf{C}$ becomes a functor, with

$$\varprojlim_{\mathbf{D}} f: \varprojlim_{\mathbf{D}} F \rightarrow \varprojlim_{\mathbf{D}} G$$

characterized by

$$(7.6.5) \quad p_G(X)(\varprojlim_{\mathbf{D}} f) = f(X)p_F(X) \quad (X \in \text{Ob}(\mathbf{D}))$$

where $p_F(X): \varprojlim_{\mathbf{D}} F \rightarrow F(x)$ and $p_G(X): \varprojlim_{\mathbf{D}} G \rightarrow G(x)$ are projection maps.

In drawing a picture of a morphism $\Delta(M) \rightarrow F$ or $F \rightarrow \Delta(M)$ ($M \in \text{Ob}(\mathbf{C})$), we can for convenience collapse the copies of the object M and the identity arrows among these into a single “ M ”. (I.e., we can collapse the picture representing Proposition 7.6.3 into the picture representing Definition 7.6.1.) What we have then looks like a “cone” of maps, with M at the apex. Hence a morphism of functors $\Delta(M) \rightarrow F$ or $F \rightarrow \Delta(M)$ is often called a “cone” from the object M to the functor F , or from the functor F to the object M ; and the limit or colimit of a functor F may be described as an object with a “universal cone” to or from F .

Exercise 7.6.3. Let \mathbf{C} and \mathbf{D} be categories. By Lemma 6.10.1 (“Law of Exponents for Functors”), the functor $\Delta: \mathbf{C} \rightarrow \mathbf{C}^{\mathbf{D}}$ corresponds to some functor $\mathbf{D} \times \mathbf{C} \rightarrow \mathbf{C}$. Describe this functor.

Our construction in Lemma 7.5.3 of the inverse limit of an inverse system of sets (X_i, f_{ij}) as the subset of $\prod X_i$ determined by “compatibility” conditions can be generalized to give a construction of general limits in any category having appropriate products and equalizers, and it dualizes to a construction of colimits in categories with appropriate coproducts and coequalizers. (The latter construction may be thought of as generalizing our construction of the direct limit of a directed system of sets as the quotient of a disjoint union by an equivalence relation, though the simple way that equivalence relation could be described when \mathbf{D} was a directed partially ordered set and \mathbf{C} was **Set** does not go over to the general situation.) In the case of inverse limits, the compatibility conditions said that for all $i < j$ in I , the pair of maps $(p_j, f_{ij}p_i)$ had to agree on elements of our subset of $\prod X_i$. Such a family of conditions can in fact be translated to a condition saying that a single pair of maps into an appropriate product object should agree. Using this construction, we get

Proposition 7.6.6. *Let \mathbf{C} be a category and \mathbf{D} a small category, and let α be an infinite cardinal such that \mathbf{D} has $< \alpha$ objects and $< \alpha$ morphisms.*

Then if \mathbf{C} has products of all families of $< \alpha$ objects, and has equalizers, then every functor $F: \mathbf{D} \rightarrow \mathbf{C}$ has a limit.

Likewise, if \mathbf{C} has coproducts of families of $< \alpha$ objects, and has coequalizers, then every functor $F: \mathbf{D} \rightarrow \mathbf{C}$ has a colimit.

Proof. Under the hypotheses of the first assertion, let

$$P = \prod_{X \in \text{Ob}(\mathbf{D})} F(X),$$

$$P' = \prod_{X, Y \in \text{Ob}(\mathbf{D}), f \in \mathbf{D}(X, Y)} F(Y).$$

(If we required categories to have disjoint hom-sets, we could write the latter definition more simply as $P' = \prod_{f \in \text{Ar}(\mathbf{D})} F(\text{cod}(f))$.) Denote the projection morphisms associated with these two product objects by $p_X: P \rightarrow F(X)$ ($X \in \text{Ob}(\mathbf{D})$) and $p'_{X, Y, f}: P' \rightarrow F(Y)$ ($X, Y \in \text{Ob}(\mathbf{D}), f \in \mathbf{D}(X, Y)$). We shall construct L as the equalizer of two maps $a, b: P \rightarrow P'$. Since a and b are to be morphisms into the direct product object P' , they may be defined by specifying their composites with the projection morphisms $p'_{X, Y, f}: P' \rightarrow F(Y)$. Define them so that

$$p'_{X, Y, f} a = p_Y, \quad p'_{X, Y, f} b = F(f)p_X.$$

If L is the equalizer of a and b , and $k: L \rightarrow P$ the canonical morphism, we see that the universal property of L as an equalizer is equivalent to the statement that the morphisms $p_X k: L \rightarrow F(X)$ form commuting triangles with the morphisms $F(f)$ and are universal for this

property. Thus, the object L together with the morphisms $p_X k$ has the property characterizing $\varprojlim F$.

The result for colimits follows by duality. \square

Exercise 7.6:4. Verify the assertion following the phrase “we see that”, near the end of the above proof.

Of course, some limits or colimits may exist even if the category does not have enough (co)products and (co)equalizers to obtain them by the above lemma. Such a case is noted in point (iv) of the next exercise. (But the most useful part of this exercise is (i), and the most difficult, surprisingly, is (ii).)

Exercise 7.6:5. Let \mathbf{C} be a category.

- (i) Show that an initial object of \mathbf{C} is equivalent to a *colimit* of the unique functor from the empty category into \mathbf{C} .
- (ii) Show that such an initial object is also equivalent to a *limit* of the identity functor of \mathbf{C} .
- (iii) State the corresponding results for a terminal object.
- (iv) Give an example where the limit of (ii) exists, but \mathbf{C} does not satisfy the hypotheses needed to get this from Proposition 7.6.6.

Here is another degenerate case of the concept of limit:

Exercise 7.6:6. Characterize the categories \mathbf{D} with the property that every constant functor from \mathbf{D} to any category \mathbf{C} , i.e., any functor of the form $\Delta(C): \mathbf{D} \rightarrow \mathbf{C}$ ($C \in \text{Ob}(\mathbf{C})$) has a limit given by the object C itself, with universal cone consisting of identity morphisms of C . State the corresponding result for colimits.

We have seen that a product or coproduct of objects in a category may or may not coincide with their product or coproduct in a subcategory to which they also belong. E.g., the coproduct of two abelian groups in the category of all groups and their coproduct in the category of all abelian groups are different, since the former is generally nonabelian. We note below that for full subcategories, such phenomena occur if and only if the constructed object in the larger category fails to lie in the subcategory.

Lemma 7.6.7. Let \mathbf{C} be a category, \mathbf{B} a full subcategory of \mathbf{C} , $I: \mathbf{B} \rightarrow \mathbf{C}$ the inclusion functor, and $F: \mathbf{D} \rightarrow \mathbf{B}$ a functor from an arbitrary category into \mathbf{B} .

If $\varprojlim IF$ exists (loosely, if there exists “a limit of the system of objects $F(X)$ in the larger category \mathbf{C} ”), and if as an object it belongs to \mathbf{B} , then this same object, with the same cone to the objects $F(X)$, constitutes a limit $\varprojlim F$ (loosely, it is also “a limit of the given system within the subcategory \mathbf{B} ”).

The same is true for colimits $\varinjlim IF$ and $\varinjlim F$. \square

Exercise 7.6:7. (i) Prove the above lemma.

- (ii) Does the above result remain true if the hypothesis that the subcategory \mathbf{B} is full in \mathbf{C} is deleted? If not, does it help to add the hypothesis that not only the object $\varprojlim IF$, but also the projection maps from this object to the values of F belong to \mathbf{B} ?

The example of the coproduct of two abelian groups in \mathbf{Ab} and in \mathbf{Group} shows that if we merely assume in the above lemma that $\varinjlim IF$ exists, this does not guarantee that it belongs to \mathbf{B} . By duality, one can get from this colimit example an analogous example for limits. (Using

what category and subcategory?)

Another way that the same object can be a limit of two related functors is examined in

Exercise 7.6:8. Given functors $\mathbf{D}_0 \xrightarrow{D} \mathbf{D}_1 \xrightarrow{F} \mathbf{C}$, we see that if F has a limit $L = \varprojlim F$, then the cone from L to F induces a cone from L to FD , and we can look for conditions under which L , with this cone, is also a limit of FD . In particular, we can ask which functors $D: \mathbf{D}_0 \rightarrow \mathbf{D}_1$ have the property that this is true for all functors F with domain \mathbf{D}_1 which have limits.

Exercise 7.5:1 answered this question for functors $D: (P_0)_{\text{cat}} \rightarrow (P_1)_{\text{cat}}$ induced by inclusions of partially ordered sets $P_0 \subseteq P_1$. Investigate the same question for functors D between general (i.e., arbitrary small, or perhaps legitimate) categories; that is, look for necessary and/or sufficient conditions on a functor D for this property to hold.

We indicated in the last two paragraphs of §6.10 that if a category \mathbf{C} has finite products, then any functor category $\mathbf{C}^{\mathbf{E}}$ will also have such products, which can be computed “objectwise”. To formulate the analogous result for general limits and colimits, suppose \mathbf{C} and \mathbf{E} are categories and E an object of \mathbf{E} ; then let us write $c_E: \mathbf{C}^{\mathbf{E}} \rightarrow \mathbf{C}$ for the “ E th coordinate functor”, taking functors and morphisms of functors to their values at the object E . Likewise, if $f: E_1 \rightarrow E_2$ is a morphism in \mathbf{E} , then $c_f: c_{E_1} \rightarrow c_{E_2}$ will denote the induced morphism of coordinate functors. You should find it easy to verify

Lemma 7.6.8. *Let \mathbf{C} , \mathbf{D} and \mathbf{E} be categories. Then if all functors $\mathbf{D} \rightarrow \mathbf{C}$ have limits, so do all functors $\mathbf{D} \rightarrow \mathbf{C}^{\mathbf{E}}$. Namely, given $F: \mathbf{D} \rightarrow \mathbf{C}^{\mathbf{E}}$, the object $L = \varprojlim_{\mathbf{D}} F$ of $\mathbf{C}^{\mathbf{E}}$ can be described as the functor taking each $E \in \text{Ob}(\mathbf{E})$ to $\varprojlim_{\mathbf{D}} c_E F$, and each $f \in \mathbf{E}(E_1, E_2)$ to $\varprojlim_{\mathbf{D}} c_f \circ F$.*

Likewise, if all functors $\mathbf{D} \rightarrow \mathbf{C}$ have colimits, then all functors $\mathbf{D} \rightarrow \mathbf{C}^{\mathbf{E}}$ have colimits, which are similarly constructed “object- and morphism-wise”. \square

Exercise 7.6:9. Prove Lemma 7.6.8 for the case of limits.

7.7. What respects what. It is natural to ask what one can say about *limits* and *colimits* of systems of objects constructed by *adjoint functors*, about the values of *adjoint functors* on objects constructed by *limits* and *colimits*, and similar questions for other sorts of universal constructions.

Some quick examples: It is not hard to see that the free group on a disjoint union of sets, $X \sqcup Y$, is the coproduct of the free groups on X and Y . If we look similarly at the free group on the coequalizer of a pair of set maps, $f, g: X \rightrightarrows Y$ we find that it is the coequalizer of the induced maps of free groups, $F(f), F(g): F(X) \rightrightarrows F(Y)$. On the other hand, a direct product of free groups is in general not a free group, in particular not the free group on the direct product set. So the *free group* construction seems to respect *colimits*, but not *limits*.

If we look at its right adjoint, the underlying set functor, we find the opposite: The underlying set of a product or equalizer of groups is the product or equalizer of the underlying sets of the groups (that is how we constructed products and equalizers of groups), but the underlying set of a coproduct of groups is not the coproduct (disjoint union) of their underlying sets, both because the group operation within this coproduct generally produces new elements from the elements of the two given groups, and because the images of the two identity elements fall together. Similarly, when we take a coequalizer of two group homomorphisms $f, g: G \rightrightarrows H$, more identifications of elements are forced than in the set-theoretic coequalizer; not only must pairs of elements $f(a)$ and

$g(a)$ ($a \in |G|$) fall together, but also pairs such as $f(a)b$ and $g(a)b$ ($a \in |G|$, $b \in |H|$).

These examples suggest the general principle that “left universal constructions respect left universal constructions, and right universal constructions respect right universal constructions”. We shall prove a series of theorems of that form in this and the next section.

We have seen left universal constructions in four guises: initial objects, representing objects for covariant set-valued functors, left adjoint functors, and colimits. Since an initial object of a category may be described as the object representing a certain trivial set-valued functor (Exercise 7.2:8) or as the colimit of a functor from a certain trivial category (Exercise 7.6:5(i)), let us focus on relations among the remaining three types of constructions. These give us six unordered pairs of constructions to consider. But I see no way that one can speak of the construction of an object representing a covariant set-valued functor U “respecting” the construction of an object representing another such set-valued functor V , so let us move on to the next case, the relation between left adjoint functors and representing objects for covariant set-valued functors. We give this, along with its dual, as

Theorem 7.7.1. Suppose $\mathbf{D} \begin{array}{c} \xrightarrow{U} \\ \xleftarrow{F} \end{array} \mathbf{C}$ are adjoint functors, with U the right adjoint and F the left adjoint, and with unit η and counit ε .

If $A: \mathbf{C} \rightarrow \mathbf{Set}$ is a representable functor, with representing object $R \in \text{Ob}(\mathbf{C})$ and universal element $u \in A(R)$, then $AU: \mathbf{D} \rightarrow \mathbf{Set}$ is also representable, with representing object $F(R)$ and universal element $A(\eta(R))(u) \in A(U(F(R)))$.

Likewise, if $B: \mathbf{D}^{\text{op}} \rightarrow \mathbf{Set}$ is representable, with representing object $R \in \text{Ob}(\mathbf{D})$ and universal element $u \in B(R)$, then $BF: \mathbf{C}^{\text{op}} \rightarrow \mathbf{Set}$ is representable, with representing object $U(R)$ and universal element $B(\varepsilon(R))(u) \in B(F(U(R)))$.

Proof. In the first situation, $AU(-) \cong \mathbf{C}(R, U(-)) \cong \mathbf{D}(F(R), -)$, showing that AU is represented by $F(R)$. The identification of the universal element, corresponding to the identity morphism in $\mathbf{D}(F(R), F(R))$ is straightforward. The second situation is the dual of the first. \square

As an example, suppose we wish to construct the ring with a universal pair of elements x, y satisfying the relation $xy = yx^2$. We notice that this ring-theoretic relation is “actually a monoid relation”; the formal statement is that the functor we want to represent can be written AU , where U is the forgetful functor from \mathbf{Ring}^1 to \mathbf{Monoid} , and A the functor associating to any monoid S the set of pairs (x, y) of elements of S satisfying $xy = yx^2$. It is not hard to see that we can construct our ring by first forming the monoid R presented by these generators and relation, and then passing to the monoid ring $\mathbb{Z}R$, i.e., applying the left adjoint to U , and that this is an instance of the above theorem. Note that the universal ring elements x and y satisfying the given equation are the images of the corresponding universal monoid elements, under the canonical map $\eta(R): R \rightarrow U(F(R))$ (informally, the inclusion map $R \rightarrow \mathbb{Z}R$). Applying $\eta(R)$ to this pair of elements of R corresponds to applying $A(\eta(R))$ to the element $(x, y) \in A(R)$, as in the statement of the theorem.

The above example makes it clear that Theorem 7.7.1 is a powerful tool, and that it indeed deserves to be described as saying that “left adjoint functors respect the construction of objects representing covariant set-valued functors”.

Note, however, that, the sense in which the latter statement is true is rather idiosyncratic; the formulation involves both the left adjoint functor and its right adjoint, and it does not appear to be a special case of any natural concept of a left adjoint functor respecting a general construction, or of a general functor respecting the construction of representing objects. There is a similarly

idiosyncratic sense in which “left adjoint functors respect other left adjoint functors”; this is Theorem 7.3.9, already proved, which says that the composite of the left adjoints of two functors is the left adjoint of their composite (in the opposite order).

In contrast, when one looks at questions of how one of our three sorts of left universal construction interacts with colimits (these three cases being all we have left to consider of our six possible sorts of interaction), one finds that there *is* a natural definition of an arbitrary functor’s respecting a colimit. We will examine that concept in the next section, and verify the remaining cases of our observation that left universal constructions respect left universal constructions, and hence, dually, that right universal constructions respect right universal constructions.

Exercise 7.7:1. Prove the following converse to the first assertion of Theorem 7.7.1: If $U: \mathbf{D} \rightarrow \mathbf{C}$ is a functor such that for every representable functor $A: \mathbf{C} \rightarrow \mathbf{Set}$, the composite functor $AU: \mathbf{D} \rightarrow \mathbf{Set}$ is representable, then U has a left adjoint. Also state the dual result.

7.8. Functors respecting limits and colimits. Here is the definition of a functor “respecting” a limit or colimit.

Definition 7.8.1. Let \mathbf{C}, \mathbf{C}' be categories, and $F: \mathbf{C} \rightarrow \mathbf{C}'$ a functor.

Then if $S: \mathbf{E} \rightarrow \mathbf{C}$ is a functor into \mathbf{C} , having a limit $\varprojlim S$, with cone of projection maps $p_E: \varprojlim S \rightarrow S(E)$ ($E \in \text{Ob}(\mathbf{E})$), one says that F respects the limit of S if the object $F(\varprojlim S)$, together with the cone of morphisms $F(p_E): F(\varprojlim S) \rightarrow F(S(E))$ from this object to the functor $FS: \mathbf{E} \rightarrow \mathbf{C}'$, is a limit of the functor FS .

We shall say that F respects small limits if for every functor S from a small category \mathbf{E} to \mathbf{C} which has a limit, F respects the limit of S . We shall say that F respects possibly large limits if this is true without the restriction that \mathbf{E} be small. Likewise, we shall say that F respects pullbacks, terminal objects, small products, possibly large products, small inverse limits, possibly large inverse limits, etc., if it respects all instances of the sort of limit named.

Dually, if $S: \mathbf{E} \rightarrow \mathbf{C}$ is a functor having a colimit $\varinjlim S$, with cone of coprojection maps $q_E: S(E) \rightarrow \varinjlim S$, then we shall say that F respects the colimit of S if the object $F(\varinjlim S)$, with the cone from FS given by the morphisms $F(q_E): F(S(E)) \rightarrow F(\varinjlim S)$, is a colimit of FS ; and we will say that F respects small colimits, possibly large colimits, pushouts, initial objects, small or possibly large direct limits, etc., if it respects all colimits having these respective descriptions.

In all of these situations, we may use “commutes with” as a synonym for “respects”.

(Many authors, e.g., Mac Lane [17], again following the topological associations of the word “limit”, call a functor respecting limits “continuous”, and one respecting colimits “cocontinuous”. But we will not use these terms here.)

The distinctions between the “small” and “possibly large” cases of the above definition are technically necessary, but there are situations where they can be ignored:

Observation 7.8.2. Suppose it can be proved that all functors F having a certain property P respect small limits (respectively small colimits; or small limits or colimits of a particular sort, such as products or coproducts). Suppose, moreover, that any functor that satisfies P with respect to a given universe \mathbb{U} (i.e., with all occurrences of “set” in the statement of P taken to mean “ \mathbb{U} -small set”, any occurrences of “category” taken to mean “ \mathbb{U} -legitimate category”, etc.) continues to satisfy P with respect to all larger universes. (The simplest and commonest case

where this is true is if P does not refer to any categories other than the domain and codomain of F , nor to any sets other than those obtained in straightforward ways from those categories; e.g., if P is the property “ F is a left adjoint functor”.) Then any functor satisfying P in fact respects possibly large limits (respectively, possibly large colimits, products, coproducts, etc.).

Hence, in discussing properties P which are preserved under enlarging the universe, if we make assertions that functors satisfying P “respect limits” etc., we need not specify “small” or “possibly large”. \square

The above observation will in fact allow us to ignore the small/possibly-large distinction in formulating the results of this section. For instance, since the properties of being left and right adjoints do not depend on the universe, we do not need to worry about smallness in stating

Theorem 7.8.3. *Left adjoint functors respect colimits, and right adjoint functors respect limits.*

Proof. Let $\mathbf{D} \begin{array}{c} \xrightarrow{U} \\ \xleftarrow{F} \end{array} \mathbf{C}$ be adjoint functors, with U the right and F the left adjoint, and suppose $S: \mathbf{E} \rightarrow \mathbf{C}$ has a colimit L , with coprojection maps q_E ($E \in \text{Ob}(\mathbf{E})$). Recall that L represents the functor $\mathbf{C}^{\mathbf{E}}(S, \Delta(-)): \mathbf{C} \rightarrow \mathbf{Set}$, i.e., the construction taking each object $C \in \text{Ob}(\mathbf{C})$ to the set of cones $\mathbf{C}^{\mathbf{E}}(S, \Delta(C))$ and acting correspondingly on morphisms, and that the cone $(q_E)_{E \in \text{Ob}(\mathbf{E})}$ is the universal element for this representing object.

Applying Theorem 7.7.1, we see that $F(L)$ will represent the functor $\mathbf{D} \rightarrow \mathbf{Set}$ given by

$$(7.8.4) \quad \mathbf{C}^{\mathbf{E}}(S, \Delta(U(-))) = \mathbf{C}^{\mathbf{E}}(S, U\Delta(-)) \cong \mathbf{D}^{\mathbf{E}}(FS, \Delta(-));$$

in other words, it will be a colimit of FS .

The universal cone could hardly be anything but $(F(q_E))$; but we need to check this formally. By Theorem 7.7.1, to get this universal element we apply to L the unit η of our adjunction, getting a morphism $\eta(L): L \rightarrow UF(L)$, apply the functor $\mathbf{C}^{\mathbf{E}}(S, \Delta(-))$ to it, getting a set map

$$\mathbf{C}^{\mathbf{E}}(S, \Delta(\eta(L))): \mathbf{C}^{\mathbf{E}}(S, \Delta(L)) \rightarrow \mathbf{C}^{\mathbf{E}}(S, \Delta(UF(L))),$$

and apply this set map to our original universal cone. Now the above set map is given by left composition with $\eta(L)$, so it transforms our original cone (q_E) from S to L into the cone $(\eta(L)q_E)$ from S to $UF(L)$. Following (7.8.4), we identify cones from the functor S to objects $U(D)$ ($D \in \text{Ob}(\mathbf{D})$) with cones from FS to the objects D by use of the given adjunction. This identification works by applying F to the given morphisms, then applying the counit of the adjunction to the codomains of the resulting morphisms. So the morphisms $\eta(L)q_E$ of our cone are first transformed to $F(\eta(L)q_E) = F(\eta(L))F(q_E)$, then composed on the left with $\varepsilon(F(L))$. By Theorem 7.3.7(iii), the latter morphism is left inverse to $F(\eta(L))$, so the composite is $F(q_E)$, as claimed.

The assertion about right adjoint functors and limits follows by duality. \square

For example, suppose (\mathbf{C}, U) is a concrete category having free objects on all sets, i.e., such that U has a left adjoint F . Then we see by applying the above theorem to appropriate colimits in \mathbf{Set} that a free object in \mathbf{C} on a disjoint union of sets is a coproduct of the free objects on the given sets, and that a free object on the empty set is an initial object. (These facts were noted for particular cases in Chapter 3.) The fact that right adjoints respect limits tells us, likewise, that for \mathbf{C} and U as above, if we call $U(X)$ the “underlying set” of $X \in \text{Ob}(\mathbf{C})$, then underlying sets of product objects, terminal objects, equalizers, and inverse limits are, respectively, direct products of underlying sets, the one-element set, equalizers of underlying sets, and inverse limits of

underlying sets. This explains why, in so many familiar cases, the construction of the latter objects begins by applying the corresponding construction to underlying sets. (The perceptive reader may note that what this actually does is reduce these many facts to the one unexplained fact that the underlying set functors of the categories arising in algebra tend to have left adjoints – though they rarely have right adjoints.)

Exercise 7.8.1. (i) Combining the above theorem with Lemma 6.8.10, obtain results on how left and adjoint functors behave with respect to epimorphisms and monomorphisms.

(ii) These results will *not* say that both left and right adjoint functors preserve both epimorphisms and monomorphisms. Find examples showing that the implications *not* proved in part (i) do not, in general, hold.

(For some related observations, positive and negative, cf. Exercise 6.7:9 and Exercise 6.8:7(v)-(vi).)

Let us look next at how limits and colimits interact with objects that represent functors. In this form, there is not an obvious question to ask; but we can ask whether *representable functors* respect limits and colimits. However, our definition of what it means for a functor F to respect a limit or colimit assumed F covariant; so to include the case of contravariant representable functors, we need to adapt that definition.

Definition 7.8.5. Let \mathbf{C} , \mathbf{C}' be categories, and F a contravariant functor from \mathbf{C} to \mathbf{C}' , i.e., a functor $\mathbf{C}^{\text{op}} \rightarrow \mathbf{C}'$.

Then if $S: \mathbf{E} \rightarrow \mathbf{C}$ is a functor having a limit $\varprojlim S$, with projection maps $p_E: \varprojlim S \rightarrow S(E)$, one says that F turns the limit of S into a colimit if the object $F(\varprojlim S)$, together with the cone from the functor $FS: \mathbf{E}^{\text{op}} \rightarrow \mathbf{C}'$ to this object given by the morphisms $F(p_E): F(S(E)) \rightarrow F(\varprojlim S)$, is a colimit of that functor (equivalently, if, viewing $\varprojlim S$ and (p_E) as an object and a cone of morphisms in \mathbf{C}^{op} which comprise a colimit of the functor $S^{\text{op}}: \mathbf{E}^{\text{op}} \rightarrow \mathbf{C}^{\text{op}}$, the functor F respects this colimit).

This yields the obvious definitions of statements such as that F “turns small limits into colimits”, “turns pullbacks into pushouts”, “turns terminal objects into initial objects”, etc..

We define analogously the concept of F turning the colimit of a functor S into a limit (and thus the concepts of turning coproducts into products, pushouts into pullbacks, etc.).

We can now state

Theorem 7.8.6. Let \mathbf{C} be a category. Then covariant representable functors $V: \mathbf{C} \rightarrow \mathbf{Set}$ respect limits, and contravariant representable functors on \mathbf{C} , $W: \mathbf{C}^{\text{op}} \rightarrow \mathbf{Set}$, turn colimits (in \mathbf{C}) into limits (in \mathbf{Set}).

Sketch of Proof. The second statement is equivalent to the first applied to the category \mathbf{C}^{op} , so it suffices to prove the first assertion.

Without loss of generality we may take $V = h_R$ where $R \in \text{Ob}(\mathbf{C})$. Let L be the limit of a functor $S: \mathbf{E} \rightarrow \mathbf{C}$. Thus, L is given with a universal cone to S , i.e., a universal $\text{Ob}(\mathbf{E})$ -tuple of morphisms $p_E: L \rightarrow S(E)$ making commuting diagrams with the morphisms $S(f)$ arising from morphisms in \mathbf{E} . Applying h_R , we get a cone of set maps from the set $h_R(L) = \mathbf{C}(R, L)$ to the sets $\mathbf{C}(R, S(E))$. The fact that it is a cone tells us that each element of $\mathbf{C}(R, L)$ determines, under these maps, a family of elements of the sets $\mathbf{C}(R, S(E))$ that is respected by the maps $h_R(S(f))$.

Moreover, the universal property of L tells us that each system of elements of the $\mathbf{C}(R, S(E))$

that is respected by the maps $h_R(S(f))$ arises in this way from a unique element of $\mathbf{C}(R, L)$. On the other hand, limits over \mathbf{E} in \mathbf{Set} are given by $\text{Ob}(\mathbf{E})$ -tuples of elements satisfying just these compatibility conditions; so we see that $\varprojlim_{\mathbf{E}} h_R(S(E))$ and its universal cone to the sets $h_R(S(E))$ can be identified with $h_R(L)$ and its universal cone to these same sets. Thus, the functors agree on objects; the behavior on morphisms is determined by compatibility with maps among cones, and so also agrees. \square

Exercise 7.8:2. (i) Show by example that covariant representable functors $\mathbf{Ab} \rightarrow \mathbf{Set}$ need not respect colimits. In fact, give examples of failure to respect coproducts, failure to respect coequalizers, and failure to respect direct limits.

(ii) Similarly show by examples that contravariant representable functors on \mathbf{Ab} in general fail to turn products, equalizers, and inverse limits into coproducts, coequalizers and direct limits respectively.

Finally, we come to the interaction of colimits with colimits, and of limits with limits. Suppose $B: \mathbf{D} \times \mathbf{E} \rightarrow \mathbf{C}$ is a bifunctor. Then each object D of \mathbf{D} induces a functor $B(D, -): \mathbf{E} \rightarrow \mathbf{C}$, and each morphism $f: D \rightarrow D'$ in \mathbf{D} yields a morphism of functors, $B(f, -): B(D, -) \rightarrow B(D', -)$. (Cf. Lemma 6.10.1 and preceding discussion.) If for each D the functor $B(D, -)$ has a colimit, let us write these objects $\varinjlim_{\mathbf{E}} B(D, E) \in \text{Ob}(\mathbf{C})$. The morphisms between functors $B(D, -)$ induce morphisms among these colimit objects (cf. (7.6.4) and preceding display), so that the construction of $\varinjlim_{\mathbf{E}} B(D, E)$ from D becomes a functor $\varinjlim_{\mathbf{E}} B(-, E): \mathbf{D} \rightarrow \mathbf{C}$. Suppose this functor in turn has a colimit, which we write $\varinjlim_{\mathbf{D}} (\varinjlim_{\mathbf{E}} B(D, E))$. Then the composites of coprojections

$$(7.8.7) \quad B(D_0, E_0) \rightarrow \varinjlim_{\mathbf{E}} B(D_0, E) \rightarrow \varinjlim_{\mathbf{D}} (\varinjlim_{\mathbf{E}} B(D, E)) \quad (D_0 \in \text{Ob}(\mathbf{D}), E_0 \in \text{Ob}(\mathbf{E}))$$

constitute a cone of morphisms from the $B(D_0, E_0)$ to our iterated colimit, and it is straightforward to verify that the latter object, together with this cone, has the universal property of $\varinjlim_{\mathbf{D} \times \mathbf{E}} B(D, E)$.

Exercise 7.8:3. (i) Prove the above claim, that if $\varinjlim_{\mathbf{D}} (\varinjlim_{\mathbf{E}} B(D, E))$ exists, the morphisms (7.8.7) form a cone with respect to which the right hand object satisfies the universal property of $\varinjlim_{\mathbf{D} \times \mathbf{E}} B(D, E)$.

(ii) Give an example where $\varinjlim_{\mathbf{D} \times \mathbf{E}} B(D, E)$ exists, but $\varinjlim_{\mathbf{D}} (\varinjlim_{\mathbf{E}} B(D, E))$ does not.

Thus we have the first isomorphism in the first display of the next theorem. By symmetry, we likewise have the second isomorphism of that display if the rightmost colimit exists. The isomorphisms of the second display similarly hold under the dual hypotheses.

Theorem 7.8.8. *Colimits commute with colimits, and limits commute with limits.*

Precisely, let $B: \mathbf{D} \times \mathbf{E} \rightarrow \mathbf{C}$ be a bifunctor. Then

$$(7.8.9) \quad \varinjlim_{\mathbf{D}} (\varinjlim_{\mathbf{E}} B(D, E)) \cong \varinjlim_{\mathbf{D} \times \mathbf{E}} B(D, E) \cong \varinjlim_{\mathbf{E}} (\varinjlim_{\mathbf{D}} B(D, E)),$$

in the sense that if the left side of the above display is defined, then this object also has the universal property of the middle object, via the cone of morphisms (7.8.7), and similarly, if the right side is defined, it has the property of the middle object via the analogous cone. Hence, if both sides are defined, they are isomorphic.

Likewise

$$(7.8.10) \quad \varprojlim_{\mathbf{D}} (\varprojlim_{\mathbf{E}} B(D, E)) \cong \varprojlim_{\mathbf{D} \times \mathbf{E}} B(D, E) \cong \varprojlim_{\mathbf{E}} (\varprojlim_{\mathbf{D}} B(D, E))$$

in the same sense. \square

As formulated, (7.8.9) is not an instance of a functor “respecting” colimits in the precise sense of Definition 7.8.1, because the minimalist hypotheses we assumed do not make $\varinjlim_{\mathbf{D}}$ a functor on all of $\mathbf{C}^{\mathbf{D}}$. If we in fact assume that all functors from \mathbf{D} to \mathbf{C} have colimits (e.g., if \mathbf{C} has small colimits and \mathbf{D} is small), then the isomorphism $\varinjlim_{\mathbf{D}} (\varinjlim_{\mathbf{E}} (B(D, E))) \cong \varinjlim_{\mathbf{E}} (\varinjlim_{\mathbf{D}} (B(D, E)))$ becomes a case of Theorem 7.8.3, since $\varinjlim_{\mathbf{D}}$ becomes a left adjoint functor $\mathbf{C}^{\mathbf{D}} \rightarrow \mathbf{C}$. However, the identification of the common value of the two iterated colimits as $\varinjlim_{\mathbf{D} \times \mathbf{E}} B(D, E)$ must still be stated and proved separately. (In the same spirit, if \mathbf{C} has small coproducts, the covariant case of Theorem 7.8.6 follows from Theorem 7.8.3 and Exercise 7.3:3.)

The case of (7.8.9) where \mathbf{E} is the empty category says that colimits respect initial objects; i.e., that if I is an initial object of \mathbf{C} , then the diagram $\Delta(I) \in \mathbf{C}^{\mathbf{D}}$ has colimit I . For instance, the coproduct in \mathbf{Ring}^1 of two copies of \mathbb{Z} is again \mathbb{Z} . The next exercise examines variants of this result.

Exercise 7.8:4. (i) Show, conversely, that if an object I of a category \mathbf{C} has the property that for all small categories \mathbf{D} , the functor $\Delta(I) \in \mathbf{C}^{\mathbf{D}}$ has a colimit isomorphic to I , then I is an initial object of \mathbf{C} .

(Contrast Exercise 7.6:6, which asks for a description of those categories \mathbf{D} such that this property holds for *all* objects of \mathbf{C} .)

(ii) Can you characterize those objects I of a category \mathbf{C} for which the hypothesis of (i) holds for all *nonempty* small categories \mathbf{D} ?

(iii) Show that in \mathbf{Ring}^1 (or if you prefer, $\mathbf{CommRing}^1$), every ring of the form \mathbb{Z}_n has the property of (ii).

Despite the similar nomenclature, category-theoretic double limits behave quite differently from double limits in topology. The contrast is explored in

Exercise 7.8:5. (i) For nonnegative integers i, j , define b_{ij} to be 1 if $i > j$, 2 if $i \leq j$. Show that as limits of real-valued functions, $\lim_{i \rightarrow \infty} (\lim_{j \rightarrow \infty} b_{ij})$ and $\lim_{j \rightarrow \infty} (\lim_{i \rightarrow \infty} b_{ij})$ exist and are unequal.

(ii) Let the set $\omega \times \omega$ be partially ordered by setting $(i, j) \leq (i', j')$ if and only if $i \leq i'$ and $j \leq j'$. Show that there exist functors (directed systems) $B: (\omega \times \omega)_{\mathbf{cat}} \rightarrow \mathbf{Set}$ satisfying $\text{card}(B(i, j)) = b_{ij}$, for the function b_{ij} defined in (i).

(iii) Deduce from Theorem 7.8.8 that a functor as in (ii) can never have the property that for each i , the given morphisms $B(i, j) \rightarrow B(i, j+1)$ and $B(i, j) \rightarrow B(i+1, j)$ are isomorphisms for all sufficiently large j .

(iv) Establish the result of (iii) directly, without using the concept of category-theoretic limit.

In earlier sections, there were several exercises asking you to determine whether functors were representable or had right or left adjoints. If you go back over the cases where the functors turned out *not* to be representable, or not to have an adjoint, you will find that, whatever ad hoc arguments you may have used at the time, each of these negative results can be deduced from Theorem 7.8.6 or 7.8.3 by noting that the functor in question fails to respect some limit or colimit.

Since limits and colimits come in many shapes and sizes, it is useful to note that to test whether a functor respects these constructions, it suffices to check two basic cases.

Corollary 7.8.11 (to proof of Proposition 7.6.6). *Let \mathbf{C} , \mathbf{D} be categories and $F: \mathbf{C} \rightarrow \mathbf{D}$ a functor.*

If \mathbf{C} has small colimits, then F respects such colimits if and only if it respects coequalizers and respects coproducts of small families of objects.

Likewise, if \mathbf{C} has small limits, then F respects these if and only if it respects equalizers, and products of small families. \square

One can break things down further, if one wishes:

Exercise 7.8:6. (i) Let \mathbf{C} be a category having coproducts of pairs of objects, and hence of finite nonempty families of objects. Show that the universal property of a coproduct of an arbitrary family $\coprod_I X_i$ is equivalent to that of a direct limit, over the directed partially ordered set of finite nonempty subsets $I_0 \subseteq I$, of the finite coproducts $\coprod_{I_0} X_i$.

(ii) Deduce that a category has small colimits if and only if it has coequalizers, finite coproducts, and colimits over directed partially ordered sets; and that a functor on such a category will respect small colimits if and only if it respects those three constructions.

State the corresponding result for *limits*.

(iii) For every two of the three conditions “respects equalizers”, “respects finite products”, “respects inverse limits over inversely directed partially ordered sets” (the conditions occurring in the dual to the result of (ii)), try to find an example of a functor among categories having small limits which satisfies those two conditions but not the third. As far as possible, use naturally occurring examples.

You might look at further similar questions; e.g., whether you can find an example respecting both finite and infinite products, but not inverse limits; or whether you can still get a full set of examples if you break the condition of respecting finite products into the two conditions of respecting pairwise products and respecting the terminal object (the product of the empty family).

One can go into this more deeply. I do not know the answers to most of the questions raised in

Exercise 7.8:7. Let A denote the (large) set of all small categories, and B the (large) set of all legitimate categories. Define a relation $R \subseteq A \times B$ by putting $(\mathbf{E}, \mathbf{C}) \in R$ if all functors $\mathbf{E} \rightarrow \mathbf{C}$ have colimits.

(i) The above relation induces a Galois connection between A and B . Translate results proved about existence of colimits in Proposition 7.6.6 and part (ii) of the preceding exercise into statements about the closure operator $**$ on A .

(ii) Investigate further the properties of the lattice of closed subsets of A . Is it finite, or infinite? Can you characterize the induced closure operator on the subclass of A or of B consisting of categories $P_{\mathbf{cat}}$ for partially ordered sets P ?

The above questions concerned *existence* of colimits. To study preservation of colimits, let C denote the class of functors F whose domain and codomain are legitimate categories having small colimits, and let us define a relation $S \subseteq A \times C$ by putting $(\mathbf{E}, F) \in S$ if $F: \mathbf{C} \rightarrow \mathbf{D}$ respects the colimits of all functors $\mathbf{E} \rightarrow \mathbf{C}$. This relation likewise induces a Galois connection between A and C ; so let us ask

(iii) Can you obtain results relating the lattice of closed subsets of A under this new Galois connection and the lattice of subsets of A closed under the Galois connection of part (i)? If they are not identical, investigate the structure of this new lattice. (You will have to use a notation that distinguishes between the two Galois connections.)

In studying situations where we do not know whether one functor respects the (co)limit of another, but where the two (co)limits in question both exist, there is a natural way to compare them:

Definition 7.8.12. If $\mathbf{E} \xrightarrow{S} \mathbf{C} \xrightarrow{F} \mathbf{D}$ are functors such that $\varinjlim S$ and $\varinjlim FS$ both exist, then by the comparison morphism

$$\varinjlim FS \rightarrow F(\varinjlim S)$$

we shall mean the unique morphism from the former to the latter object which makes a commuting diagram with the natural cones of maps from the functor FS to these two objects (namely, the universal cone from FS to $\varinjlim FS$, and the cone obtained by applying F to the universal cone from S to $\varinjlim S$. The existence and uniqueness of this map follow from the universal property of the former cone.)

Likewise, if $\varprojlim S$ and $\varprojlim FS$ both exist, then by the comparison morphism

$$F(\varprojlim S) \rightarrow \varprojlim FS$$

we shall mean the unique morphism which makes a commuting diagram with the obvious cones from these two objects to the functor FS .

In particular, we will use the term “comparison morphism” in connection with coproducts, products, coequalizers, equalizers, etc., regarding these as colimits and limits.

It is clear that these comparison morphisms measure whether the functor F respects these colimits and limits, i.e., comparing Definitions 7.8.12 and 7.8.1 we have

Lemma 7.8.13. Given S and F as in the first paragraph of Definition 7.8.12, the functor F respects the colimit of S if and only if the comparison morphism $\varinjlim FS \rightarrow F(\varinjlim S)$ is an isomorphism. Likewise, under the assumptions of the second paragraph of that definition, F respects the limit of S if and only if the comparison morphism $F(\varprojlim S) \rightarrow \varprojlim FS$ is an isomorphism. \square

Exercise 7.8:8. Suppose \mathbf{C} , \mathbf{D} and \mathbf{E} are categories such that \mathbf{C} has colimits of all functors $\mathbf{D} \rightarrow \mathbf{C}$, and also of all functors $\mathbf{E} \rightarrow \mathbf{C}$, so that $\varinjlim_{\mathbf{D}}$ becomes a functor $\mathbf{C}^{\mathbf{D}} \rightarrow \mathbf{C}$ and $\varinjlim_{\mathbf{E}}$ a functor $\mathbf{C}^{\mathbf{E}} \rightarrow \mathbf{C}$. Show that for any bifunctor $B: \mathbf{D} \times \mathbf{E} \rightarrow \mathbf{C}$, the above definition yields comparison morphisms $\varinjlim_{\mathbf{D}} (\varinjlim_{\mathbf{E}} B(D, E)) \rightarrow \varinjlim_{\mathbf{E}} (\varinjlim_{\mathbf{D}} B(D, E))$ and also $\varinjlim_{\mathbf{E}} (\varinjlim_{\mathbf{D}} B(D, E)) \rightarrow \varinjlim_{\mathbf{D}} (\varinjlim_{\mathbf{E}} B(D, E))$, and that these are inverse to one another. This gives yet another proof of the isomorphism between the two sides of (7.8.9) under these hypotheses.

Earlier in this section, I said that there was no obvious way to talk about limits or colimits “respecting” the construction of objects representing functors, and we looked instead at the subject of representable functors respecting limits and colimits. But there are actually some not-so-obvious results one can get on limits and colimits of objects that represent functors. Conveniently, these reduce to statements that certain functors respect limits and colimits. You can develop these in

Exercise 7.8:9. (i) Show that the covariant Yoneda embedding $\mathbf{C} \rightarrow \mathbf{Set}^{\mathbf{C}^{\text{op}}}$ respects small limits, and that the contravariant Yoneda embedding $\mathbf{C}^{\text{op}} \rightarrow \mathbf{Set}^{\mathbf{C}}$ turns small colimits into limits. (Idea: combine Lemma 7.6.8 and Theorem 7.8.6.)

(ii) Turn the above results into statements on the representability of set-valued functors which are limits or colimits of other representable functors, and characterizations of the objects that represent these.

(iii) Deduce the characterization, noted near the beginning of §3.6, of pairwise coproducts of groups defined by presentations, and the assertion of Exercise 7.5:7, that every group is a direct limit of finitely presented groups.

- (iv) Show by example that the covariant Yoneda embedding of a category need not respect small colimits, and that the contravariant Yoneda embedding need not turn small colimits into limits.
- (v) Suppose \mathbf{C} , \mathbf{D} , \mathbf{E} are categories, with \mathbf{E} small, and $U: \mathbf{E} \rightarrow \mathbf{C}^{\mathbf{D}}$ a functor such that each of the functors $U(E): \mathbf{D} \rightarrow \mathbf{C}$ has a left adjoint $F(E)$. Under appropriate assumptions on existence of small limits and/or colimits in one or more of these categories, deduce from preceding parts of this exercise that $\varprojlim_{\mathbf{E}} U(E)$ exists (as an object of $\mathbf{C}^{\mathbf{D}}$), and (as a functor $\mathbf{D} \rightarrow \mathbf{C}$) has a left adjoint, constructible from the $F(E)$.
- (vi) Show by example that the analogous statement about *colimits* of functors which have left adjoints is false.

7.9. Interaction between limits and colimits. Since limits are right universal constructions and colimits are left universal, these two sorts of constructions cannot be expected to respect one another in general. However, there are important special cases where they do. We observed in §7.5 (and will prove formally in the next chapter) that one can form the direct limit of any directed system of algebras with finitary operations by taking the direct limit of their underlying sets, and putting operations on this set in a natural manner. The essential reason for this is that algebra structures are given by operations $|A| \times \dots \times |A| \rightarrow |A|$ on sets, and that in **Set**, direct limits commute with finite products – although generally colimits do not.

When we ask whether a given limit and a given colimit commute, there are potentially two comparison morphisms to consider, one a case of the comparison morphism for a limit and a general functor, the other of the comparison morphism for a colimit and a general functor. A priori, one of these might be an isomorphism and the other not, or they might give different isomorphisms between the same objects. Fortunately, these anomalies cannot occur; as we shall now prove, the two comparison morphisms coincide. (Note that these morphisms go in the same direction, because the comparison morphism for limits goes into the limit object, while the comparison morphism for colimits comes out of the colimit object. Contrast the interaction between limits and limits or between colimits and colimits, where the two comparison morphisms go in opposite directions, and, as shown in Exercise 7.8:8, are inverse to one another.)

Lemma 7.9.1. *Suppose \mathbf{C} , \mathbf{D} and \mathbf{E} are categories such that \mathbf{C} has colimits of all functors with domain \mathbf{D} and has limits of all functors with domain \mathbf{E} , and let $B: \mathbf{D} \times \mathbf{E} \rightarrow \mathbf{C}$ be a bifunctor. Then the two comparison morphisms*

$$\varinjlim_{\mathbf{D}} \varprojlim_{\mathbf{E}} B(D, E) \rightarrow \varprojlim_{\mathbf{E}} \varinjlim_{\mathbf{D}} B(D, E)$$

coincide, their common value being characterizable as the unique morphism c_B such that for every $D_0 \in \text{Ob}(\mathbf{D})$ and $E_0 \in \text{Ob}(\mathbf{E})$, the following diagram commutes:

$$(7.9.2) \quad \begin{array}{ccccc} \varprojlim_{\mathbf{E}} B(D_0, E) & \xrightarrow{p(D_0, E_0)} & B(D_0, E_0) & \xrightarrow{q(D_0, E_0)} & \varinjlim_{\mathbf{D}} B(D, E_0) \\ & & \downarrow q(D_0) & & \uparrow p(E_0) \\ & & \varinjlim_{\mathbf{D}} \varprojlim_{\mathbf{E}} B(D, E) & \xrightarrow{c_B} & \varprojlim_{\mathbf{E}} \varinjlim_{\mathbf{D}} B(D, E) \end{array}$$

Here $p(D_0, E_0)$ and $p(E_0)$ denote the E_0 th projection maps out of the respective limits

$\varprojlim_{\mathbf{E}} B(D_0, E)$ and $\varprojlim_{\mathbf{E}} \varinjlim_{\mathbf{D}} B(D, E)$, and $q(D_0, E_0)$ and $q(D_0)$, the D_0 th coprojection morphisms into the colimits $\varinjlim_{\mathbf{D}} B(D, E_0)$ and $\varinjlim_{\mathbf{D}} \varprojlim_{\mathbf{E}} B(D, E)$.

Proof. Let c_B denote the comparison map between the objects at the bottom of (7.9.2) which tests whether $\varprojlim_{\mathbf{E}} : \mathbf{C}^{\mathbf{E}} \rightarrow \mathbf{C}$, regarded as a functor (and not specifically as a limit), respects the indicated colimit over \mathbf{D} . We shall verify that this is the unique morphism making that family of diagrams commute. The dual argument shows the same for the other comparison map, proving the lemma.

The defining property of the colimit-comparison morphism c_B is that it respect the cones from the family of objects $\varprojlim_{\mathbf{E}} B(D_0, E)$ ($D_0 \in \text{Ob}(\mathbf{D})$) to the two objects in the bottom line of (7.9.2), where the cone to the left-hand object is the universal one for the colimit over \mathbf{D} , and consists of the left-hand vertical arrows of the diagram, while the cone to the right-hand object consists of morphisms going diagonally across the diagram, the map for each D_0 being obtained by applying $\varprojlim_{\mathbf{E}}(-, E)$ to the family of coprojection maps $(q(D_0, E))_{E \in \text{Ob}(\mathbf{E})}$ (see top right-hand arrow in (7.9.2) above). Now when we apply $\varprojlim_{\mathbf{E}}(-, E)$ to such a family, the resulting morphism is characterized by the condition that for each E_0 , it form a commuting square with the projection maps to the objects indexed by E_0 (cf. (7.6.5)). In our case, the sides of this commuting square are the p 's in (7.9.2); thus the condition is that for all E_0 , our diagonal map should commute with the top and right-hand arrows of (7.9.2). Hence c_B makes (7.9.2) commute for all D_0 and E_0 , and we see from the universal properties involved that it will be the unique morphism with this property. \square

Before proving that in certain cases the above comparison morphism is an isomorphism, let us note some easy examples where it is not.

Exercise 7.9:1. Let \mathbf{D} and \mathbf{E} each be the category with object-set $\{0, 1\}$, and no morphisms other than identity morphisms.

- (i) Suppose L is a lattice, $U(L)$ its underlying partially ordered set, and $\mathbf{C} = U(L)_{\text{cat}}$. For these choices of \mathbf{C} , \mathbf{D} and \mathbf{E} , say what it means to give a bifunctor B as in Lemma 7.9.1, verify that the indicated limits and colimits exist, and identify the morphism c_B of the lemma. Show that even if \mathbf{C} is the 2-element lattice, this morphism can fail to be an isomorphism.
- (ii) Analyze similarly the case where $\mathbf{C} = \mathbf{Set}$, and \mathbf{D} and \mathbf{E} are as above.

Here, however, is a positive result, generalizing the claim in the first paragraph of this section about direct limits of finite products in \mathbf{Set} . The proof will involve chasing elements in objects of \mathbf{Set} , and we shall see subsequently that the corresponding statement with \mathbf{Set} replaced by a general category \mathbf{C} is false. In thinking about what the result says, you might begin with the cases where $\mathbf{D} = \omega_{\text{cat}}$ (ω the partially ordered set of natural numbers), and \mathbf{E} is the two- or three-object category such that limits over \mathbf{E} are equalizers, respectively pullbacks, or the one-object category G_{cat} for G a finitely generated group, and think through the verification of the assertion of the proposition in this case before reading the general arguments given.

Let us note a convenient piece of notation that will be used in the proof. If $B: \mathbf{D} \times \mathbf{E} \rightarrow \mathbf{C}$ is a bifunctor, D an object of \mathbf{D} , and $f: E_1 \rightarrow E_2$ a morphism of \mathbf{E} , then one often writes $B(D, f)$ for the induced morphism $B(D, E_1) \rightarrow B(D, E_2)$, which is, strictly, $B(\text{id}_D, f)$. Similarly, given a morphism g of \mathbf{D} and an object E of \mathbf{E} , one may write $B(g, E)$ for $B(g, \text{id}_E)$.

Proposition 7.9.3. *If \mathbf{D} is a category of the form $P_{\mathbf{cat}}$, for P a directed partially ordered set, and \mathbf{E} is a nonempty category which has only finitely many objects, and whose morphism-set is finitely generated under composition, then for any bifunctor $B: \mathbf{D} \times \mathbf{E} \rightarrow \mathbf{Set}$, the morphism c_B of Lemma 7.9.1 is an isomorphism. (Briefly: “In \mathbf{Set} , direct limits commute with finite limits.”)*

Proof. Let E_0, \dots, E_{m-1} be the objects of \mathbf{E} , and f_0, \dots, f_{n-1} a generating set for the morphisms of \mathbf{E} , with $f_j \in \mathbf{E}(E_{u(j)}, E_{v(j)})$. Given elements $D \leq D'$ in the partially ordered set P , let us write $g_{D, D'}$ for the unique morphism $D \rightarrow D'$ in $P_{\mathbf{cat}} = \mathbf{D}$. Projection and coprojection morphisms associated to limits and colimits of our system will be named as in (7.9.2).

To show surjectivity of c_B , let x be any element of $\varprojlim_{\mathbf{E}} \varinjlim_{\mathbf{D}} B(D, E)$. For each of the finitely many objects E_i of \mathbf{E} , consider $p(E_i)(x) \in \varinjlim_{\mathbf{D}} B(D, E_i)$. By the construction of direct limits in \mathbf{Set} (second paragraph of Lemma 7.5.3; cf. discussion preceding Proposition 7.6.6) there must exist for each i a $D(i) \in P = \text{Ob}(\mathbf{D})$, such that this element arises from some $x_i \in B(D(i), E_i)$, i.e.,

$$p(E_i)(x) = q(D(i), E_i)(x_i) \quad (i = 0, \dots, m-1).$$

Since the partially ordered set P is directed, we can find $D_0 \in P$ majorizing all the $D(i)$. Thus we have images of all the x_i at the “ D_0 level”; let us denote these

$$x'_i = B(g_{D(i), D_0}, E_i)(x_i) \in B(D_0, E_i) \quad (i = 0, \dots, m-1).$$

Thus,

$$(7.9.4) \quad p(E_i)(x) = q(D_0, E_i)(x'_i) \quad (i = 0, \dots, m-1).$$

Now the definition of $\varprojlim_{\mathbf{E}} \varinjlim_{\mathbf{D}} B(D, E)$ as a limit tells us that the system of elements on the left-hand side of (7.9.4) is “respected” by all morphisms of \mathbf{E} , equivalently, by the generating family of morphisms f_j . That is,

$$(7.9.5) \quad \begin{aligned} \varinjlim_{\mathbf{D}} B(D, f_j): \varinjlim_{\mathbf{D}} B(D, E_{u(j)}) &\rightarrow \varinjlim_{\mathbf{D}} B(D, E_{v(j)}) \\ \text{carries } p(E_{u(j)})(x) &\text{ to } p(E_{v(j)})(x) \quad (j = 0, \dots, n-1). \end{aligned}$$

It is not necessarily true that the system of preimages $x'_i \in B(D_0, E_i)$ that we have found for these elements satisfy the corresponding relations, i.e., that $B(D_0, f_j)$ carries $x'_{u(j)}$ to $x'_{v(j)}$; but by the construction of direct limits in \mathbf{Set} referred to earlier, applied to the direct limit objects of (7.9.5), we see that for each j , there is some $D'(j) \geq D_0$ such that the corresponding relation holds, namely

$$B(D'(j), f_j)(B(g_{D_0, D'(j)}, E_{u(j)})(x'_{u(j)})) = B(g_{D_0, D'(j)}, E_{v(j)})(x'_{v(j)}) \quad (j = 0, \dots, n-1).$$

Hence taking D_1 majorizing all the $D'(j)$'s, and letting

$$x''_i = B(g_{D_0, D_1}, E_i)(x'_i) \in B(D_1, E_i) \quad (i = 0, \dots, m-1)$$

we have the desired “lifting” of the system of equations (7.9.5):

$$B(D_1, f_j)(x''_{u(j)}) = x''_{v(j)} \quad (j = 0, \dots, n-1).$$

That is, the f 's respect the x''_i . Hence, since every morphism of \mathbf{E} is a composite of the f_j , every morphism of \mathbf{E} respects the x''_i ; so the x''_i define an element $x'' \in \varprojlim_{\mathbf{E}} B(D_1, E)$. The element $q(D_1)(x'') \in \varinjlim_{\mathbf{D}} \varprojlim_{\mathbf{E}} B(D, E)$ is the required inverse image of x under c_B . (Cf. (7.9.2).)

The proof that c_B is one-to-one is similar, but easier; indeed, it does not need any hypothesis on the morphisms of \mathbf{E} , but only the finiteness of the object-set. Suppose $x, y \in \varinjlim_{\mathbf{D}} \varprojlim_{\mathbf{E}} B(D, E)$ with $c_B(x) = c_B(y)$. Since \mathbf{D} is directed, there will exist $D_0 \in \text{Ob}(\mathbf{D})$ such that we can write x and y as the images of some $x_0, y_0 \in \varprojlim_{\mathbf{E}} B(D_0, E)$. By assumption, these elements fall together when mapped into $\varprojlim_{\mathbf{E}} \varinjlim_{\mathbf{D}} B(D, E)$, which means that for each i , the projections $p(D_0, E_i)(x_0)$ and $p(D_0, E_i)(y_0)$ fall together in $\varinjlim_{\mathbf{D}} B(D, E_i)$. By the construction of direct limits in \mathbf{Set} , this means that for each i there is some $D(i) \geq D_0$ such that the images of these elements already agree in $B(D(i), E_i)$. Let $D_1 \in \text{Ob}(\mathbf{D})$ majorize all these $D(i)$. Thus the images of x_0 and y_0 fall together in all the $B(D_1, E_i)$, hence in $\varprojlim_{\mathbf{E}} B(D_1, E)$. Hence in $\varinjlim_{\mathbf{D}} \varprojlim_{\mathbf{E}} B(D, E)$, $x = y$. \square

Exercise 7.9:2. Show that the above proposition remains true if the condition that \mathbf{E} be nonempty is replaced by the condition that \mathbf{D} be nonempty, but fails when both are empty. The proof we gave for the proposition does not explicitly refer to the nonemptiness of \mathbf{E} ; where is it used implicitly? (Note that a statement that something is true “for all E_i ” does not require that the set of E_i be nonempty – it is vacuously true if the set is empty. So you need to find something less obvious than that.)

In the above proposition, neither the assumption that \mathbf{E} has finite object-set nor the assumption that its morphism-set is finitely generated can be dropped, as the next exercise shows.

Exercise 7.9:3. (i) Show that direct limits in \mathbf{Set} do not commute with infinite products. In fact, give examples both of failure of one-one-ness and of failure of surjectivity.

Now, a product over a set X is a limit over the category $X_{\mathbf{cat}}$ having object-set X and only identity morphisms; thus, the morphism-set of that category may be regarded as generated by the empty set. Hence in the examples you constructed for (i), \mathbf{E} has infinite object-set, but finitely generated morphism-set.

(ii) To show that finite generation of the morphism-set cannot be dropped either, let $\mathbf{E} = G_{\mathbf{cat}}$ for G a non-finitely-generated group, and let P be the partially ordered set of all finitely generated subgroups $H \subseteq G$. Take the direct limit over P of the G -sets G/H , examine the action of $\varprojlim_{\mathbf{E}}$ on this direct limit, and show that this gives the desired counterexample.

The above examples show the need for our hypotheses on \mathbf{E} . What about the condition that \mathbf{D} have the form $P_{\mathbf{cat}}$ for P a directed partially ordered set? A simple example of a partially ordered set that is not directed is \sphericalangle , while some examples of categories not of the form $P_{\mathbf{cat}}$ for any partially ordered set are the two-object category $\cdot \rightrightarrows \cdot$, and the one-object category $\mathbb{Z}_{\mathbf{cat}}$ where \mathbb{Z} is the infinite cyclic group. So

Exercise 7.9:4. Give examples showing that the fixed-point-set construction on \mathbb{Z} -sets (a limit over a one-object category with finitely generated morphism set) respects neither pushouts, nor coequalizers, nor orbit-sets of actions of \mathbb{Z} .

Moreover, part (i) of the next exercise shows that we cannot interchange the hypotheses on \mathbf{D} and \mathbf{E} . As one can see from part (iii), this is equivalent to saying that Proposition 7.9.3 does not remain true if we replace the category \mathbf{Set} by \mathbf{Set}^{op} ; in particular, we cannot replace \mathbf{Set} in that proposition by a general category having small limits and colimits.

Exercise 7.9:5. (i) Show that inverse limits in \mathbf{Set} do *not* commute with coequalizers.

(ii) Show, on the other hand, that inverse limits in \mathbf{Set} *do* commute with small coproducts.

(iii) Translate the results of (i) and (ii) into statements about constructions in \mathbf{Set}^{op} .

Though we saw in Exercise 7.9:3(ii) that the finite generation hypothesis of Proposition 7.9.3

cannot be dropped, the next exercise shows that it can sometimes be weakened.

Exercise 7.9:6. If S is a monoid, then a *left congruence* on S means an equivalence relation \sim on $|S|$ such that $y \sim z \Rightarrow xy \sim xz$. It is easy to verify that an equivalence relation \sim is a left congruence if and only if the natural structure of left S -set on $|S|$ induces a structure of left S -set on $|S|/\sim$. Given any subset $R \subseteq |S| \times |S|$, there is a least left congruence on $|S|$ containing R , the left congruence “generated by” R . The set $|S| \times |S|$ itself may be called the improper left congruence on S .

(i) Show that the following conditions on a monoid S are equivalent: (a) The improper left congruence on S is finitely generated. (b) The 1-element S -set is finitely presented. (c) The fixed-point-set functor $S\text{-Set} \rightarrow \mathbf{Set}$ respects direct limits.

(ii) Show that the monoids satisfying the equivalent conditions of (i) include all finitely generated monoids, and all monoids having a right zero element (an element z such that $xz = z$ for all x).

(iii) Find a monoid S which satisfies the equivalent conditions of (i), but such that S^{op} does not.

(iv) Deduce from (ii) or (iii), or preferably from each of them, that the class of categories \mathbf{E} with finitely many objects such that limits over \mathbf{E} respect direct limits of sets is strictly larger than the class of such categories with finitely generated morphism sets.

(v) Can you generalize the result of (i) to get a necessary and sufficient condition on a category \mathbf{E} (perhaps under the assumption that it has only finitely many objects, or some weaker condition) for colimits over \mathbf{E} to respect direct limits of sets?

We noted in the last paragraph of the proof of Proposition 7.9.3 that the one-one-ness part of the conclusion did not require finite generation of the morphism set of \mathbf{E} . It also does not require the non-emptiness assumption on the object-set; moreover, even the assumption that the object-set be finite can be weakened, using the idea of Lemma 7.6.2, to say that it contains a “good” finite subset. Thus, you can easily verify

Corollary 7.9.6 (to proofs of Proposition 7.9.3 and Lemma 7.6.2). *Let \mathbf{D} be a category of the form P_{cat} , for P a directed partially ordered set, and let \mathbf{E} be a category with only finitely many objects, or more generally, having a finite family of objects E_0, \dots, E_{m-1} such that every object E admits a morphism $E_i \rightarrow E$ for some i . Then for any bifunctor $B: \mathbf{D} \times \mathbf{E} \rightarrow \mathbf{Set}$, the comparison morphism c_B of Lemma 7.9.1 is one-to-one. \square*

Let us note next that the role of finiteness in all the above considerations is easily generalized. The reader will find that under the next definition, the proofs of our proposition and corollary yield the proposition stated below.

Definition 7.9.7. *If α is a cardinal and P a partially ordered set, then P will be called $< \alpha$ -directed if every subset of P of cardinality $< \alpha$ has an upper bound in P .*

Proposition 7.9.8. *Let α be an infinite cardinal. If \mathbf{D} is a category of the form P_{cat} , for P a $< \alpha$ -directed partially ordered set, and \mathbf{E} is a nonempty category which has $< \alpha$ objects, and whose morphism-set is generated under composition by a set of $< \alpha$ morphisms (which, except in the case $\alpha = \omega$, is equivalent to saying that \mathbf{E} has $< \alpha$ morphisms), then for any bifunctor $B: \mathbf{D} \times \mathbf{E} \rightarrow \mathbf{Set}$, the morphism c_B of Lemma 7.9.1 is an isomorphism. (Briefly: “In \mathbf{Set} , $< \alpha$ -directed direct limits commute with limits over $< \alpha$ -generated categories.”)*

Further, the one-one-ness of c_B continues to hold if we weaken the hypotheses on \mathbf{E} to say

merely that there is a set S of $< \alpha$ objects of \mathbf{E} such that every object of \mathbf{E} admits a morphism from a member of S . \square

Here are a few more exercises on commuting limits and colimits, some of them open-ended.

Exercise 7.9:7. Generalizing part (ii) of Exercise 7.9:5, determine the class of all small categories \mathbf{E} such that limits over \mathbf{E} in \mathbf{Set} commute with coproducts.

Exercise 7.9:8. Let G be a group or monoid, let $(X_i)_{i \in P}$ be an inverse system of G -sets, and let $c_X: \varinjlim_{G\text{cat}} \varprojlim_{i \in P} X_i \rightarrow \varprojlim_{i \in P} \varinjlim_{G\text{cat}} X_i$ be the associated comparison morphism. (In the case where G is a group, recall that $\varinjlim_{G\text{cat}}$ is the orbit-set construction of Exercise 7.6:1.)

(i) Show that if G is a group and P is countable, then c_X is surjective. (Hint: Use Exercise 7.5:6(ii).)

(ii) Does the result of (i) remain true for G a monoid? For P not necessarily countable? If either of these generalizations fails, can you find any additional conditions under which it again becomes true?

(iii) What can you say (positive or negative) about conditions under which c_X will be one-to-one?

Exercise 7.9:9. (i) In the spirit of Exercise 7.8:7, investigate the Galois connection between the set of small categories \mathbf{D} and the set of small categories \mathbf{E} determined by the relation “colimits over \mathbf{D} commute with limits over \mathbf{E} in \mathbf{Set} ”.

(ii) Investigate the Galois connections (still on the class of all small categories) obtained by replacing “ \mathbf{Set} ” in (i) with one or more other natural categories; e.g., \mathbf{Ab} .

We have been considering the interaction between limits and colimits. One can also look at the interaction between *limits* and *left adjoint* functors, and between right adjoint functors and colimits. For example

Exercise 7.9:10. Does the abelianization functor $(\)^{\text{ab}}: \mathbf{Group} \rightarrow \mathbf{Ab}$ respect inverse limits? Products? Equalizers? In each case where the answer is negative, is it one-one-ness, surjectivity, or both properties of the comparison morphism that can fail? (Cf. Exercise 3.7:1.)

A different sort of “comparison morphism” is considered in

Exercise 7.9:11. Given functors $\mathbf{D} \xrightarrow{F} \mathbf{E} \xrightarrow{S} \mathbf{C}$ such that S and SF both have colimits in \mathbf{C} , describe a natural morphism (in one direction or the other) between these objects, and examine conditions under which these morphisms will or will not be invertible. (Cf. Exercise 7.5:1.) Also state the corresponding results for limits.

7.10. Some existence theorems. Having defined several sorts of universal objects, and established facts about them, it would be nice to have some general results on which such objects exist.

Basic results on the existence of *algebras* with universal properties must wait for the next chapter, where we will set up a general theory of algebras. What we can prove before then are relative results, to the effect that if in a category one can perform certain constructions, then one can perform others; for instance, Proposition 7.6.6 was of this sort. With this limitation in mind, can we abstract any of the methods by which we proved the existence of *free groups* in Chapter 2?

The construction by *terms* modulo consequences of the *identities* clearly depends on the fact that one is considering algebras. Generalizing this will be one of the first things we do in Chapter 8.

The *normal form* description is still more specialized. As mentioned toward the end of §2.4,

different sorts of algebras vary widely as to whether such results hold. I hope to develop some methods for obtaining normal forms in a later (as yet unwritten) chapter; we are not ready to do anything along that line here.

But the *subobject of a big direct product* approach of §2.3 seems amenable to a category-theoretic development, and we shall in fact obtain below several results that have evolved from that construction. The approach is due to Peter Freyd.

We know how to translate the concept of direct product into category theoretic terms. There were two other key ideas in the construction of §2.3: a cardinality estimate, which allowed us to find a *small* set of groups to take the direct product of, and the passage to “the subgroup of the product generated by the given family”. The first of these will simply be made a hypothesis – that there exists a small set of objects with an appropriate property. What about the concept of “subalgebra generated”? We know that there is not a canonical concept of “subobject” in category theory, but is there one that is appropriate to this proof?

We saw at various points in Chapters 2 and 3 that if we had an object satisfying one of our left universal properties, except possibly for the *uniqueness* of the factoring maps, then the added condition of uniqueness was equivalent to the object being generated by the appropriate set (e.g., Exercise 2.1:2, and end of proof of Proposition 3.3.3). To put things negatively, in the case of the universal property of a free group on X , we saw in Exercise 2.1:1 that if our candidate F for a free group was not generated by the image of X , then we could get a pair of group homomorphisms from F into some group which agreed on the elements of X , but were *not* equal on all of F . This suggests that the subgroup generated by X may be obtainable as an *equalizer*, using pairs of morphisms having equal composites with the image of X . That is the idea which we shall abstract below.

Repeating the approach of the first half of this chapter, let us start with an existence result for initial objects. In reading the next lemma and its proof, you might think of the case where \mathbf{C} is the category of 4-tuples (G, a, b, c) with G a group and $a, b, c \in |G|$, and of the principle that guided us to the subgroup-of-a-product construction, that if one such object (G, a, b, c) is mappable to another such object (H, a', b', c') , then the set of relations satisfied by a, b, c in G is contained in the set of relations satisfied by a', b', c' in H .

Lemma 7.10.1. *Let \mathbf{C} be a (legitimate) category having small limits. Suppose there exists a small set of objects $S \subseteq \text{Ob}(\mathbf{C})$, such that for every $X \in \text{Ob}(\mathbf{C})$ there is a $Y \in S$ with $\mathbf{C}(Y, X)$ nonempty. Then \mathbf{C} has an initial object.*

Proof. Let $J = \prod_{Y \in S} Y \in \text{Ob}(\mathbf{C})$. For every $X \in \text{Ob}(\mathbf{C})$ there is at least one morphism from J to X , since we can compose the projection of J to some $Y \in S$ with a morphism $Y \rightarrow X$. Hence our hypothesis on the set of objects S has been concentrated in this one object J , and we may henceforth forget S and work with J .

We wish to form the “intersection of the equalizers of all pairs of maps from J into objects of \mathbf{C} ”. If we were working in a category of algebras, this would make sense, for even though all such pairs of maps do not form a small set, the underlying set of J would be small, and hence the set of subalgebras that are equalizers of such pairs of maps would be small, and we could take its intersection. That argument is not available here; but it turns out that, just as we were able to use the family S as a substitute for the class of all objects in forming our product J , so it can serve as a substitute for the class of all objects in this second capacity, though a less obvious argument will be needed. But since the hypothesis on S has been concentrated in the object J , let us again use J in place of S in this function.

So let us form a product $\prod_{(u,v)} J$ of copies of J indexed by the set of all pairs of morphisms $u, v \in \mathbf{C}(J, J)$. (Since \mathbf{C} is legitimate, such pairs form a small set.) Let $a, b: J \rightrightarrows \prod_{(u,v)} J$ be defined by the conditions that for all $u, v \in \mathbf{C}(J, J)$, a followed by the projection of the product onto the (u, v) component gives u , and b followed by that projection gives v . Let us form the equalizer $i: I \rightarrow J$ of this pair of morphisms. Note that by the universal property of I , $ui = vi$ for any two endomorphisms u, v of J .

Since J can be mapped to every object of \mathbf{C} , we can find a morphism $x: J \rightarrow I$. Now suppose c is any endomorphism of I . By our preceding observation, the morphisms $I \rightarrow J$ given by i, ix_i , and icx_i are equal. But by Lemma 7.6.2, i is a monomorphism; hence we can cancel it on the left and conclude that id_I, x_i , and cx_i are equal. Substituting the equation $x_i = \text{id}_I$ into $cx_i = x_i$, we get $c = \text{id}_I$; i.e., I has no nonidentity endomorphisms.

I also inherits from J the property of having morphisms into every object of \mathbf{C} , so we can now forget J and work with I only.

We claim that I is an initial object of \mathbf{C} . We know it has morphisms into every $X \in \text{Ob}(\mathbf{C})$; consider two such morphisms $u, v \in \mathbf{C}(I, X)$. We may form their equalizer, $k: K \rightarrow I$, and take an arbitrary morphism the other way, $d: I \rightarrow K$. Then kd is an endomorphism of I , hence it is the identity. By choice of k , $uk = vk$, hence $ukd = vkd$, i.e., $u = v$; so I has exactly one morphism into each object of \mathbf{C} , as claimed. \square

Exercise 7.10:1. The final part of the proof of the above lemma used the facts that (a) the object I of \mathbf{C} had morphisms into all objects, (b) I had no nonidentity endomorphism, and (c) \mathbf{C} had equalizers. Do (a) and (b) alone imply that I is initial in \mathbf{C} ?

For some perspective on the above result, recall Exercise 7.6:5, which showed that an initial object of a category \mathbf{C} is equivalent to a *colimit* of the unique functor from the empty category to \mathbf{C} , and also to a *limit* of the *identity functor* of \mathbf{C} . Now in the study of categories of algebraic objects (for instance, the category of groups with 3-tuples of distinguished elements), one does not have, to begin with, any easy way of constructing colimits, even for as trivial a functor as the one from the empty category! One can, however, construct products and equalizers using the corresponding constructions on the underlying sets of one's algebras; hence one can get all small limits. This suggests trying to construct an initial object as a limit of the identity functor of the whole category. The difficulty is, of course, that the domain of that functor is not small. Hence one looks for a small set S of objects of \mathbf{C} which “get around enough” to serve in place of the set of *all* objects.

In fact, if this had been used as our motivation for the above lemma, we would have gotten a proof in which the initial object I was constructed in one step, as the limit of the inclusion functor of the full subcategory with object-set S into \mathbf{C} . But I preferred the present approach because the characterization of limits of identity functors is itself not easy to prove. In [17] you can find both versions of the proof, as Theorem 1 on p. 116 and Theorem 1 on p. 231 respectively.

In results such as the above, the assumption that there exists a small set S which, for the purposes in question, is “as good as” the set of all objects is known as the “solution-set condition”.

On, now, to the next result in this family. Since a representing object for a functor $U: \mathbf{C} \rightarrow \mathbf{Set}$ is equivalent to an initial object in an appropriate auxiliary category \mathbf{C}' , let us see under what conditions we can apply Lemma 7.10.1 to such an auxiliary category to get a representability result. By Theorem 7.8.6, if U is representable it must respect limits, so the

condition of respecting limits must somehow be a precondition for the application of Lemma 7.10.1 in this way. The next result shows that for the auxiliary category \mathbf{C}' to have small limits is equivalent to U respecting such limits.

Lemma 7.10.2. *Let \mathbf{C} be a category, $U: \mathbf{C} \rightarrow \mathbf{Set}$ any functor, and \mathbf{C}' the category whose objects are pairs (X, x) with $X \in \text{Ob}(\mathbf{C})$ and $x \in U(X)$, and whose morphisms are morphisms of first components respecting second components (in the notation of Exercise 6.8:26, the comma category $(1 \downarrow U)$). Let $V: \mathbf{C}' \rightarrow \mathbf{C}$ denote the forgetful functor taking (X, x) to X . Then*

(a) *If \mathbf{D} is a small category and $G: \mathbf{D} \rightarrow \mathbf{C}$ a functor having a limit in \mathbf{C} , the following conditions are equivalent:*

(i) *U respects the limit of G ; i.e., the comparison morphism $c: U(\varprojlim G) \rightarrow \varprojlim UG$ is a bijection of sets.*

(ii) *Every functor $F: \mathbf{D} \rightarrow \mathbf{C}'$ satisfying $VF = G$ has a limit in \mathbf{C}' .*

Hence,

(b) *If \mathbf{C} has small limits, then \mathbf{C}' will have small limits if and only if U respects small limits.*

Sketch of Proof. We shall prove (a), from which (b) will clearly follow.

Note that a functor F that ‘‘lifts G ’’ as in (ii) is essentially a compatible way of choosing for each $X \in \text{Ob}(\mathbf{D})$ an element $x \in UG(X)$; hence it corresponds to an element $y \in \varprojlim UG$. Now assuming (i), such an element y has the form $c(z)$ for a unique $z \in U(\varprojlim G)$, and it is immediate that the pair $(\varprojlim G, z)$ is a limit of F in \mathbf{C}' , giving (ii).

Conversely, assuming (ii), let y be any element of $\varprojlim UG$. As noted, this corresponds to a functor $F: \mathbf{D} \rightarrow \mathbf{C}'$, and by (ii) F has a limit (Z, z) in \mathbf{C}' . The cone from this limit object to F , applied to first components, gives a cone from Z to the objects $G(X)$, under which the second component, z is carried to the components of y ; hence the map $Z \rightarrow \varprojlim G$ induced by this cone carries $z \in U(Z)$ to an element $w \in U(\varprojlim G)$, which is taken by c to $y \in \varprojlim UG$. This establishes the surjectivity of c .

Suppose now that c also takes another element $w' \in U(\varprojlim G)$ to y . By the universal property of (Z, z) , there is a morphism $\varprojlim G \rightarrow Z$ carrying w' to z ; composing this with our morphism $Z \rightarrow \varprojlim G$ we get an endomorphism of $\varprojlim G$ carrying w' to w . But all these morphisms, and hence this endomorphism in particular, respect cones to G in \mathbf{C} , hence by the universal property of $\varprojlim G$, this endomorphism must be the identity morphism of $\varprojlim G$. This shows that $w' = w$, proving one-one-ness of c . \square

Exercise 7.10:2. Give the details of the proof of (i) \Rightarrow (ii) above.

Exercise 7.10:3. In part (a) of the above lemma, we assumed that the functor G had a limit. We may ask whether this assumption is needed in proving (ii) \Rightarrow (i), or whether the existence of the limits assumed in (ii) implies this.

To answer this question, let \mathbf{C} be the category whose objects are pairs (G, S) where G is a group and S a cyclic subgroup of G (a subgroup generated by one element), and where a morphism $(G, S) \rightarrow (H, T)$ means a homomorphism $G \rightarrow H$ which carries the subgroup S onto the subgroup T . Let $U: \mathbf{C} \rightarrow \mathbf{Set}$ be the functor which carries each pair (G, S) to the set of generating elements of S .

Show how to define U on morphisms. Show that \mathbf{C} does not, in general, have products of pairs of objects, but that the category \mathbf{C}' , defined as in the above lemma, has all small limits, hence, in particular, pairwise products. Apply this example to answer the original question.

The reader should verify that Lemmas 7.10.1 and 7.10.2 now give the desired criterion for representability, namely

Proposition 7.10.3. *Let \mathbf{C} be a category with small limits, and $U: \mathbf{C} \rightarrow \mathbf{Set}$ a functor. Then U is representable if and only if*

- (a) U respects small limits, and
- (b) *there exists a small set S of objects of \mathbf{C} such that for every object Y of \mathbf{C} and $y \in U(Y)$, there exist $X \in S$, $x \in U(X)$, and $f \in \mathbf{C}(X, Y)$ such that $y = U(f)(x)$. \square*

Finally, let us get from this a condition for the existence of *adjoints*. (In reading the next result, note that for $\mathbf{D} = \mathbf{Group}$ and U its underlying-set functor, condition (b) below was precisely what we had to come up with in showing the existence of free groups on arbitrary sets Z .)

Theorem 7.10.4 (Freyd’s Adjoint Functor Theorem). *Let \mathbf{C} and \mathbf{D} be categories such that \mathbf{D} has small limits. Then a functor $U: \mathbf{D} \rightarrow \mathbf{C}$ has a left adjoint $F: \mathbf{C} \rightarrow \mathbf{D}$ if and only if*

- (a) U respects small limits, and
- (b) *for every $Z \in \text{Ob}(\mathbf{C})$ there exists a small set $S \subseteq \text{Ob}(\mathbf{D})$ such that for every $Y \in \text{Ob}(\mathbf{D})$ and $y \in \mathbf{C}(Z, U(Y))$, there exist $X \in S$, $x \in \mathbf{C}(Z, U(X))$ and $f \in \mathbf{D}(X, Y)$ such that $y = U(f)x$.*

Proof. The existence of a left adjoint to U is equivalent by Theorem 7.3.7(ii) to the representability, for every $Z \in \text{Ob}(\mathbf{C})$, of the functor $\mathbf{C}(Z, U(-)): \mathbf{D} \rightarrow \mathbf{Set}$. Condition (b) is clearly a translation of condition (b) of the preceding proposition. As for condition (a), we know by Theorem 7.8.3 that it, too, is necessary for the existence of a left adjoint, so it suffices to show that it implies that each set-valued functor $\mathbf{C}(Z, U(-))$ respects limits. If we write this functor as $h_Z U$, and recall that covariant representable functors h_Z respect limits, this is immediate. \square

Exercise 7.10:4. Show the converse of the observation used in the last step of the above proof: If \mathbf{C} and \mathbf{D} are categories, and $U: \mathbf{D} \rightarrow \mathbf{C}$ a functor such that for every $Z \in \text{Ob}(\mathbf{D})$, $h_Z U$ respects limits, then U respects limits.

I remarked in §7.8 that for every example we *had seen* of a functor that was not representable or did not have a left adjoint, the nonrepresentability or nonexistence of an adjoint could be proved by showing that the functor did not respect some limit. We can now understand this better. On a category having small limits, the only way a functor respecting these limits can fail to have a left adjoint or a representing object is if the solution-set condition fails. Since the solution-set condition says “a *small* set is sufficient”, its failure must involve uncircumventable set-theoretic difficulties, which are rare in algebraic contexts. However, knowing now that this is what we should look for, we can find examples. The next exercise gives a simple, if somewhat artificial example. The example in the exercise after that is more complicated, but more relevant to constructions we are interested in.

Exercise 7.10:5. Let \mathbf{D} be the subcategory of \mathbf{Set} whose objects are all sets (or if you prefer, all ordinals; in either case, “small” is understood, since by definition \mathbf{Set} is the category of all small sets), and whose morphisms are the *inclusion maps* of subsets. Show that \mathbf{D} has small colimits (and has limits over all nonempty categories, though this will not be needed), but has no terminal object.

Hence, letting $\mathbf{C} = \mathbf{D}^{\text{op}}$, the category \mathbf{C} has small limits (and colimits over nonempty

categories) but no initial object. Translate the nonexistence of an initial object for \mathbf{C} to the nonrepresentability of a certain functor $U: \mathbf{C} \rightarrow \mathbf{Set}$ which respects limits (cf. Exercise 7.2:8).

The results of this section would imply the existence of an initial object of \mathbf{C} , and of a representing object for U , if a certain solution-set condition held. State this condition, and note why it does not hold.

The next exercise illustrates the comment made in §5.2, that because the class of *complete lattices* is not defined by a small set of operations it fails in some ways to behave like classes of “ordinary” algebras. We shall see that the solution-set condition required for the existence of the free complete lattice on 3 generators fails, and indeed, that there is no such free object.

Exercise 7.10:6. First, a preparatory observation:

(i) Show that every ordinal has a unique decomposition $\alpha = \beta + n$, where β is a limit ordinal (possibly 0) and $n \in \omega$. Let us call α *even* or *odd* respectively according as the summand n in this decomposition is even or odd.

Now let α be an arbitrary ordinal, let $S = \alpha \cup \{x, y\}$ where x, y are two elements that are not ordinals, and let L be the lattice of all subsets $T \subseteq S$ such that (a) if T contains x and all ordinals less than an odd ordinal $\beta \in \alpha$, then it contains β , and (b) if T contains y and all ordinals less than an even ordinal $\beta \in \alpha$, then it contains β .

(ii) Show that the complete sublattice of L generated by the three elements $\{x\}$, $\{0, y\}$ and α (i.e., the closure of this set of three elements under arbitrary meets and joins within L) has cardinality $\geq \text{card}(\alpha)$. (This is an extension of the trick of Exercise 5.3:9.)

(iii) Deduce that there can be no free complete lattice on 3 generators.

(This was first proved in [72], by a different construction. Three proofs of the similar result that there is no free complete Boolean algebra on countably many generators are given in [67], [72] and [113].)

But this does not mean that a class of algebras having a large set of primitive operations *cannot* have free objects on all sets. The next exercise gives an example of one that does.

Exercise 7.10:7. Complete \vee -semilattices with least elements, like complete lattices, have an α -fold join operation for every cardinal α . Nevertheless:

(i) Show that a complete \vee -semilattice with least element generated by an X -tuple of elements has at most $\text{card}(\mathbf{P}(X))$ elements.

(ii) Deduce from Freyd’s Adjoint Functor Theorem that there exist free complete \vee -semilattices with least elements on all sets. (This despite the fact that complete \vee -semilattices with least elements are, as partially ordered sets, the same objects as nonempty complete lattices!)

(iii) Does the category of \vee -complete *lattices* with least element behave, in this respect, like that of complete \vee -semilattices with least element, or like that of complete lattices? I.e., does it have free objects on all sets or not?

In §3.17, where we constructed the Stone-Čech compactification of a topological space, we obtained the solution-set condition using the fact that in a compact Hausdorff space, continuous maps to the unit interval $[0, 1]$ separate points. Freyd [9, Exercise 3-M, p.89] cf. [17, §V.8] abstracts this observation to give a variant of Theorem 7.10.4, called the “special adjoint functor theorem”, in which the solution-set hypothesis is replaced by an assumption that there exists such an object, called a “cogenerator” of the category, together with a smallness assumption on sets of monomorphisms. However, since the existence of cogenerators is not as common in algebra as the direct verifiability of the solution set condition, we will not develop that result here.

You may have noticed that in this section, I have not followed my usual practice of stating every result both for left and for right universal constructions. That practice is, of course, logically

unnecessary, since one result can always be deduced immediately from the other by putting \mathbf{C}^{op} for \mathbf{C} and making appropriate notational translations. In earlier sections I nonetheless gave dual pairs of formulations, because both statements were generally of comparable importance. However, when one studies categories of algebras, objects characterized by right universal properties are usually easier to construct directly than those characterized by left universal properties, so we have little need for results obtaining the former from the latter; hence my one-sided presentation. (It is also true that short-term generalizations about what cases are important may fail in the longer run! However, we can always call on the duals of the results of this section if we should need them.)

Here is a somewhat vague question, to which I don't know an answer.

Exercise 7.10:8. Suppose a functor U has a left adjoint F , which in turn has a left adjoint G . Can one conclude more about U itself than the results that we have shown to follow from the existence of F ? In other words, are there any nice necessary conditions for existence of *double* left adjoints, comparable to the property of respecting limits as a condition for existence of a single left adjoint?

7.11. Morphisms involving adjunctions. I am not planning on using the results of this section in subsequent chapters, so the reader may excuse a little sketchiness. (However, the material in the *next* section *will* be referred to in subsequent chapters, and should be read with your usual vigilance.)

Let \mathbf{C} and \mathbf{D} be categories, and $\mathbf{D} \begin{array}{c} \xleftarrow{U} \\ \xrightarrow{F} \end{array} \mathbf{C}$ adjoint functors. We recall the isomorphism which characterizes their adjointness:

$$(7.11.1) \quad \mathbf{C}(-, U(-)) \cong \mathbf{D}(F(-), -).$$

Suppose now that we have functors from a third category into each of these categories, $P: \mathbf{E} \rightarrow \mathbf{C}$ and $Q: \mathbf{E} \rightarrow \mathbf{D}$. It is not hard to verify that if we formally “substitute P and Q into the blanks” in (7.11.1), we get a bijection between sets of morphisms of functors:

$$\mathbf{C}^{\mathbf{E}}(P, UQ) \longleftrightarrow \mathbf{D}^{\mathbf{E}}(FP, Q).$$

As one would expect, this bijection is functorial in P and Q , i.e., respects morphisms $P \rightarrow P'$, $Q \rightarrow Q'$; in other words, writing F° and U° for the operations of composing on the left with F and U respectively, the above bijection gives an isomorphism of bifunctors $\mathbf{C}^{\mathbf{E}} \times \mathbf{D}^{\mathbf{E}} \rightarrow \mathbf{Set}$:

$$\mathbf{C}^{\mathbf{E}}(-, U^\circ-) \cong \mathbf{D}^{\mathbf{E}}(F^\circ-, -).$$

This means that we have an adjoint pair of functors on functor categories, $\mathbf{D}^{\mathbf{E}} \begin{array}{c} \xleftarrow{U^\circ} \\ \xrightarrow{F^\circ} \end{array} \mathbf{C}^{\mathbf{E}}$. We can also describe this adjunction in terms of its unit and counit; these will be $\eta^\circ: \text{Id}_{(\mathbf{C}^{\mathbf{E}})} \rightarrow (UF)^\circ$ and $\varepsilon^\circ: (FU)^\circ \rightarrow \text{Id}_{(\mathbf{D}^{\mathbf{E}})}$, where η and ε are the unit and counit of the adjunction between U and F . In fact, the quickest way to prove that U° and F° are adjoint is to note that the equations in η and ε which establish the adjointness of U and F (Theorem 7.3.7(iii)) give equations in η° and ε° establishing the adjointness of U° and F° .

The above fits with our comment at the end of §6.9 that a functor category such as $\mathbf{C}^{\mathbf{E}}$ or $\mathbf{D}^{\mathbf{E}}$ behaves very much like its codomain category, \mathbf{C} or \mathbf{D} . What that observation does not prepare us for is that analogous results hold for composition on the *right* with adjoint functors. Given adjoint functors U and F , still as in (7.11.1) above, let us take a category \mathbf{B} and functors $R: \mathbf{D} \rightarrow \mathbf{B}$, $S: \mathbf{C} \rightarrow \mathbf{B}$. I claim we get a bijection

$$\mathbf{B}^{\mathbf{D}}(SU, R) \longleftrightarrow \mathbf{B}^{\mathbf{C}}(S, RF)$$

and thus an isomorphism

$$\mathbf{B}^{\mathbf{D}}(-\circ U, -) \cong \mathbf{B}^{\mathbf{C}}(-, -\circ F),$$

i.e., a pair of adjoint functors, $\mathbf{B}^{\mathbf{D}} \xrightleftharpoons[\circ U]{\circ F} \mathbf{B}^{\mathbf{C}}$, where this time $\circ U$ is the left adjoint and $\circ F$ the right adjoint. I don't know a way of seeing this directly from (7.11.1), but it comes out easily if we check the formal properties of the unit and counit $\circ \eta$ and $\circ \varepsilon$.

Let us cook up a random example. We shall take for U and F the familiar case of the underlying set functor on groups and the free group functor. To avoid overlap with the result we proved earlier about composition of adjoints (Theorem 7.3.9), let us take for R and S functors which are not adjoints on either side: Let $R: \mathbf{Group} \rightarrow \vee\text{-Semilattice}^0$ take a group G to the upper semilattice of *subgroups* of G , and let $S: \mathbf{Set} \rightarrow \vee\text{-Semilattice}^0$ take a set X to the upper semilattice of *equivalence relations* on X . (By $\vee\text{-Semilattice}^0$ we mean the category of upper semilattices with least elements 0 , i.e., with arbitrary finite joins, including the empty join. Given a group homomorphism $h: G \rightarrow H$, $R(h)$ takes each subgroup of G to its image under h , while given a set-map $f: X \rightarrow Y$, $S(f)$ takes each equivalence relation e on X to the equivalence relation on Y generated by $\{(f(x), f(y)) \mid (x, y) \in e\}$.) A morphism from SU to R thus means a way of associating to every equivalence relation on the underlying set of a group a subgroup of that group, in a way that respects joins (including the empty join), and also respects maps induced by group homomorphisms. Though I truly chose the functors without specific examples of such morphisms in mind, there turn out to exist several constructions with these properties: Given an equivalence relation E on the underlying set of a group G , one can form (a) the subgroup of G generated by the elements xy^{-1} for $(x, y) \in E$, (b) the subgroup generated by the elements $y^{-1}x$, as well as the subgroups generated by (c) both types of elements and (d) neither (the trivial subgroup); and these constructions can easily be seen to have the required properties.

On the other hand, a morphism from S to RF means a way of associating to every equivalence relation on a set X a subgroup of the free group $F(X)$, again respecting joins and morphisms. The adjointness result stated above implies that there should be such a morphism $S \rightarrow RF$ corresponding to each of the morphisms $SU \rightarrow R$ just listed; and indeed, these can be described as associating to an equivalence relation E on X the subgroup of $F(X)$ generated by the elements xy^{-1} respectively $y^{-1}x$, respectively both, respectively neither, for $(x, y) \in E$. To get these morphisms formally from the morphisms (a)-(d) above, we look at any equivalence relation $E \in S(X)$, use it and the natural map $X \rightarrow U(F(X))$ to induce an equivalence relation on $U(F(X))$, then apply the chosen morphism $SU \rightarrow R$.

The above example is studied further in

Exercise 7.11:1. Let U, F, S and R be as in the above example. Given any set of nonzero integers, $I \subseteq \mathbb{Z} - \{0\}$, let $m_I: SU \rightarrow R$ associate to each equivalence relation E on the underlying set of a group G the subgroup of G generated by all the elements $x^i y^{-i}$ ($(x, y) \in E, i \in I$).

- (i) Show that the m_I are morphisms of functors, and are all distinct.
- (ii) Try to determine whether these are all the morphisms $SU \rightarrow R$. Are there any morphisms which respect finite joins (including empty joins) but not infinite joins?

Returning to the question of why adjointness is preserved not only by the construction $(-)^{\mathbf{E}}$

but also (with roles of right and left reversed) by the construction $\mathbf{B}^{(-)}$, the explanation seems to be that the definition of adjointness can be expressed as the condition that certain equations hold among given functors and morphisms in the \mathbf{Cat} -enriched structure (§6.11) of \mathbf{Cat} , namely those of Theorem 7.3.7(iii), and that these equations will be preserved by any functor preserving \mathbf{Cat} -enriched structure. And $(-)^{\mathbf{E}}$ and $\mathbf{B}^{(-)}$ both do so, one covariantly and the other contravariantly. (For an analogous but simpler situation, observe that, although conditions on a morphism a in a category such as being an epimorphism or a monomorphism are not preserved by arbitrary functors, the conditions of left, right and two-sided invertibility are preserved, because they come down to the existence of another morphism b satisfying one or both of the equations $ab = \text{id}_X$, $ba = \text{id}_Y$, and these conditions are preserved by functors. The formulation of adjointness in terms of unit and counit morphisms in Theorem 7.3.7 is similarly “robust”.)

To complicate things a bit further, consider next any two functors P and Q (the vertical arrows below), any adjoint pair of functors between their domain categories, and any adjoint pair of functors between their codomain categories:

$$(7.11.2) \quad \begin{array}{ccc} \mathbf{B} & \begin{array}{c} \xrightarrow{U} \\ \xleftarrow{F} \end{array} & \mathbf{C} \\ \downarrow P & & \downarrow Q \\ \mathbf{D} & \begin{array}{c} \xrightarrow{V} \\ \xleftarrow{G} \end{array} & \mathbf{E} \end{array}$$

(No commutativity conditions are assumed in this diagram!) Now we may apply on the one hand our isomorphisms involving composition on the right with adjoint pairs of functors, and on the other hand our isomorphisms involving composition on the left with such pairs, getting four bijections of morphism-sets

$$(7.11.3) \quad \begin{array}{ccc} \mathbf{E}^{\mathbf{B}}(QU, VP) & \longleftrightarrow & \mathbf{D}^{\mathbf{B}}(GQU, P) \\ \updownarrow & & \updownarrow \\ \mathbf{E}^{\mathbf{C}}(Q, VPF) & \longleftrightarrow & \mathbf{D}^{\mathbf{C}}(GQ, PF) \end{array}$$

Because composition with functors on the left commutes with composition with other functors on the right, the above diagram of bijections commutes. This result is statement (iii) of the next proposition; the preceding observations comprise statements (i) and (ii).

Proposition 7.11.4. *Suppose $\mathbf{D} \begin{array}{c} \xrightarrow{U} \\ \xleftarrow{F} \end{array} \mathbf{C}$ are adjoint functors, with F the left adjoint and U the right adjoint, and with unit $\eta: \text{Id}_{\mathbf{C}} \rightarrow UF$ and counit $\varepsilon: FU \rightarrow \text{Id}_{\mathbf{D}}$. Then*

- (i) *For any category \mathbf{E} , the functors $\mathbf{D}^{\mathbf{E}} \begin{array}{c} \xrightarrow{U^\circ} \\ \xleftarrow{F^\circ} \end{array} \mathbf{C}^{\mathbf{E}}$ are adjoint, with F° the left adjoint, U° the right adjoint, unit $\eta^\circ: \text{Id}_{\mathbf{C}^{\mathbf{E}}} \rightarrow UF^\circ$ and counit $\varepsilon^\circ: FU^\circ \rightarrow \text{Id}_{\mathbf{D}^{\mathbf{E}}}$.*

(ii) For any category \mathbf{B} , the functors $\mathbf{B}^{\mathbf{D}} \xrightleftharpoons[\circ U]{\circ F} \mathbf{B}^{\mathbf{C}}$ are adjoint, with $\circ U$ the left adjoint, $\circ F$ the right adjoint, unit $\circ \eta: \text{Id}_{\mathbf{B}^{\mathbf{C}}} \rightarrow \circ U F$ and counit $\circ \varepsilon: \circ F U \rightarrow \text{Id}_{\mathbf{B}^{\mathbf{D}}}$.

(iii) Given two pairs of adjoint functors as in (7.11.2), the square of isomorphisms of bifunctors $\mathbf{E}^{\mathbf{C}} \times \mathbf{D}^{\mathbf{B}} \rightarrow \mathbf{Set}$

$$(7.11.5) \quad \begin{array}{ccc} \mathbf{E}^{\mathbf{B}}(- \circ U, V \circ -) & \cong & \mathbf{D}^{\mathbf{B}}(G \circ - \circ U, -) \\ & \parallel \S & \parallel \S \\ \mathbf{E}^{\mathbf{C}}(-, V \circ - \circ F) & \cong & \mathbf{D}^{\mathbf{C}}(G \circ -, - \circ F) \end{array}$$

commutes. \square

Exercise 7.11:2. Give the details of the proof of parts (i) and/or (ii) of the above proposition.

My reason for setting down the above observations is to help understand a better known result, which we can get from (7.11.3) by taking $\mathbf{B} = \mathbf{D}$, $\mathbf{C} = \mathbf{E}$, and for P , Q the identity functors of these categories.

Corollary 7.11.6. Suppose $\mathbf{D} \xrightleftharpoons[F]{U} \mathbf{C}$ and $\mathbf{D} \xrightleftharpoons[G]{V} \mathbf{C}$ are two pairs of adjoint functors between the same two categories \mathbf{C} and \mathbf{D} (F and G the left adjoints, U and V the right adjoints). Then there is a natural bijection $i: \mathbf{D}^{\mathbf{C}}(G, F) \longleftrightarrow \mathbf{C}^{\mathbf{D}}(U, V)$ (an instance of the diagonal bijection of (7.11.5) above, described explicitly below). In other words, morphisms in one direction between left adjoints correspond to morphisms in the other direction between right adjoints.

Description of the bijection. Given $f \in \mathbf{D}^{\mathbf{C}}(G, F)$, one may apply $U: \mathbf{D} \rightarrow \mathbf{C}$ on the right to get

$$f \circ U \in \mathbf{D}^{\mathbf{D}}(GU, FU).$$

Composing with the counit morphism $\varepsilon_{U, F}: FU \rightarrow \text{Id}_{\mathbf{D}}$ we get $(\varepsilon_{U, F})(f \circ U) \in \mathbf{D}^{\mathbf{D}}(GU, \text{Id}_{\mathbf{D}})$. Finally, using the adjunction between G and V in a manner analogous to our above use of the adjunction between F and U , we turn this into the desired member of $\mathbf{C}^{\mathbf{D}}(U, V)$, namely

$$i(f) = (V \circ \varepsilon_{U, F})(V \circ f \circ U)(\eta_{V, G} \circ U). \quad \square$$

As an example, let U and V both be the underlying set functor $\mathbf{Group} \rightarrow \mathbf{Set}$, so that F and G are both the free group functor $\mathbf{Set} \rightarrow \mathbf{Group}$. Then the above result says that there is a natural bijection between endomorphisms of these adjoint functors. We have already looked at endomorphisms of U ; in the language of Exercise 2.3:6 they are the ‘‘functorial generalized group-theoretic operations in one variable’’, which we found were just the derived group-theoretic operations in one variable, i.e., the operations of exponentiation by arbitrary integers n . (Cf. also Exercises 6.9:5(iii), 7.2:11.)

As for endomorphisms of F , it is not hard to see that such an endomorphism is determined by the endomorphism it induces on the free group on one generator. That endomorphism will send the generator x to x^n for some integer n ; conversely, we easily verify that for each n , an endomorphism of the whole functor F with this behavior on the free group on one generator exists; hence endomorphisms of F also correspond to exponentiation by arbitrary integers n .

In the above example, we cannot see that the direction of the morphisms has been reversed. For a less degenerate case, let $\mathbf{C} = \mathbf{Group}$ and $\mathbf{D} = \mathbf{CommRing}^1$. Let U be the functor taking each commutative ring with 1, R , to the group $GL(n, R)$ of $n \times n$ invertible matrices over R , and V the functor taking R to its group of invertible elements (units). Clearly there is an important morphism $a: U \rightarrow V$, taking every invertible matrix over a ring to its *determinant*. The left adjoint F of U takes every group A to the commutative ring $F(A)$ presented by generators and relations that create a universal image of A in the group of $n \times n$ invertible matrices over $F(A)$, and likewise the left adjoint G of V will take a group A to the commutative ring $G(A)$ with a universal image of A in its group of units. (The latter is easily seen to be the *group ring* of the *abelianization* of A .) If we look at the *determinants* of the matrices over $F(A)$ comprising the universal $n \times n$ matrix representation of A , we see that these give a homomorphism of A into the group of units of $F(A)$, which by the universal property of $G(A)$ is equivalent to a ring homomorphism $G(A) \rightarrow F(A)$. This gives the morphism of functors $G \rightarrow F$ in $(\mathbf{CommRing}^1)^{\mathbf{Group}}$ corresponding to our determinant morphism $U \rightarrow V$ in $\mathbf{Group}^{\mathbf{CommRing}^1}$.

Mac Lane [17, p.98, top] calls a pair of morphisms of functors related under the bijection of Corollary 7.11.6 *conjugate*. Of course, we should have proved more about this phenomenon than we have stated in Corollary 7.11.6; in particular, that the conjugate of the composite of two morphisms between three adjoint pairs of functors $\mathbf{C} \rightleftarrows \mathbf{D}$ is the composite of their conjugates in reversed order, i.e., that conjugation constitutes a contravariant equivalence between the category of all functors $\mathbf{C} \rightarrow \mathbf{D}$ having right adjoints and the category of all functors $\mathbf{D} \rightarrow \mathbf{C}$ which have left adjoints; and likewise that conjugacy behaves properly with respect to composition of adjoint functors. These results can be looked at as follows: Suppose that within the \mathbf{Cat} -based category \mathbf{Cat} , we define the subcategory $\mathbf{RightAdj}$ to have the same objects as \mathbf{Cat} , and the same morphisms-of-morphisms (all morphisms between functors in this subcategory), but let the intermediate-level entities, the functors, be restricted to those which are right adjoints (equivalently, have left adjoints) in \mathbf{Cat} . Suppose we likewise form the subcategory $\mathbf{LeftAdj}$, as above except that the functors are those that are left adjoints in \mathbf{Cat} . Then we get an equivalence of \mathbf{Cat} -based categories $\mathbf{RightAdj} \approx \mathbf{LeftAdj}^{\text{op}}$. (Actually, one needs a notation to show that there is a “double ^{op}” here, applying both to composition of functors and to composition of morphisms of functors!) One might most elegantly consider a third \mathbf{Cat} -category, \mathbf{Adj} , isomorphic to these two and defined to have adjoint pairs of functors for its morphisms, and conjugate pairs of morphisms of functors for its morphisms of morphisms. For more details, see [17, pp.97-102].

We could also have brought into the statement of Corollary 7.11.6 the upper right-hand and lower left-hand corners of (7.11.3). For instance, in the case involving groups and commutative rings discussed above, the reader can easily describe a morphism $\text{Id}_{\mathbf{Group}} \rightarrow VF$, i.e., a functorial way of mapping each group A into the group of units of the commutative ring with a universal $n \times n$ representation of A , again based on the determinant function, and a morphism $GU \rightarrow \text{Id}_{\mathbf{CommRing}^1}$, i.e., a functorial way of mapping the group ring on the abelianization of the group of invertible $n \times n$ matrices over a ring R into R , yet again based on the determinant.

7.12. Contravariant adjunctions. The concept of an adjoint pair of functors is *self-dual*, in the sense that if we write down the definition of adjointness of $\mathbf{D} \begin{matrix} \xrightarrow{U} \\ \xleftarrow{F} \end{matrix} \mathbf{C}$, put \mathbf{C}^{op} and \mathbf{D}^{op} in place of \mathbf{C} and \mathbf{D} , and translate the resulting structure into language natural for our new \mathbf{C} and \mathbf{D} , the result has the same form as the original definition, though with the roles of \mathbf{C} and \mathbf{D} interchanged, and hence likewise U and F , and η and ε .

But since the concept of adjunction involves more than one category, it also has “partial dualizations”. For instance, if in the definition of adjunction we only replace \mathbf{C} by \mathbf{C}^{op} , we get a condition on a pair of functors $\mathbf{C}^{\text{op}} \rightleftarrows \mathbf{D}$. Note that the one going to the right is a contravariant functor from \mathbf{C} to \mathbf{D} , and the other is *equivalent to* a contravariant functor from \mathbf{D} to \mathbf{C} , i.e., a functor $\mathbf{D}^{\text{op}} \rightarrow \mathbf{C}$. Writing it in the latter form, we arrive at a setup which is symmetric, that is, in which the two categories and the two functors play equivalent roles – but which is *not* self-dual. We describe this construction and its dual in the definition below.

When we defined ordinary adjunctions, we wrote the isomorphism of bifunctors “ $\mathbf{C}(-, U(-)) \cong \mathbf{D}(F(-), -)$ ”, with the tacit understanding that the first argument “ $-$ ” on the left matched the first argument on the right, and similarly for second arguments. But below, the first argument on one side of our isomorphism will represent the same variable as the second argument on the other side. To make this clear, I will use distinct place-holders, “ $-$ ” and “ \sim ”, for the two arguments.

Definition 7.12.1. *Let $U: \mathbf{C}^{\text{op}} \rightarrow \mathbf{D}$ and $V: \mathbf{D}^{\text{op}} \rightarrow \mathbf{C}$ be contravariant functors between categories \mathbf{C} and \mathbf{D} .*

Then a contravariant right adjunction between U and V means an isomorphism

$$\mathbf{C}(-, V(\sim)) \cong \mathbf{D}(\sim, U(-))$$

of bifunctors $\mathbf{C}^{\text{op}} \times \mathbf{D}^{\text{op}} \rightarrow \mathbf{Set}$, where “ $-$ ” denotes the \mathbf{C} -valued argument and “ \sim ” the \mathbf{D} -valued argument; equivalently, an adjunction between $U: \mathbf{C}^{\text{op}} \rightarrow \mathbf{D}$ and the functor $V^{\text{op}}: \mathbf{D} \rightarrow \mathbf{C}^{\text{op}}$ corresponding to V , with U the right and V^{op} the left adjoint; equivalently, an adjunction between $V: \mathbf{D}^{\text{op}} \rightarrow \mathbf{C}$ and $U^{\text{op}}: \mathbf{C} \rightarrow \mathbf{D}^{\text{op}}$, with V the right and U^{op} the left adjoint.

Likewise, a contravariant left adjunction between U and V means an isomorphism

$$\mathbf{C}(V(\sim), -) \cong \mathbf{D}(U(-), \sim)$$

of bifunctors $\mathbf{C} \times \mathbf{D} \rightarrow \mathbf{Set}$, equivalently, an adjunction between V (left) and U^{op} (right); equivalently, an adjunction between U (left) and V^{op} (right).

Of course, these two new kinds of adjointness also have descriptions corresponding to the other ways of describing adjoint functors noted in Theorem 7.3.7. For instance, given $U: \mathbf{C}^{\text{op}} \rightarrow \mathbf{D}$, to find a contravariant right adjoint to U is equivalent to finding, for each object D of \mathbf{D} , a representing object for the contravariant functor $\mathbf{D}(D, U(-)): \mathbf{C}^{\text{op}} \rightarrow \mathbf{Set}$; in other words, an object R_D of \mathbf{C} with a map $D \rightarrow U(R_D)$, which is universal among objects of \mathbf{C} with such maps.

An example of such an adjunction is the formation of universal objects with respect to the “direction-reversing construction” investigated in Exercise 4.1:5 and generalized in Exercise 5.1:11(ii). For a variant of these, recall that in Exercise 6.6:5 we showed that for every partially ordered set P , the hom-set $h^2(P)$ has a natural structure of lattice with least element 0 and greatest element 1, $A(P)$. Given an arbitrary lattice with least and greatest element, L , one can find a partially ordered set P with a universal homomorphism of L into the lattice $A(P)$. One finds that this is the functor called B in that exercise, and that the universal property in question makes the given functor $A: \mathbf{POSet}^{\text{op}} \rightarrow \mathbf{Lattice}^{0,1}$ and the new functor $B: (\mathbf{Lattice}^{0,1})^{\text{op}} \rightarrow \mathbf{POSet}$ mutually right adjoint. We will look more closely at this and similar examples in §9.12.

Exercise 7.12:1. Show that if P and Q are partially ordered sets, then a contravariant right adjunction between $P_{\mathbf{cat}}$ and $Q_{\mathbf{cat}}$ is equivalent to a *Galois connection* between P and Q , in the generalized sense noted at the end of Exercise 5.5:2.

Contravariant *left* adjunctions rarely come up in algebra. In fact, it is shown in [42] that all such adjunctions among the kind of categories of algebras we will be studying in this course must be very degenerate.

It may seem peculiar that we got *three* phenomena – covariant adjointness, contravariant right adjointness, and contravariant left adjointness – as the orbit of one phenomenon (the first of these) under a group of symmetries (interchanging \mathbf{C} and \mathbf{C}^{op} and interchanging \mathbf{D} and \mathbf{D}^{op}) that seems to have the structure $\mathbb{Z}_2 \times \mathbb{Z}_2$. A closer look at the situation shows the following: The orbit of our original adjointness concept under the natural action of $\mathbb{Z}_2 \times \mathbb{Z}_2$ has four elements. However, in listing “distinct phenomena”, we form the orbit-space of this 4-element set under another action of \mathbb{Z}_2 , the interchange of \mathbf{C} and \mathbf{D} , since phenomena interchanged by this action are the same phenomenon, just differently labeled. This action interchanges the two covariant adjointness situations, but fixes each of the contravariant situations, leading to our set of three phenomena.

There is yet another sort of symmetry we might consider: that given by reversing the direction (and hence order of composition) of the *functors* in our statements. In general, results of category theory are *not* preserved by this symmetry, because \mathbf{Cat} is not equivalent to \mathbf{Cat}^{op} . But concepts and results which are not specific to \mathbf{Cat} , but can be formulated or proved for arbitrary \mathbf{Cat} -based categories, may be dualized in this way. We noted in the preceding section that the concept of adjointness is meaningful in an arbitrary \mathbf{Cat} -based category; hence we can apply this duality to it. It turns out to take each of the three kinds of adjointness to itself, leaving the roles of \mathbf{C} , \mathbf{D} , ε and η unchanged, but interchanging U and F . Indeed, the invariance of adjointness under this symmetry is the reason for the unexpected result, Proposition 7.11.4(ii).

Exercise 7.12:2. Prove the claim made above that \mathbf{Cat} is not equivalent to \mathbf{Cat}^{op} . (You can do this by finding an appropriate statement which holds for \mathbf{Cat} but whose dual does not.)

One might ask why, if \mathbf{Cat} is not equivalent to \mathbf{Cat}^{op} , the concept of \mathbf{Cat} -based category should be invariant under reversing order of composition. Briefly, this is because in applying that reversal to a statement about a \mathbf{Cat} -based category \mathbf{X} , one does not replace \mathbf{Cat} by \mathbf{Cat}^{op} in the definition of the categories occurring in the statement. Rather, one replaces composition maps $\mathbf{X}(\mathbf{D}, \mathbf{E}) \times \mathbf{X}(\mathbf{C}, \mathbf{D}) \rightarrow \mathbf{X}(\mathbf{C}, \mathbf{E})$ by maps in which the order of the product is reversed; in other words, one uses the symmetry of the product bifunctor on \mathbf{Cat} . (Replacing \mathbf{Cat} by \mathbf{Cat}^{op} would instead redefine composition as being given by functors $\mathbf{X}(\mathbf{C}, \mathbf{E}) \rightarrow \mathbf{X}(\mathbf{D}, \mathbf{E}) \amalg \mathbf{X}(\mathbf{C}, \mathbf{D})$.)

This is similar to the fact that though \mathbf{Set} is non-self-dual, the symmetry of its product bifunctor allows us to define a functor $(-)^{\text{op}}: \mathbf{Cat} \rightarrow \mathbf{Cat}$, and use this in ordinary (i.e., \mathbf{Set} -based) category theory to prove the dual of any true result.