



## Chapter 8. Varieties of algebras.

We are at last ready to set up a general theory of algebras!

We recall our convention that a fixed universe  $\mathbb{U}$  is assumed chosen, and that when the contrary is not stated, a “set” (or for emphasis, “small set”) means a set which is a member of  $\mathbb{U}$ , while a “category” means a  $\mathbb{U}$ -legitimate category.

We will begin by formalizing some of the ideas we sketched in §§1.4-1.7. (The reader who was not previously familiar with them might review those sections before beginning this formal development.)

**8.1. The category  $\Omega$ -Alg.** In studying structures consisting of a set  $|A|$  given with some operations, we will want to say that two such structures are of the same *type* if we have indexed their operations in the same way, with corresponding operations having the same arities (cf. §1.4). Hence, below, we shall define a “type” to mean an index set for the operations, with an arity associated to each operation-symbol.

Without loss of generality one could index the operations by a cardinal, and also take the arities to be cardinals; and indeed, one or both of these assumptions is usually made. But allowing more general index sets and arities in our definition involves no complication, so let us do so.

**Definition 8.1.1.** A type will mean a pair  $\Omega = (|\Omega|, \text{ari}_\Omega)$ , where  $|\Omega|$  is a set, and  $\text{ari}_\Omega$  (written  $\text{ari}$  when there is no danger of ambiguity), is a map from  $|\Omega|$  to sets. The elements  $s \in |\Omega|$  are called the operation-symbols of  $\Omega$ , and for each such  $s$ , the set  $\text{ari}(s)$  is called the arity of the operation-symbol  $s$ . (As mentioned in §1.4, a more common notation in the literature for the arity of  $s$  is  $n(s)$ .)

$\Omega$  is called finitary if all of its operation-symbols have finite arity, i.e., if for all  $s \in |\Omega|$ ,  $\text{card}(\text{ari}(s)) < \omega$ .

We will call a type  $\Omega$  conventional if  $|\Omega|$  is a cardinal, and for each  $s \in |\Omega|$ ,  $\text{ari}(s)$  is a cardinal. In this situation,  $\Omega$  may be expressed by giving the arity function as a tuple of cardinals,  $(\text{ari}(0), \text{ari}(1), \dots)$ .

**Definition 8.1.2.** If  $\Omega$  is a type, then an algebra of type  $\Omega$ , or  $\Omega$ -algebra, will mean a pair  $A = (|A|, (s_A)_{s \in |\Omega|})$ , where  $|A|$  is a set, and for each  $s \in |\Omega|$ ,  $s_A$  is an  $\text{ari}(s)$ -ary operation on  $|A|$ , i.e., a map  $|A|^{\text{ari}(s)} \rightarrow |A|$ .

For example, the type  $\Omega$  which indexes the operations of *groups* has three operation-symbols, which we may write  $\mu$ ,  $\iota$ ,  $\varepsilon$ , with  $\text{ari}(\mu) = 2$ ,  $\text{ari}(\iota) = 1$ ,  $\text{ari}(\varepsilon) = 0$ . Every group is an algebra of this type, but not every algebra of this type is a group, since there are algebras of this type not satisfying the associative, inverse and identity laws. If we replaced this by a “conventional type” and followed the usage that represents a type by its arity function, we would say that groups are certain algebras “of type  $(2, 1, 0)$ ”.

If  $R$  is a ring, then right or left  $R$ -modules can be described as certain algebras of type  $\Omega$ , where  $|\Omega| = \{+, -, 0\} \sqcup |R|$ , and all these operation-symbols are unary except  $+$ , which is binary, and  $0$ , which is zeroary. Here the first three operations specify an additive group structure, while the remaining, generally infinite family of operations gives the scalar multiplications by all members of  $R$ . To translate this type into conventional notation, one would

index  $|R|$  by a cardinal  $\alpha$ , and let  $|\Omega|$  be the cardinal  $3 + \alpha$ ; here the convenience of allowing more general sets for  $|\Omega|$  is clear!

For an example in which it is natural to regard some operations as having for their *arities* sets other than cardinals, let  $n$  be a fixed positive integer, and for every commutative ring  $R$ , let  $d$  denote the determinant function taking  $n \times n$  matrices over  $R$  to elements of  $R$ . Suppose one wishes to construct from each commutative ring  $R = (|R|, +, -, 0, \cdot, 1)$  the object  $(|R|, +, -, 0, d)$ , i.e., to study the set of elements of  $R$  as an additive group with an  $n \times n$  “determinant” operation. Now one would conventionally consider  $d$  as  $n^2$ -ary, which would mean writing a typical value as  $d(x_0, \dots, x_{n^2-1})$ . But it is more natural to treat  $d$  as an  $(n \times n)$ -ary operation, and write  $d(x_{00}, x_{01}, \dots, x_{n-1, n-1})$ , i.e., to call the typical argument of  $d$  the  $(i, j)$  argument where  $0 \leq i, j < n$ , rather than the  $m$ th argument where  $0 \leq m < n^2$ .

If there were a significant advantage in restricting ourselves to conventional algebra-types, then we might say, “Let us use conventional types in our formal development. We can always *translate* our results into the form appropriate to a particular area of mathematics when we make our applications.” But I see no advantage in such a restriction. At some points we will indeed find it convenient to restrict attention to cardinal-valued arities, but we will still put no restriction on the set of operation-symbols.

Let us note here the unfortunate ambiguity of the word “algebra” – there is the ring-theoretic concept of “an algebra over a commutative ring”, and the present much broader concept used in General Algebra. It would be desirable if a new word could be coined to replace one of them; but there is a large literature in both fields, so it would be hard to get such a change accepted. Since the literature in ring theory is the more enormous of the two, I suppose it is the General Algebra definition that would have to change.

In situations where there is a danger of misunderstanding, authors generally specify “an algebra over a commutative ring  $k$ ” on the one hand, or “an algebra in the sense of Universal Algebra” on the other. The Russians shorten the latter phrase to “a universal algebra”, which is easier to say, but somewhat inappropriate, since it suggests an object with a universal property. (The term “algebra in the sense of Universal Algebra” should now presumably be changed to “algebra in the sense of General Algebra”, for the reasons mentioned in §0.5.)

Incidentally, what is the original source of the word “algebra”? It goes back to a 9th century Arabic text, *Al-jabr w'al-muqābalaḥ*; this title is composed of two technical terms concerning the solving of equations, whose literal meanings are something like “restoration and comparison”. This title was transliterated, rather than translated, into medieval Latin, so that the book became known as *Algebra*, which eventually became the name of the subject. Not only this work but also its author, abu-Ja‘far Muḥammed ibn-Mūsā, has entered mathematical language: He was known as *Al-Khuwārizmi*, “the person from Khuwarizm”; this name was rendered as *algorism*, and, further distorted in English, has become the word *algorithm*.

Of course, we want to make the set of  $\Omega$ -algebras into a category, so:

**Definition 8.1.3.** A homomorphism between algebras of the same type means a map of underlying sets which respects operations.

Precisely, if  $A$  and  $B$  are algebras of type  $\Omega$ , a homomorphism  $A \rightarrow B$  means a set map  $f: |A| \rightarrow |B|$  such that for all  $s \in |\Omega|$  and  $(x_i)_{i \in \text{ari}(s)} \in |A|^{\text{ari}(s)}$ , one has

$$f(s_A((x_i)_{i \in \text{ari}(s)})) = s_B((f(x_i))_{i \in \text{ari}(s)}).$$

For each type  $\Omega$ , the category of all  $\Omega$ -algebras, with homomorphisms for the morphisms, will be denoted  $\Omega\text{-Alg}$ .

Note that when applying a set map to a tuple of elements, one generally drops one pair of parentheses, e.g., shortens  $f((x_1, x_2, x_3))$  to  $f(x_1, x_2, x_3)$ , or  $u((a_i)_{i \in I})$  to  $u(a_i)$ . So the above equation saying that  $f$  respects  $s$  can be simplified to  $f(s(x_i)) = s(f(x_i))$ . If one abbreviates the  $\text{ari}(s)$ -tuple  $(x_i)$  to  $x$  and uses parenthesis-free notation for functions, one can still further shorten this to  $fsx = sfx$ , or, distinguishing between  $f$ , which acts on elements of  $|A|$ , and the induced map on  $\text{ari}(s)$ -tuples of such elements,  $fsx = sf^{\text{ari}(s)}x$ .

**Definition 8.1.4.** Let  $A$  be an  $\Omega$ -algebra.

Then a subalgebra of  $A$  means an  $\Omega$ -algebra  $B$  such that  $|B| \subseteq |A|$ , and such that the operations of  $B$  are restrictions of the corresponding operations of  $A$ ; equivalently, such that the inclusion map  $|B| \rightarrow |A|$  is a homomorphism  $B \rightarrow A$ . Thus, the subalgebras of  $A$  correspond to the subsets of  $|A|$  closed under the operation of  $A$ . If  $B$  is a subalgebra of  $A$  we will, by a slight abuse of notation, write " $B \subseteq A$ ". We shall consider the set of subalgebras of  $A$  to be partially ordered by inclusion (of underlying sets).

A homomorphic image of  $A$  means an algebra  $B$  given with a homomorphism  $f: A \rightarrow B$  which is surjective on underlying sets.

Another notational problem: If  $A$  is an algebra, and if we have shown that some subset  $S \subseteq |A|$  is closed under the operations of  $A$ , we have no simple notation for "the subalgebra of  $A$  whose underlying set is  $S$ ". We shall give such algebras ad hoc names when we refer to them, though it would be tempting to fall back on the sloppy usage which does not distinguish between an algebra and its underlying set.

**Lemma 8.1.5.** If  $A$  is any  $\Omega$ -algebra, the class of subalgebras of  $A$  is "closed under intersections"; i.e., for every set of subalgebras  $B_i$  of  $A$  ( $i \in I$ ), the intersection of the underlying sets,  $\bigcap_I |B_i|$ , is the underlying set of a subalgebra, which we may loosely call  $\bigcap_I B_i$ . Hence the subalgebras of  $A$  form a complete lattice, with meets given by intersections of underlying sets.

If  $X$  is any subset of  $|A|$ , the intersection of the underlying sets of all subalgebras of  $A$  containing  $X$  will be the underlying set of the least subalgebra containing  $X$ , called the subalgebra generated by  $X$ . We say that  $A$  is generated by a subset  $X \subseteq |A|$  if the subalgebra of  $A$  generated by  $X$  is all of  $A$ .  $\square$

As we observed in Chapter 1, a zeroary operation on a set is equivalent to a choice of a distinguished element of that set. Note that if  $\Omega$  is a type without zeroary operation-symbols, then the empty set can be made an  $\Omega$ -algebra in a unique way. On the other hand, the empty set does not admit any zeroary operations, so if  $\Omega$  has any operation-symbols of arity 0, all  $\Omega$ -algebras are nonempty. The least element of the subalgebra lattice of an algebra  $A$  of any type  $\Omega$  will be the subalgebra generated by the empty set; this can also be described as the subalgebra generated, under the operations of positive arity, by the values of the zeroary operations. So if the type has zeroary operations, this least subalgebra is nonempty, while if it does not, it is empty.

Empty algebras sometimes constitute special cases in algebraic considerations, and many general algebraists avoid this "problem" by requiring in their definitions that an algebra have a nonempty underlying set. But the problem gets back at them: For instance, they can no longer define subalgebra lattices as above, since when an algebra has no zeroary operations, an intersection

of nonempty subalgebras can be empty. Thus they make definitions such as “the subalgebra lattice of an algebra  $A$  consists of all subalgebras of  $A$ , and also the empty set if  $A$  has no zeroary operations.” I feel strongly that it is best *not* to exclude empty algebras, but to allow them when dealing with a type without zeroary operations, and accept the need to occasionally give special arguments for them and to mention them as exceptions to some statements.

Let us note that in the category  $\Omega\text{-Alg}$  we can construct products in the manner to which we have become accustomed: If  $(A_i)_{i \in I}$  is a family of  $\Omega$ -algebras, then the set  $\prod_I |A_i|$  becomes an  $\Omega$ -algebra  $P$  under componentwise operations; that is, for each  $s \in |\Omega|$  and  $\text{ari}(s)$ -tuple of elements of  $\prod_I |A_i|$ , say

$$(a_j)_{j \in \text{ari}(s)} = ((a_{ij})_{i \in I})_{j \in \text{ari}(s)} \in |P|^{\text{ari}(s)} = (\prod_I |A_i|)^{\text{ari}(s)},$$

we define

$$s_P(a_j) = (s_{A_i}((a_{ij})_{j \in \text{ari}(s)}))_{i \in I}.$$

The resulting algebra  $P$  is easily seen to have the universal property of the product  $\prod_I A_i$  in  $\Omega\text{-Alg}$ . Products in  $\Omega\text{-Alg}$  are often called by the traditional term, *direct products*.

Similarly, given a pair of homomorphisms of  $\Omega$ -algebras  $f, g: A \rightarrow B$ , their equalizer as set maps will be the underlying set of a subalgebra of  $A$ , which will constitute an equalizer of  $f$  and  $g$  in  $\Omega\text{-Alg}$ .

Since general *limits* can be constructed from products and equalizers (Proposition 7.6.6), we have

**Proposition 8.1.6.** *Let  $\Omega$  be any type. Then the category  $\Omega\text{-Alg}$  has small limits, which can be constructed by taking the limits of the underlying sets and making them  $\Omega$ -algebras under pointwise operations.*

*Explicitly, if  $\mathbf{D}$  is a small category and  $F: \mathbf{D} \rightarrow \Omega\text{-Alg}$  a functor, then the set*

$$\varprojlim_{\mathbf{D}} |F(D)| = \{(a_D) \in \prod_{D \in \text{Ob}(\mathbf{D})} |F(D)| \mid (\forall f \in \mathbf{D}(D_1, D_2)) \ a_{D_2} = F(f)(a_{D_1})\}$$

*is the underlying set of a subalgebra of  $\prod_{\mathbf{D}} F(D)$ , which constitutes a limit of  $F$  in  $\Omega\text{-Alg}$ .  $\square$*

**Exercise 8.1:1.** Show that if empty algebras are excluded from  $\Omega\text{-Alg}$ , the resulting category can fail to have small limits.

On the other hand, *colimits* and other *left-universal* constructions are not, in general, the same in  $\Omega\text{-Alg}$  as in **Set**. We will construct general colimits in §8.3; but there are two cases that we can obtain now. We first need to note

**Lemma 8.1.7.** *Let  $A$  be an  $\Omega$ -algebra and  $E \subseteq |A| \times |A|$  an equivalence relation on  $|A|$ . Then the following conditions are equivalent:*

- (i) *The set  $|A|/E$  can be made the underlying set of an  $\Omega$ -algebra  $A/E$  in such a way that the canonical map  $|A| \rightarrow |A|/E$  is a homomorphism  $A \rightarrow A/E$ .*
- (ii)  *$E$  is the equivalence relation on  $|A|$  induced by a homomorphism of  $\Omega$ -algebras with domain  $A$ . (I.e., there exists an  $\Omega$ -algebra  $B$  and a homomorphism  $f: A \rightarrow B$  such that  $E = \{(x, y) \in |A| \times |A| \mid f(x) = f(y)\}$ .)*

(iii)  $E$  is the underlying set of a subalgebra of  $A \times A$ .

Further, if  $R$  is any subset of  $|A| \times |A|$ , and  $E$  the intersection of all underlying sets of subalgebras of  $A \times A$  which contain  $R$ , and which form equivalence relations on  $|A|$ , then  $A/E$  will be universal (initial) among algebras  $B$  given with homomorphisms  $f: A \rightarrow B$  such that for all  $(r, s) \in R$ ,  $f(r) = f(s)$ .  $\square$

**Definition 8.1.8.** If  $A$  is an  $\Omega$ -algebra, then an equivalence relation  $E$  on  $|A|$  which is the underlying set of a subalgebra of  $A \times A$  (as in condition (iii) of the above lemma) will be called a congruence on the algebra  $A$ , and  $A/E$  (defined as in condition (i) thereof) will be called the quotient algebra (or factor-algebra) of  $A$  by the congruence  $E$ .

The complete lattice of all congruences on  $A$  is called the congruence lattice of  $A$ . The least congruence containing a given subset  $R \subseteq |A| \times |A|$  is called the congruence on  $A$  generated by  $R$ , and the quotient of  $A$  by this congruence is often called the algebra obtained by imposing on  $A$  the family of relations  $R$ , or loosely, the family of relations  $(x = y)_{(x, y) \in R}$ .

I say “loosely” in the last sentence because (as we noted in passing in §3.3), there is an abuse of notation in writing such a relation as “ $x = y$ ”. The symbol  $x = y$  usually denotes a proposition, i.e., an assertion about elements of  $A$ , and this proposition is generally false in the case where the relation is one we wish to impose on  $A$ ! What is true is that in our quotient algebra the images of  $x$  and  $y$  satisfy the corresponding relation; and when there is no danger of ambiguity, one may denote these images by the same symbols  $x$  and  $y$  as the original elements of  $A$ , so that  $x = y$  becomes a true statement in that quotient algebra. But in more precise notation, the statement which is true in the latter algebra must be written  $\bar{x} = \bar{y}$  or  $[x] = [y]$ . We will be precise about this here, but in informal algebraic use, the language of “imposing the relation  $x = y$  on  $A$ ” is very convenient.

Many workers in General Algebra and Logic make a convention half-way between these extremes, defining “relations” or “identities” to be symbols of the form “ $x \approx y$ ”. (E.g., [18, p.234].) These are essentially just our ordered pairs  $(x, y)$ , written in a more suggestive form. A notation that allows one to avoid ambiguity while using the same symbols for elements of different algebras is that of Model Theory, where one writes  $A/E \models x = y$  to mean “ $x = y$  holds in  $A/E$ ”, so that this is distinguishable from  $A \models x = y$ .

Using the quotient construction, we immediately get

**Lemma 8.1.9.** For any type  $\Omega$ , the category  $\Omega\text{-Alg}$  has coequalizers. Namely, the coequalizer of a pair of maps  $f, g: A \rightrightarrows B$  may be constructed as the quotient  $B/E$  where  $E$  is the congruence on  $B$  generated by  $\{(f(x), g(x)) \mid x \in |A|\}$ .  $\square$

The other left universal construction that we can get easily is that of direct limit, assuming appropriate restrictions on the arities of our operations:

**Lemma 8.1.10.** If  $\Omega$  is a finitary type, then  $\Omega\text{-Alg}$  has direct limits, i.e., colimits over directed partially ordered sets. Namely, suppose  $J$  is a directed partially ordered set and  $A: J_{\text{cat}} \rightarrow \Omega\text{-Alg}$  a functor, whose values at objects and morphisms of  $J_{\text{cat}}$  will be written  $A_j$  ( $j \in J$ ) and  $A(j, j')$  ( $j \leq j' \in J$ ) respectively. Then the  $\Omega$ -algebra structures of the algebras  $A_j$  induce an  $\Omega$ -algebra structure on the set-theoretic direct limit  $\varinjlim_J |A_j|$  which makes it a direct limit algebra,  $\varinjlim_J A_j$ .

More generally, if  $\alpha$  is an infinite cardinal, and  $\Omega$  a type in which all arities have cardinality  $< \alpha$ , then the category  $\Omega\text{-Alg}$  has direct limits over all  $< \alpha$ -directed partially ordered sets (Definition 7.9.7), which may be constructed by giving an  $\Omega$ -algebra structure to the direct limit of the underlying sets.

**Proof.** We will prove the general case. Let  $|L| = \varinjlim_J |A_j|$ , and let  $q_j: |A_j| \rightarrow |L|$  ( $j \in J$ ) be the coprojection maps. We wish to define an  $\Omega$ -algebra structure on  $|L|$ . Given  $s \in |\Omega|$  and an  $\text{ari}(s)$ -tuple  $(x_i)_{i \in \text{ari}(s)}$  of elements of  $|L|$ , let us write each  $x_i$  as  $q_{j(i)}(y_i)$  for some  $j(i) \in J$  and  $y_i \in |A_{j(i)}|$ . Because  $J$  is  $< \alpha$ -directed and  $\text{card}(\text{ari}(s)) < \alpha$ , we can choose  $j \in J$  majorizing all the  $j(i)$ . Taking such a  $j$ , and letting  $z_i = A(j(i), j)(y_i) \in A(j)$  for each  $i$ , we have

$$(8.1.11) \quad x_i = q_j(z_i) \quad \text{for all } i \in \text{ari}(s).$$

To define  $s_L$ , let us say that whenever we have a family  $(x_i) \in |L|^{\text{ari}(s)}$  expressed as in (8.1.11) for some  $j \in J$ , we will let

$$s_L(x_i) = q_j(s_{A_j}(z_i)) \in |L|.$$

The verification that these operations  $s_L$  are well-defined, and that the resulting  $\Omega$ -algebra  $L$  has the universal property of  $\varinjlim A$ , are straightforward, again by the method of “going far enough out along the  $< \alpha$ -directed set  $J$ ”.  $\square$

**Exercise 8.1:2.** Write out these final verifications.

As noted at the beginning of §7.9, the “reason” the above lemma holds is that in **Set**, direct limits respect finite products (a case of Proposition 7.9.3) and more generally, that direct limits over  $< \alpha$ -directed partially ordered sets respect  $\alpha$ -fold products (Proposition 7.9.8). Similarly, Proposition 8.1.6 holds because arbitrary products in **Set** (indeed, in any category) respect arbitrary limits (Theorem 7.8.8).

Since we shall prove in §8.3 that  $\Omega\text{-Alg}$  has general colimits, the arity-restrictions of the above lemma are not needed for the existence statements to hold. But they are needed for the direct limits in question to have the descriptions given. Indeed

**Exercise 8.1:3.** Show by example that the last sentence of the first paragraph of Lemma 8.1.10 fails if the assumption that  $\Omega$  is finitary is dropped. Specifically, show that there may not exist an algebra with underlying set the direct limit of the  $|A_j|$ , and having the universal property of  $\varinjlim A_j$ .

**Exercise 8.1:4.** (i) Let  $\Omega$  be a finitary type, and  $A$  an  $\Omega$ -algebra. Show that a subalgebra of  $A$  is finitely generated if and only if it is *compact* as an element of the lattice of subalgebras. (Cf. Lemma 5.3.6.)

Deduce that a congruence on  $A$  is finitely generated if and only if it is a compact element of the lattice of congruences. Deduce, in turn, that the subalgebra lattice, respectively the congruence lattice, has ascending chain condition if and only if every subalgebra of  $A$ , respectively every congruence on  $A$ , is finitely generated.

(ii) Show that the one direction of the preceding results can fail if  $\Omega$  is not finitary, while the other will continue to hold.

**Exercise 8.1:5.** Let  $M$  be a monoid. As mentioned in Exercise 7.9:6, a *left congruence* on  $M$  means an equivalence relation  $\sim$  on  $|M|$  such that for all  $a, b, c \in M$  one has  $a \sim b \Rightarrow ca \sim cb$ . (We will not call on that exercise here, though point (i) below asks you to prove a slight strengthening of a result there referred to as “easy to verify”.)

(i) Show that a binary relation  $\sim$  on  $|M|$  is a left congruence if and only if there exists a left action of  $M$  on a set  $X$ , and an element  $x \in X$ , such that  $\sim = \{(a, b) \in |M| \times |M| \mid ax = bx\}$ .

(ii) If  $M$  is a monoid of the form  $G_{\text{md}}$  for  $G$  a group (i.e., is the monoid obtained by “forgetting” the inverse operation of  $G$ , as in §3.11), show that the left congruences on  $M$  are in natural bijective correspondence with the subgroups of  $G$ . (Not, as would seem more natural, with the submonoids of  $M$ .)

(iii) (Open question of E. Hotzel [76].) If a monoid  $M$  has ascending chain on left congruences, must  $M$  be finitely generated?

It has been shown [88] that if  $M$  has ascending chain condition on both left and right congruences, then it is indeed finitely generated. (Students wishing to read that paper might want to have [59] in hand; semigroup theorists use some rather arcane terminology.)

**Exercise 8.1:6.** Let us call an  $\Omega$ -algebra “just infinite” if it is infinite, but every proper homomorphic image (i.e., every image under a non-one-to-one homomorphism) is finite.

(i) Show that if  $\Omega$  has only finitely many operations, and all are finitary, then every infinite finitely generated  $\Omega$ -algebra has a just-infinite homomorphic image.

(ii) Can any of the above hypotheses be dropped? Can “infinite” be replaced by “ $\geq \alpha$ ” for a more general cardinal  $\alpha$ ?

**8.2. Generating subalgebras from below.** We want to construct other left universal objects in  $\Omega\text{-Alg}$  – free algebras, coproducts, arbitrary small colimits, etc.. In general, these will contain elements created by applying operations of  $\Omega$  to tuples of the elements we start with, further elements obtained by applying the operations to elements we get in this way, and so on. Whatever methods we use to justify these constructions must involve showing that this iteration process “eventually ends”.

“Eventually” does not mean in a finite number of steps, of course – even in constructing algebras with operations of finite arity such as groups, we needed countably many iterations to get the full set of such elements. When we have infinitary operations, we may have to continue the process longer than that.

To see how long, let us examine the process by which a subset of an algebra generates a subalgebra. Let  $\Omega$  be an arbitrary type and  $A$  an  $\Omega$ -algebra. Given a subset  $X \subseteq |A|$ , define a sequence of subsets of  $|A|$  indexed by the ordinals:

$$(8.2.1) \quad \begin{aligned} S^{(0)} &= X, \\ S^{(\alpha+1)} &= S^{(\alpha)} \cup \{s_A(x_i) \mid s \in |\Omega| \text{ and } x_i \in S^{(\alpha)} \text{ for all } i \in \text{ari}(s)\}, \\ S^{(\alpha)} &= \bigcup_{\beta < \alpha} S^{(\beta)} \text{ for } \alpha \text{ a limit ordinal } > 0. \end{aligned}$$

We see by induction that the  $S^{(\alpha)}$ 's increase monotonically. Since  $|A|$  is a small set,  $S^{(\alpha)}$  and  $S^{(\alpha+1)}$  cannot be distinct for all cardinals  $\alpha$ , and clearly as soon as two of them are equal, the chain will become constant. The constant value  $S$  that it assumes will contain  $S^{(0)} = X$  and be closed under the operations  $s_A$ ; moreover, by induction on  $\alpha$ , each  $S^{(\alpha)}$ , and so in particular,  $S$ , is contained in every subalgebra of  $A$  containing  $X$ . Hence  $S$  is the underlying set of the least subalgebra of  $A$  containing  $X$ , i.e., the subalgebra generated by  $X$ .

We now want to bound in terms of properties of  $\Omega$  the least value of  $\alpha$  for which  $S^{(\alpha)} = S$ . (Above, we implicitly bounded it in terms of  $\text{card } |A|$ .)

We know how to show that if  $\Omega$  is finitary,  $S = S^{(\omega)}$ . Namely, given finitely many elements  $s_0, \dots, s_{n-1} \in S^{(\omega)}$ , all of the  $s_i$  will have been reached by some finite step  $S^{(N)}$ , hence the

value of any operation of  $A$  on this family lies in  $S^{(N+1)}$ , and hence is in  $S^{(\omega)}$ .

The next case is that of a type  $\Omega$  in which all operations have arity of cardinality  $\leq \omega$ , equivalently,  $< \omega_1$ . (Recall that  $\omega_1$  denotes the first uncountable ordinal.) It is then no longer true that the above process converges by the  $\omega$ th step: If  $s \in |\Omega|$  is  $\omega$ -ary, and we can find for each nonnegative  $n$  an element  $x_n$  which first appears in  $S^{(n)}$ , then  $S^{(\omega)}$  will not in general contain  $s_A(x_0, x_1, \dots, x_n, \dots)$ . Rather, this element will appear in  $S^{(\omega+1)}$ , and further elements obtained from it under the operations of  $A$  will in general appear at still later steps. However, I claim that this process stabilizes by the  $\omega_1$ st step. Indeed, given a countable (possibly finite) family of elements  $x_i \in S^{(\omega_1)}$ , each occurs in some  $S^{(\alpha_i)}$  for a countable ordinal  $\alpha_i \in \omega_1$ , hence all the  $x_i$  will occur in  $S^{(\alpha)}$  where  $\alpha = \sup(\alpha_i)$ , and this ordinal  $\alpha$  is still  $< \omega_1$ , since  $\sup(\alpha_i)$  is  $\leq$  the ordinal sum of the  $\alpha_i$  (defined as in (4.5.10)), which has cardinality equal to the cardinal sum of the  $\text{card}(\alpha_i)$ , which is a countable sum of countable cardinals, hence countable. For  $\alpha$  so defined, the value at  $(x_1, \dots, x_n, \dots)$  of any operation of countable arity lies in  $S^{(\alpha+1)} \subseteq S^{(\omega_1)}$ , showing that  $S^{(\omega_1)}$  is closed under the operations of  $A$ , and hence that (8.2.1) stabilizes by the  $\omega_1$ st step. The next exercise shows that in this statement we cannot replace the estimate  $\omega_1$  by any smaller ordinal (such as  $\omega^2$  or  $\omega^\omega$ ).

**Exercise 8.2:1.** Let  $\gamma$  be any uncountable ordinal, and let  $A$  be an algebra with underlying set  $\gamma$  and three operations: the zeroary operation taking the value  $0 \in \gamma$ , the unary operation taking  $\alpha \in \gamma$  to  $\alpha+1$  if  $\alpha+1 < \gamma$ , or to  $0$  if  $\alpha+1 = \gamma$ , and the  $\omega$ -ary operation taking  $(\alpha_0, \alpha_1, \dots)$  to  $\cup \alpha_i$  if this is  $< \gamma$ , to  $0$  otherwise. Taking  $X = \emptyset \subseteq |A|$ , determine explicitly the sequence of subsets  $S^{(\alpha)}$ , and show that this sequence does not become constant until  $S^{(\omega_1)}$ .

The same argument will show that if all members of  $|\Omega|$  have arity  $\leq \omega_1$ , then we get our desired algebra as  $S^{(\omega_2)}$ , that if all arities are  $\leq \omega_2$ , we get it as  $S^{(\omega_3)}$ , etc.; and it might appear that the proper general statement is that if  $\alpha$  is any infinite ordinal of cardinality *greater* than the arities of all members of  $|\Omega|$ , then  $S^{(\alpha)}$  is closed under the operations of  $\Omega$ .

But this is not quite correct. The first value of  $\alpha$  for which it fails is  $\omega_\omega$ . If  $A$  has operations of arities  $\omega, \omega_1, \omega_2$  etc. (all the infinite cardinals  $< \omega_\omega$ ), then by the above considerations the chain of subalgebras  $S^{(\omega)} \subseteq S^{(\omega_1)} \subseteq S^{(\omega_2)} \subseteq \dots$  can be strictly increasing. If we now choose an element  $x_i \in S^{(\omega_{i+1})} - S^{(\omega_i)}$  for each  $i$ , we get a countable family of elements of  $S^{(\omega_\omega)}$ , and we see that the value of an operation of (merely!) countable arity on this family cannot be expected to lie in  $S^{(\omega_\omega)}$ .

**Exercise 8.2:2.** Construct an explicit example with the properties sketched above, i.e., an algebra  $A$  all of whose operations have arities  $< \omega_\omega$ , and a subset  $X \subseteq |A|$ , such that the chain of subsets  $S^{(\alpha)}$  does not reach its maximum value till  $S^{(\omega_{\omega+1})}$ . (Suggestion: Adapt the idea of the preceding exercise.)

To state the right choice of  $\alpha$ , we recall from Definition 4.5.17 that an infinite cardinal  $\alpha$  is called *regular* if, as a partially ordered set,  $\alpha$  has no cofinal subset of cardinality  $< \alpha$ , and that a cardinal that is not regular is called *singular*. What we have run into is the first singular infinite cardinal,  $\omega_\omega$ . Fortunately, *regular* cardinals are quite abundant: as shown in Exercise 4.5:13, the cardinal  $\omega$  is regular, and every infinite successor cardinal, i.e., every cardinal of the form  $\omega_{\alpha+1}$  for  $\alpha$  an ordinal, is also regular. We can now show

**Lemma 8.2.2.** *Let  $\Omega$  be a type, and  $\gamma$  a regular infinite cardinal such that  $\text{card}(\text{ari}(s)) < \gamma$  for all  $s \in |\Omega|$  (e.g., the least such regular cardinal). Then for any  $\Omega$ -algebra  $A$ , and any subset  $X \subseteq |A|$ , if we define the chain of sets  $S^{(\alpha)}$  by (8.2.1), then  $S^{(\gamma)}$  is closed under the operations of  $A$ , hence is the underlying set of the subalgebra of  $A$  generated by  $X$ .*

**Proof.** Consider any  $s \in |\Omega|$  and elements  $x_i \in S^{(\gamma)}$  ( $i \in \text{ari}(s)$ ). Since  $\gamma$  is a limit ordinal,  $S^{(\gamma)} = \bigcup_{\beta < \gamma} S^{(\beta)}$ , hence each  $x_i$  lies in some  $S^{(\beta_i)}$  ( $\beta_i \in \gamma$ ). Since  $\text{card}(\text{ari}(s)) < \gamma$  and  $\gamma$  is regular, the set  $\{\beta_i \mid i \in \text{ari}(s)\}$  is not cofinal in  $\gamma$ , hence that set is majorized by some  $\beta < \gamma$ . For this choice of  $\beta$ , all  $x_i$  lie in  $S^{(\beta)}$ , hence  $s(x_i) \in S^{(\beta+1)} \subseteq S^{(\gamma)}$ , as required.  $\square$

In the next section we will apply the above result to the construction of left universal objects.

For later use, we record the following generalization of the familiar observation that if an algebra with finitary operations is generated by a set  $X$ , each element of the algebra can be expressed in terms of finitely many elements of  $X$ .

**Lemma 8.2.3.** *Let  $\Omega$  be a type, and  $\gamma$  a regular infinite cardinal such that  $\text{card}(\text{ari}(s)) < \gamma$  for all  $s \in |\Omega|$ . Let  $A$  be any  $\Omega$ -algebra, and  $X$  any generating set for  $A$ . Then each element of  $|A|$  belongs to the subalgebra of  $A$  generated by a subset  $X_0 \subseteq X$  of cardinality  $< \gamma$ .*

**Sketch of Proof.** It is easy to verify that under the given hypothesis the set of elements of  $|A|$  belonging to subalgebras generated by  $< \gamma$  elements of  $X$  forms a subalgebra. As it contains  $X$ , it must be all of  $|A|$ .  $\square$

**Exercise 8.2:3.** Write out the easy verification referred to. Show that the result becomes false if the regularity assumption on  $\gamma$  is deleted.

It may now seem anomalous that in our results on direct limits over  $< \alpha$ -directed partially ordered sets, Proposition 7.9.8 and Lemma 8.1.10, we did *not* have to assume  $\alpha$  regular! This is explained by

**Exercise 8.2:4.** Show that if  $\alpha$  is a singular infinite cardinal and  $J$  a  $< \alpha$ -directed partially ordered set, then  $J$  is also  $< \alpha'$ -directed, where  $\alpha'$  is the successor cardinal to  $\alpha$ .

Thus, if  $J$  is  $< \alpha$ -directed for a cardinal  $\alpha$  greater than the arities of all operations of  $\Omega$ , it is in fact  $< \alpha'$ -directed for a *regular* cardinal  $\alpha'$  greater than the arities of those operations.

We could have avoided using the concept of regular cardinal in this section by taking  $\gamma$  in our results to be “the successor cardinal of the least infinite upper bound of the arities of the operation-symbols of  $\Omega$ ”. However, in the case where  $\Omega$  is finitary, this would have given  $\gamma = \omega_1$ , whereas the development we have used shows that  $\omega$  suffices in that important case.

**8.3. Terms and left universal constructions.** Given a type  $\Omega$  and a set  $X$ , Lemma 8.2.2 can be used to obtain a bound on the size of an  $\Omega$ -algebra generated by an  $X$ -tuple of elements, and hence to establish the *solution set* hypotheses needed by the existence results for left universal constructions developed in §7.10. Now such a bound can be thought of as an estimate of the number of “ $\Omega$ -algebra terms in an  $X$ -tuple of variable-symbols”, and rather than just giving the existence proof suggested above, we can, with little additional work, construct such a set of terms, thus laying the groundwork for the more explicit approach to universal constructions that was sketched in §2.2.

Let us first define precisely the concept of a “term”. At the beginning of this course

(Definition 1.5.1) we described “the set of group-theoretic terms in the elements of  $X$ ” as a set  $T$  given with certain structure: a map of  $X$  into it, and a family of “formal group-theoretic operations” satisfying some further conditions. If we make the corresponding definition for  $\Omega$ -algebras, we see that the “formal operations” in fact make the set  $T$  into an  $\Omega$ -algebra. (We could not similarly say that formal operations made the set of group-theoretic terms into a group, because they did not satisfy the group identities. But in the present development, we are studying algebras of type  $\Omega$  in general, before introducing identities.) So we state the definition accordingly:

**Definition 8.3.1.** *Let  $\Omega$  be any type, and  $X$  any set. Then an “ $\Omega$ -term algebra on  $X$ ” will mean a pair  $(F, u)$ , where  $F$  is an  $\Omega$ -algebra, and  $u: X \rightarrow |F|$  a set map, such that*

- (i) *the map  $u: X \rightarrow |F|$ , and all the maps  $s_F: |F|^{\text{ari}(s)} \rightarrow |F|$  are one-to-one,*
- (ii) *the images of the above maps in  $|F|$  are disjoint,*
- (iii) *the union of these images is all of  $|F|$ , and*
- (iv)  *$F$  is generated as an  $\Omega$ -algebra by  $u(X)$ .*

Note that the first three points of the above definition can be stated as a single condition: If we write  $\sqcup$  for disjoint union of sets, and consider the map  $u$  and the operations  $s_F$  as defining a single map  $X \sqcup \bigsqcup_{s \in |\Omega|} |F|^{\text{ari}(s)} \rightarrow |F|$ , then (i)-(iii) say that this map is bijective.

Since the concept of  $\Omega$ -algebra involves no identities, the idea of constructing free objects by taking “terms modulo identities” simplifies in this case to

**Lemma 8.3.2.** *Let  $\Omega$  be any type, and  $X$  any set. Suppose there exists an  $\Omega$ -term algebra  $(F, u)$  on  $X$ . Then  $(F, u)$  is a free  $\Omega$ -algebra on  $X$ .*

**Proof.** To prove that  $(F, u)$  has the universal property of a free  $\Omega$ -algebra on  $X$ , suppose  $A$  is an  $\Omega$ -algebra and  $v: X \rightarrow |A|$  any set map. We wish to construct a homomorphism  $f: F \rightarrow A$  such that  $v = fu$ . Intuitively  $f$  should represent “substitution of the particular values  $v(x)$  for the variable-symbols  $u(x)$  in our terms”.

Let us write  $|F|$  as the union of a chain of subsets  $S^{(\alpha)}$  as in (8.2.1), starting with the generating set  $S^{(0)} = u(X)$ . Assume recursively that  $f$  has been defined on all the sets  $S^{(\beta)}$  with  $\beta < \alpha$ ; we wish to extend  $f$  to  $S^{(\alpha)}$ . If  $\alpha = 0$ ,  $S^{(\alpha)}$  consists of elements  $u(x)$  ( $x \in X$ ), all distinct, and we let  $f(u(x)) = v(x) \in |A|$ . If  $\alpha$  is a successor ordinal  $\beta+1$ , then an element which first appears in  $S^{(\alpha)}$  will have the form  $s_F(t_i)$ , where  $s \in |\Omega|$  and each  $t_i \in |S^{(\beta)}|$ . Thus the  $f(t_i)$  have already been defined, and we define  $f(s_F(t_i)) = s_A(f(t_i))$ . If  $\alpha$  is a nonzero limit ordinal, then  $S^{(\alpha)} = \bigcup_{\beta < \alpha} S^{(\beta)}$ , and having defined  $f$  consistently on  $S^{(\beta)}$  for all  $\beta < \alpha$ , we have defined it on  $S^{(\alpha)}$ .

In each case, the one-one-ness condition (i) and the disjointness condition (ii) of Definition 8.3.1 insure that if an element of  $F$  occurs at some stage as  $u(x)$  or  $s_F(t_i)$ , it cannot occur (at the same or another stage) in a different way as  $u(x')$  or  $s'_F(t'_i)$ . Hence our definition of  $f$  is unambiguous. By construction,  $f$  is a homomorphism of  $\Omega$ -algebras and satisfies  $fu = v$ ; and by (iv) it is unique for this property.  $\square$

I should mention that the technique of explicit induction or recursion on the forms of elements, as above, is one that seldom has to be used. Results to the effect that if an algebra  $A$  is generated by a set  $X$  of elements having some property  $P$ , then all elements of  $A$  satisfy  $P$ , which

beginners often prove by such an argument, can generally be obtained more simply by verifying that the set of elements satisfying  $P$  is closed under the algebra operations, hence forms a subalgebra containing  $X$ , hence is all of  $|A|$ . On the other hand, if we want to construct some homomorphism on the free algebra  $A$  on a set  $X$  starting from its values on elements of  $X$ , we can do this using the universal property of  $A$  as a free object. In the case of free objects of  $\Omega\text{-Alg}$ , we have just proved that universal property by recursion on elements, but this one application of that method frees us from having to repeat that argument in similar situations.

We have not proved the converse statement, that if a free  $\Omega$ -algebra on  $X$  exists, it will be an  $\Omega$ -term algebra on  $X$ . We would want this if we planned to prove the existence of free algebras first and deduce from this the existence of term algebras, but we shall be going the other way. However, this implication is not hard to prove; so I will make it

**Exercise 8.3.1.** Show (without assuming the existence of  $\Omega$ -term algebras) that if  $(F, u)$  is a free  $\Omega$ -algebra on  $X$ , then it is an  $\Omega$ -term algebra on  $X$ .

(Hint: If  $F$  fails to satisfy one of conditions (i)-(iv) of Definition 8.3.1, you want to find a pair  $(A, v)$  for which the universal property of  $(F, u)$  fails. If condition (iii) or (iv) fails, make  $A$  a subalgebra of  $F$ ; if (i) or (ii) fails, obtain  $A$  by replacing one element  $p$  of  $F$  by two elements  $p_1$  and  $p_2$ , and defining the operations appropriately on  $|F| - \{p\} \cup \{p_1, p_2\}$ . Since the operations of  $\Omega$ -algebras are not required to satisfy any identities, any definition of these operations yields an  $\Omega$ -algebra.)

Let us now prove

**Theorem 8.3.3.** *Let  $\Omega$  be any type, and  $X$  any set. Then there exists an  $\Omega$ -term algebra on  $X$ ; equivalently, a free  $\Omega$ -algebra on  $X$ .*

**Proof.** Let  $*$  be any element not in  $|\Omega|$ , and  $\gamma$  an infinite regular cardinal which is  $> \text{card}(\text{ari}(s))$  for all  $s \in |\Omega|$ . We define recursively a chain  $(S^{(\alpha)})_{\alpha \leq \gamma}$  of sets of ordered pairs, by taking

$$\begin{aligned} S^{(0)} &= \{(*, x) \mid x \in X\}, \\ S^{(\alpha+1)} &= S^{(\alpha)} \cup \{(s, (x_i)) \mid s \in |\Omega|, (x_i) \in (S^{(\alpha)})^{\text{ari}(s)}\}, \\ S^{(\alpha)} &= \bigcup_{\beta < \alpha} S^{(\beta)} \text{ if } \alpha \text{ is a limit ordinal } > 0. \end{aligned}$$

Let  $|F| = S^{(\gamma)}$ , and define  $u: X \rightarrow |F|$ , and maps  $s_F: |F|^{\text{ari}(s)} \rightarrow |F|$  ( $s \in |\Omega|$ ), by

$$\begin{aligned} u(x) &= (*, x) \quad (x \in X), \\ s_F(x_i) &= (s, (x_i)) \quad (s \in |\Omega|, (x_i) \in |F|^{\text{ari}(s)}). \end{aligned}$$

That the operations  $s_F$  carry  $|F| = S^{(\gamma)}$  into itself follows from our choice of  $\gamma$ , by the same argument we used in proving Lemma 8.2.2. Thus these operations make  $|F|$  an  $\Omega$ -algebra  $F$ . That  $F$  satisfies conditions (i)-(iii) follows from the set-theoretic fact that an ordered pair uniquely determines its first and second components. To get (iv), one verifies by induction that a subalgebra containing  $X$  must contain each  $S^{(\alpha)}$ .  $\square$

Since we have free  $\Omega$ -algebras on all sets  $X$ , these give a left adjoint to the underlying-set functor from  $\Omega\text{-Alg}$  to  $\mathbf{Set}$ .

**Exercise 8.3:2.** Show how we could, alternatively, have gotten the existence of such an adjoint using Freyd's Adjoint Functor Theorem (Theorem 7.10.4) and Lemma 8.2.2.

Let us fix a notation for these functors.

**Definition 8.3.4.** *The underlying-set functor of  $\Omega\text{-Alg}$  and its left adjoint, the free algebra functor, will be denoted  $U_\Omega: \Omega\text{-Alg} \rightarrow \mathbf{Set}$  and  $F_\Omega: \mathbf{Set} \rightarrow \Omega\text{-Alg}$  respectively.*

*A symbol such as  $F_\Omega(\{x_0, \dots, x_{n-1}\})$  may be abbreviated to  $F_\Omega(x_0, \dots, x_{n-1})$  when there is no danger of misunderstanding.*

The “danger of misunderstanding” referred to is that the symbol  $F_\Omega(X)$  for the free  $\Omega$ -algebra on a set  $X$  might be misinterpreted, under the above convention, as meaning the one-generator free algebra  $F_\Omega(\{X\})$ . But in context, there is almost never any doubt as to whether a given entity is meant to be treated as a free generator, or as a set of free generators.

There is another sort of looseness in our usage, which we noted in Chapter 2. Although we have formally defined free algebras to be pairs  $(F, u)$ , we also sometimes use the term for the first components of such pairs, thought of as algebras “given with” the set-maps  $u$ . (E.g., when we spoke of the free-algebra functor above, the values of the functor were algebras  $F$ , not ordered pairs  $(F, u)$ ; the maps  $u$  are the values of the unit of the adjunction,  $\eta(X): X \rightarrow U_\Omega(F_\Omega(X))$ .) At other times, we speak of an algebra  $F$  as being free on a given set of its elements, without specifying an indexing of this set by any external set (though we can always index it by its identity map to itself). Finally, we may speak of an object as being “free”, meaning that there *exists* a generating set on which it is free, without choosing a particular such set, as when we say that any subgroup of a free abelian group is free abelian. So we need to be sure it is always clear which version of the concept we are using.

The next exercise shows that in a category of the form  $\Omega\text{-Alg}$ , and in certain others, the last two of the above senses of “free algebra” essentially coincide.

**Exercise 8.3:3.** (i) Show that a free  $\Omega$ -algebra is free on a unique set of generators. That is, if  $(F, u)$  is a free  $\Omega$ -algebra, then the image in  $|F|$  of the set map  $u$ , and hence also the cardinality of the domain of  $u$ , are determined by the  $\Omega$ -algebra structure of  $F$ . (Hint: Definition 8.3.1.)

(ii) Is the analogous statement true for free groups? Free monoids? Free rings?

(iii) Same question for free upper (or lower) semilattices.

(iv) Same question for free lattices. (If you know the structure theorem for free lattices this is not hard. Even if you do not, a little ingenuity will yield the answer by a direct argument.)

**Exercise 8.3:4.** (i) Show that every subalgebra  $A$  of a free  $\Omega$ -algebra  $F$  is free.

We mentioned above the fact (proved in the standard beginning graduate algebra course) that the corresponding statement holds for free abelian groups. It is also a basic (though harder to prove) result of group theory that it holds for free groups. But

(ii) Is the analogous statement true for free monoids? Free rings? Free upper semilattices? Free lattices?

**Exercise 8.3:5.** (i) Let  $\Omega$  be a finitary type without zeroary operation symbols, and  $F_\Omega(x)$  the free  $\Omega$ -algebra on a single generator  $x$ . Show that the monoid of endomorphisms  $\text{End}(F_\Omega(x))$  (under composition) is a free monoid. If you wish, you may for simplicity assume that  $|\Omega|$  consists of a single binary operation-symbol (since even in this case, the description of the free generating set for the monoid  $\text{End}(F_\Omega(x))$  is nontrivial).

(ii) Does the result of (i) remain true if the assumption that  $\Omega$  is finitary is removed?

- (iii) Show that the corresponding result is never true if  $\Omega$  has zeroary operations. Can you describe the monoid in this case?
- (iv) If all operation-symbols of  $\Omega$  have arity 1, describe the monoid  $\text{End}(F_\Omega(x))$  precisely in terms of  $|\Omega|$ .

The next result is easily seen from the explicit description of free  $\Omega$ -algebras in our proof of Theorem 8.3.3.

**Corollary 8.3.5** (to proof of Theorem 8.3.3). *If  $a: X \rightarrow Y$  is an injective (respectively surjective) map of sets, then the induced map of free  $\Omega$ -algebras  $F_\Omega(a): F_\Omega(X) \rightarrow F_\Omega(Y)$  is likewise injective (surjective) on underlying sets.  $\square$*

We can also get the above result from a very general observation, though in that case we need a special argument to handle the free algebra on the empty set:

- Exercise 8.3:6.** (i) Show that every functor  $A: \mathbf{Set} \rightarrow \mathbf{Set}$  carries surjective maps to surjective maps, and carries injective maps with nonempty domains to injective maps. (Hint: Use right and left invertibility.)
- (ii) Show that (i) becomes false if the qualification about nonempty domains is dropped.
- (iii) Show, however, that if  $A$  has the form  $UF$ , where  $U$  is a functor from some category to  $\mathbf{Set}$ , and  $F$  is a left adjoint to  $U$ , then  $A$  carries maps with empty domain to injective maps.
- (iv) Deduce Corollary 8.3.5 without calling on an explicit description of free  $\Omega$ -algebras.

Using free algebras, we can obtain other left universal constructions. A basic tool will be

**Definition 8.3.6.** *Let  $\Omega$  be a type,  $X$  a set, and  $(F_\Omega(X), u_X)$  a free  $\Omega$ -algebra on  $X$ . An  $\Omega$ -algebra relation in an  $X$ -tuple of variables will mean an element  $(s, t) \in |F_\Omega(X)| \times |F_\Omega(X)|$  (often informally written “ $s = t$ ”). An  $X$ -tuple  $v$  of elements of an  $\Omega$ -algebra  $A$  is said to satisfy the relation  $(s, t)$  if the unique homomorphism  $f: F_\Omega(X) \rightarrow A$  such that  $fu_X = v$  has the property  $f(s) = f(t)$ .*

*If  $Y \subseteq |F_\Omega(X)| \times |F_\Omega(X)|$  is a set of relations, then an  $\Omega$ -algebra presented by generators  $X$  and relations  $Y$  will mean an initial object  $(B, w)$  in the category whose objects are pairs  $(A, v)$  with  $A$  an  $\Omega$ -algebra and  $v$  an  $X$ -tuple of elements of  $|A|$  satisfying all the relations in  $Y$ , and whose morphisms are homomorphisms of first components respecting second components; equivalently, a representing object for the functor  $\Omega\text{-Alg} \rightarrow \mathbf{Set}$  associating to every  $\Omega$ -algebra  $A$  the set of  $X$ -tuples  $v$  satisfying all the relations in  $Y$ . Such an algebra  $B$  will be denoted  $\langle X | Y \rangle_{\Omega\text{-Alg}}$ , or, when there is no danger of ambiguity,  $\langle X | Y \rangle$ .*

(If we wanted to be more precise, we might write our relations as  $(s, t, (F_\Omega(X), u_X))$ , since formally, a given pair of elements  $s$  and  $t$  can belong to underlying sets of various free algebras. But to avoid messy notation, we will assume that there is no ambiguity as to which free algebra is meant. Also, strictly speaking, the object so presented should be given as a pair  $(\langle X | Y \rangle, w)$ , where  $w$  is the canonical map  $X \rightarrow |\langle X | Y \rangle|$ . But again we will speak of it as  $\langle X | Y \rangle$ , and leave it understood that  $w$  is there if we need to refer to it.)

**Theorem 8.3.7.** *Let  $\Omega$  be a type. Then  $\Omega\text{-Alg}$  has algebras  $\langle X | Y \rangle$  presented by arbitrary sets of generators  $X$  and relations  $Y$ .*

**Proof.**  $\langle X | Y \rangle$  can be constructed as the quotient of  $F_\Omega(X)$  by the congruence generated by  $Y$  (Definition 8.1.8).  $\square$

**Exercise 8.3:7.** Give an alternative proof of the above theorem using the results of §7.10.

**Exercise 8.3:8.** At the end of §3.5 we introduced the term *residually finite* to describe a group  $G$  with the property that for any two elements  $x \neq y \in |G|$ , there exists a homomorphism  $f$  of  $G$  into a finite group such that  $f(x) \neq f(y)$ . The same definition applies to  $\Omega$ -algebras, for arbitrary  $\Omega$ .

Show that if  $\Omega$  is a finitary type, then every finitely related  $\Omega$ -algebra is residually finite.

**Exercise 8.3:9.** This exercise will show that finitely related  $\Omega$ -algebras tend to have large free subalgebras. Let  $\Omega$  be a finitary type,  $X$  a finite set, and  $(F, u_F)$  a free  $\Omega$ -algebra on  $X$ .

(i) Show that for every element  $a \in |F|$  not belonging to the subalgebra generated by the empty set, there exists a subalgebra  $A \subseteq F$  not containing  $a$ , such that  $|F| - |A|$  is finite.

(ii) Let  $Y = \{(s_0, t_0), \dots, (s_{n-1}, t_{n-1})\} \subseteq |F|^2$ . Show that an algebra  $A$  isomorphic to  $\langle X | Y \rangle_\Omega$  can be obtained from  $F$  by a construction of the following form:

(a) Define a certain algebra  $B$  with underlying set  $|F|$ , and with operations and distinguished  $X$ -tuple of elements  $u$  that differ from those of  $F$  in a total of at most  $n$  places. (I.e., such that the number of elements of  $X$  at which  $u_B$  differs from  $u_F$ , and the number of  $\text{ari}(s)$ -tuples at which the various  $s_B$  differ from the  $s_F$ , add up to at most  $n$ . The idea is to modify these maps so as to make the relations of  $Y$  hold in  $B$ .)

(b) Let  $A$  be the subalgebra of  $B$  generated by  $u_B(X)$  under the operations  $s_B$ .

(iii) Show from (i) and (ii) that if  $\Omega$  has no zeroary operations, and  $Y$  is as in part (ii) above, then  $\langle X | Y \rangle_\Omega$  has a subalgebra  $C$  such that  $|\langle X | Y \rangle_\Omega| - |C|$  is finite and  $C$  is isomorphic to a subalgebra of  $A$ , hence, by Exercise 8.3:4, is free.

After working out your proof, you might see whether you can weaken the assumption that  $\Omega$  have no zeroary operations, either to a weaker condition on  $\Omega$ , or to a condition on  $Y$ .

**Theorem 8.3.8.** *The category  $\Omega\text{-Alg}$  has all small colimits.*

**Proof.** By Proposition 7.6.6 (last statement), it is enough to show that  $\Omega\text{-Alg}$  has coequalizers of pairs of morphisms, and small coproducts. We obtained coequalizers in Lemma 8.1.9; we shall now construct the coproduct of a small family of  $\Omega$ -algebras  $(A_i)_{i \in I}$ .

We assume without loss of generality that the  $A_i$  have disjoint underlying sets (since we can replace them with disjoint isomorphic algebras if they do not). Let  $A$  be the algebra presented by the generating set  $\cup |A_i|$  and, for relations, all the relations satisfied within the separate  $A_i$ 's. (Precisely, we take for relations the images in  $|F_\Omega(\cup_I |A_i|)| \times |F_\Omega(\cup_I |A_i|)|$ , under the canonical maps  $F_\Omega(|A_j|) \rightarrow F_\Omega(\cup_I |A_i|)$ , of all the relations  $(s, t) \in |F_\Omega(|A_j|)| \times |F_\Omega(|A_j|)|$  holding in the given algebras  $A_j$ .) It is easy to verify that  $A$  is the desired coproduct.  $\square$

We end this section with two exercises which assume familiarity with point-set topology, and which concern certain algebras with a single binary operation. The first exercise sets up a general construction and establishes some of its properties, to give you the feel of things. The second restricts attention to a particular instance of this construction, and asks you to establish a seemingly bizarre universal property of that object.

**Exercise 8.3:10.** Let the set  $2^\omega$  of all sequences  $(\iota_0, \iota_1, \dots)$  of 0's and 1's be given the product topology induced by the discrete topology on  $\{0, 1\}$ . (The resulting space can be naturally identified with the Cantor set.) Let us define two continuous maps  $\alpha, \beta: 2^\omega \rightarrow 2^\omega$ , by letting

$$\alpha(\iota_0, \iota_1, \dots) = (0, \iota_0, \iota_1, \dots), \text{ and } \beta(\iota_0, \iota_1, \dots) = (1, \iota_0, \iota_1, \dots).$$

Thus,  $2^\omega$  is the disjoint union of the two copies of itself,  $\alpha(2^\omega)$  and  $\beta(2^\omega)$ .

Now let  $\Omega$  be the type determined by a single binary operation  $*$ , and let us define a covariant functor  $F$  from the category **HausTop** of Hausdorff topological spaces to  $\Omega\text{-Alg}$ . For every space  $S$ , the set  $|F(S)|$  will be **HausTop**( $2^\omega, S$ ), i.e., the space of continuous  $S$ -valued functions on  $2^\omega$ . Thus, these sets are given by the covariant hom-functor  $h_2^\omega: \mathbf{HausTop} \rightarrow \mathbf{Set}$ . To describe the binary operation, let  $u, v \in |F(S)|$ . Then we define  $u*v$  to be the function  $2^\omega \rightarrow S$  such that

$$(u*v)(\alpha(x)) = u(x) \quad (u*v)(\beta(x)) = v(x) \quad (x \in 2^\omega).$$

Thus, if we identify  $2^\omega$  with the Cantor set,  $u*v$  is the map whose graph on the first half of that set looks like the graph of  $u$  compressed horizontally, and whose graph on the second half of the Cantor set is a similarly compressed copy of the graph of  $v$ . Let  $F(S) = (|F(S)|, *)$ .

- (i) Show that for every  $S$ , the map  $*$ :  $|F(S)| \times |F(S)| \rightarrow |F(S)|$  is bijective.
- (ii) Let  $S$  be any Hausdorff topological space and  $X$  any finite subset of  $|F(S)|$ . Let  $X_0$  be the set of those  $x \in X$  which, as maps  $2^\omega \rightarrow S$ , are constant, and  $X_1$  the set of  $x \in X$  which are not constant, and such that  $x$  does not belong to the  $\Omega$ -subalgebra of  $F(S)$  generated by  $X - \{x\}$ . Show that the  $\Omega$ -subalgebra of  $F(S)$  generated by  $X$  can be presented by the generating set  $X_0 \cup X_1$ , and the relations  $x*x = x$  for  $x \in X_0$ .
- (iii) Deduce that the set of nonconstant elements of  $F(S)$  forms a subalgebra  $N$  every finitely generated subalgebra of which is free. Show, however, that if  $S$  contains a homeomorphic copy of  $2^\omega$ , then  $N$  itself is not free.  
(Can you find necessary and sufficient conditions on  $S$  for  $N$  to be free?)

Our definition above of the element  $u*v$  involved its composites on the right with  $\alpha$  and  $\beta$ . We shall now let our construction take its tail in its mouth, by applying it with  $S = 2^\omega$ . Since elements of the resulting algebra also have  $2^\omega$  as *codomain*, we can also compose them on the *left* with  $\alpha$  and  $\beta$ .

**Exercise 8.3:11.** Let  $\alpha, \beta: 2^\omega \rightarrow 2^\omega$  and  $F: \mathbf{HausTop} \rightarrow \Omega\text{-Alg}$  be defined as in the preceding exercise, and let  $A = F(2^\omega)$ , an  $\Omega$ -algebra with underlying set **HausTop**( $2^\omega, 2^\omega$ ).

- (i) Show that each of the  $\Omega$ -algebra homomorphisms  $F(\alpha), F(\beta): A \rightarrow A$  is an embedding, and that  $A$  is the coproduct in  $\Omega\text{-Alg}$  of the images of these homomorphisms.

This is equivalent to saying that  $A$  is a coproduct of two copies of itself, with coprojection maps  $F(\alpha)$  and  $F(\beta)$ ; or, fixing an arbitrary coproduct of two copies of  $A$  and calling it  $A \amalg A$ , and its coprojection maps  $q_0$  and  $q_1$ , it is equivalent to saying that the unique homomorphism  $f: A \amalg A \rightarrow A$  satisfying  $f q_0 = \alpha$  and  $f q_1 = \beta$  is an isomorphism.

We now come to the strange universal property. Let  $\mathbf{m}_A: A \rightarrow A \amalg A$  be the *inverse* of the above map  $f$ .

- (ii) Show that if  $B$  is any  $\Omega$ -algebra given with a homomorphism  $\mathbf{m}_B: B \rightarrow B \amalg B$ , there exists a unique homomorphism  $\theta: B \rightarrow A$  such that the following diagram commutes:

$$\begin{array}{ccc} B & \xrightarrow{\mathbf{m}_B} & B \amalg B \\ \downarrow \theta & & \downarrow \theta \amalg \theta \\ A & \xrightarrow{\mathbf{m}_A} & A \amalg A \end{array}$$

(Note that though our construction of  $A$  uses topology, so that the same is necessarily true of the proofs of (i) and (ii), the statements of these properties of  $A$  are purely algebraic. We will be able to make sense of the above universal property in Chapter 9.)

### 8.4. Identities and varieties.

Here is a definition that needs no introduction!

**Definition 8.4.1.** Let  $\Omega$  be a type,  $X$  a set, and  $(F_\Omega(X), u_X)$  a free  $\Omega$ -algebra on  $X$ . An identity in an  $X$ -tuple of variables will mean an element  $(s, t) \in |F_\Omega(X)| \times |F_\Omega(X)|$ , i.e., formally the same thing as a relation, and likewise often informally written “ $s = t$ ”. However an  $\Omega$ -algebra  $A$  will be said to “satisfy” the identity  $(s, t)$  if and only if every  $X$ -tuple  $v$  of elements of  $|A|$  satisfies  $(s, t)$  as a relation; that is, if and only if for every homomorphism  $f: F_\Omega(X) \rightarrow A$ , one has  $f(s) = f(t)$ .

The next lemma will relate identities in different sets of variables.

**Lemma 8.4.2.** Let  $\Omega$  be a type,  $X$  a set, and  $(s, t) \in |F_\Omega(X)| \times |F_\Omega(X)|$  an identity in an  $X$ -tuple of variables. Then if  $f: X \rightarrow Y$  is a one-to-one set map, an  $\Omega$ -algebra  $A$  satisfies the identity  $(s, t)$  if and only if it satisfies the identity in a  $Y$ -tuple of variables,  $(F_\Omega(f)(s), F_\Omega(f)(t))$ .

Hence if  $\gamma$  is an infinite cardinal such that  $\gamma \geq \text{card}(\text{ari}(s))$  for all  $s \in |\Omega|$ , every identity  $(s, t)$  in any set  $X$  of variables is equivalent to an identity  $(s', t')$  in a  $\gamma$ -tuple of variables (i.e., there is an identity  $(s', t') \in |F_\Omega(\gamma)| \times |F_\Omega(\gamma)|$  which is satisfied by an  $\Omega$ -algebra  $A$  if and only if  $A$  satisfies  $(s, t)$ ).

**Proof.** First statement: It is easy to see (without assuming the given map  $f$  one-to-one) that for any  $Y$ -tuple of elements of  $|A|$ ,  $v: Y \rightarrow |A|$ , the induced  $X$ -tuple  $vf: X \rightarrow |A|$  will satisfy the relation  $(s, t)$  if and only if  $v$  satisfies the relation  $(F_\Omega(f)(s), F_\Omega(f)(t))$ . Hence if  $A$  satisfies  $(s, t)$  as an identity it will likewise satisfy  $(F_\Omega(f)(s), F_\Omega(f)(t))$  as an identity. The converse will hold (for  $f$  a one-to-one map) if we can show that every map  $w: X \rightarrow |A|$  can be written  $vf$  for some  $v: Y \rightarrow |A|$ . It is clear how to define  $v$  on elements of the one-to-one image of  $X$  in  $Y$  under  $f$ . If  $|A|$  is nonempty, we can extend this map by giving  $v$  arbitrary values on other elements of  $Y$ . If  $|A|$  is empty, on the other hand, then there can be no homomorphisms to  $A$  from the algebra  $F_\Omega(X)$ , which is nonempty because it contains  $s$  and  $t$ , so this case is vacuous. (An empty algebra satisfies every identity  $(s, t)$ , because the hypothesis of the implication defining “satisfaction” can never hold.)

Now let  $\gamma$  be as in the second statement, and let  $\gamma'$  denote the successor cardinal to  $\gamma$ . Then  $\gamma'$  will be a regular cardinal greater than the arity of every operation of  $\Omega$ ; hence given any set  $X$  and any  $s, t \in |F_\Omega(X)|$ , Lemma 8.2.3 tells us that  $s$  and  $t$  lie in the subalgebra generated by some subset  $X_0 \subseteq X$  of cardinality  $< \gamma'$ , hence  $\leq \gamma$ ; hence the set  $X_0$  can be mapped injectively into  $\gamma$ . Hence applying the first statement of this lemma to the inclusion of  $X_0$  in  $X$  on the one hand, and to an embedding of  $X_0$  in  $\gamma$  on the other, we see that  $(s, t)$  is equivalent to some identity in a  $\gamma$ -tuple of variables.  $\square$

Thus, for the purpose of studying families of identities satisfied by  $\Omega$ -algebras, and classes of algebras determined by identities, we can restrict ourselves to identities in a  $\gamma$ -tuple of variables for  $\gamma$  as above. In particular, identities in a countable set of variables suffice for the case of finitary algebras, and even for the case of algebras all of whose operations have countable arity.

In making the second assertion of the above result, why have we looked at a cardinal such that all operations have cardinalities  $\leq \gamma$ , rather than following the pattern recommended earlier, of looking at cardinals that strictly bound the quantities we are interested in? Although generally speaking, the latter pattern gives one greater flexibility in stating conditions, in this case the conclusion we wanted was that there was a single free algebra in terms of which all our identities could be expressed, so we wanted a particular value for the cardinality of a generating set; and the

non-strict inequalities used above yielded the smallest such cardinal.

Unfortunately, for considerations below involving direct limits, we will still want a cardinal satisfying strict inequalities. Hence

(8.4.3) For the remainder of this section,  $\Omega$  will denote a fixed type,  $\gamma_0$  will denote an infinite cardinal that is  $\geq \text{ari}(s)$  for all  $s \in |\Omega|$ , and  $\gamma_1$  will denote a regular infinite cardinal that is  $> \text{ari}(s)$  for all  $s \in |\Omega|$ . An *identity* will mean an  $\Omega$ -algebra identity in a  $\gamma_0$ -tuple of variables. In writing identities, we shall often write  $x_\alpha$  for the image  $u(\alpha) \in |F_\Omega(\gamma)|$  of  $\alpha \in \gamma$ . We may also at times write  $x, y$ , etc., for  $x_0, x_1$ , etc..

Note that if the smallest possible choice for  $\gamma_1$  is a successor cardinal, then the smallest possible choice for  $\gamma_0$  is its predecessor, while if the smallest choice for  $\gamma_1$  is a limit cardinal, the smallest choice for  $\gamma_0$  is the same cardinal. In particular, in the classical case where all operations are finitary,  $\aleph_0$  can be used for both  $\gamma_0$  and  $\gamma_1$ .

The next exercise shows that when all arities are 0 or 1 one can do still better than described above (though we will not use that observation in what follows).

**Exercise 8.4:1.** Show that if all operation-symbols of  $\Omega$  are of arity  $\leq 1$ , then the statement of Lemma 8.2.3 holds with  $\gamma = 2$  (even though 2 is not a regular cardinal), and deduce that the final statement of Lemma 8.4.2 also holds for  $\gamma = 2$ . On the other hand, show by example that it does not hold for  $\gamma = 1$ .

Let us denote the set of all  $\Omega$ -algebra identities by

$$(8.4.4) \quad I_\Omega = |F_\Omega(\gamma_0)| \times |F_\Omega(\gamma_0)|.$$

Thus we have a relation of *satisfaction* (Definition 8.4.1) defined between elements of the large set  $\text{Ob}(\Omega\text{-Alg})$  of all  $\Omega$ -algebras and elements of the small set  $I_\Omega$  of all identities. If  $C$  is a (not necessarily small) set of  $\Omega$ -algebras, let us for the moment write  $C^*$  for the set of identities satisfied by all members of  $C$ , and if  $J$  is a set of identities, let us write  $J^*$  for the (large) set of  $\Omega$ -algebras that satisfy all identities in  $J$ . The theory of Galois connections (§5.5) tells us that the two composite operators  $**$  will be closure operators, that every set  $J^*$  or  $C^*$  will be closed under the appropriate closure operator  $**$ , and that the operators  $*$  give an antiisomorphism between the complete lattice of all closed sets of algebras and the complete lattice of all closed sets of identities.

In talking about this Galois connection, it is obviously not convenient to apply to sets of algebras our convention that sets are small if the contrary is not stated; so we make

**Convention 8.4.5.** *For the remainder of this chapter, we suspend for sets of algebras (as we have done from the start for object-sets of categories) the assumption that sets are small if the contrary is not stated.*

(However, we still assume that any set of algebras is a subset of our universe  $\cup$  if the contrary is not stated; i.e., the smallness convention still applies to the underlying set of each algebra.)

**Definition 8.4.6.** A variety of  $\Omega$ -algebras means a full subcategory  $\mathbf{V}$  of  $\Omega\text{-Alg}$  having for object-set the set  $J^*$  of algebras determined by some set  $J$  of identities. The variety with object-set  $J^*$  will be written  $\mathbf{V}(J)$ . A category is called a variety of algebras if it is a variety of  $\Omega$ -algebras for some type  $\Omega$ .

If  $\mathbf{V}$  is a variety, an algebra belonging to  $\mathbf{V}$  will be called a  $\mathbf{V}$ -algebra. The least variety of

$\Omega$ -algebras whose object-set contains a given set  $C$  of algebras, that is, the full subcategory of  $\Omega\text{-Alg}$  with object-set  $C^{**}$ , is called the variety generated by  $C$ , written  $\mathbf{Var}(C)$ .

An equational theory for  $\Omega$ -algebras means a subset of  $I_\Omega$  (i.e., a set of identities for  $\Omega$ -algebras) which can be written  $C^*$  for some set  $C$  of  $\Omega$ -algebras;  $C^*$  is called the equational theory of the class  $C$ . If  $\mathbf{C}$  is a full subcategory of  $\Omega\text{-Alg}$ , then the equational theory of  $\text{Ob}(\mathbf{C})$  may also be called ‘the equational theory of  $\mathbf{C}$ ’. The least equational theory containing a set  $J$  of identities, namely,  $J^{**}$ , is called the equational theory generated by  $J$ .

Examples: The categories we have named **Group**, **Ab**, **Monoid**, **Semigroup**, **Ring**<sup>1</sup>, **CommRing**<sup>1</sup>,  $\vee$ -**Semilattice**,  $\wedge$ -**Semilattice** and **Lattice** are all varieties of algebras (up to trivial notational adjustment; e.g., we originally defined an object of **Group** as a 4-tuple  $(|G|, \mu, \iota, \varepsilon)$ ; under our present definition it is a pair  $(|G|, (\mu, \iota, \varepsilon))$ ). For every group  $G$ , the category  $G\text{-Set}$  is a variety; for every ring  $R$  the category  $R\text{-Mod}$  is a variety, and for every commutative ring  $k$  the category of all associative  $k$ -algebras is a variety. For every type  $\Omega$ , the whole category  $\Omega\text{-Alg}$  is a variety (the greatest element in the complete lattice of varieties of  $\Omega$ -algebras, definable by the empty set of identities. Its equational theory consists of the tautological identities  $(s, s)$ .) Taking for  $\Omega$  the trivial type, with no operation-symbols, we see that **Set** is (up to notational adjustment) a variety.

If  $\mathbf{C}$  is the full subcategory of **Monoid** consisting of those monoids all of whose elements are invertible, then  $\mathbf{C}$  is not a variety of algebras, since invertibility is not an identity; nevertheless, this category is *equivalent* (Definition 6.9:4) to the variety **Group**.

Finally, some categories we have looked at which are not varieties, and are not in any obvious way equivalent to varieties, are **POSet**, **Top**, **Set**<sup>OP</sup>, **RelSet**, the category of *complete* lattices, the full subcategory of **CommRing**<sup>1</sup> consisting of the *integral domains*, and the category of *torsion-free* groups (groups without elements of finite order other than  $e$ ). How to determine whether or not any of these is nonetheless equivalent to some variety of algebras is a question we are not yet ready to tackle.

**Remark 8.4.7.** An algebra  $A$  satisfies the identity  $x_0 = x_1$  if and only if all its elements are equal. Hence an algebra satisfying this identity satisfies all identities; i.e.,  $\{(x_0, x_1)\}^{**} = I_\Omega$ , the greatest element of the lattice of equational theories of  $\Omega$ -algebras. The corresponding variety of  $\Omega$ -algebras is the least element of the lattice of such varieties, and consists of algebras with *at most one* element. If  $\Omega$  has any zeroary operation-symbols, then this variety consists only of one-element algebras, which are all isomorphic; thus the variety is equivalent to the category **1** with only one object and its identity morphism. If  $\Omega$  has no zeroary operations, then this least variety contains both the empty algebra and all one-element algebras, and is equivalent to the 2-object category **2**.

Let us establish some easy results about varieties.

**Proposition 8.4.8.** *Let  $\mathbf{V} \subseteq \Omega\text{-Alg}$  be a variety. Then:*

- (i) *Any subalgebra of an algebra in  $\mathbf{V}$  again lies in  $\mathbf{V}$ .*
- (ii) *The limit  $\varprojlim_{\mathbf{D}} A(D)$ , taken in  $\Omega\text{-Alg}$ , of any functor  $A$  from a small category  $\mathbf{D}$  to  $\mathbf{V} \subseteq \Omega\text{-Alg}$  again lies in  $\mathbf{V}$ .*
- (iii) *Any homomorphic image of an algebra in  $\mathbf{V}$  again lies in  $\mathbf{V}$ .*
- (iv) *The direct limit (colimit)  $\varinjlim A_j$ , taken in  $\Omega\text{-Alg}$ , of any  $<\gamma_1$ -directed system of*

$\mathbf{V}$ -algebras again lies in  $\mathbf{V}$ . (For  $\gamma_1$  see (8.4.3). Lemma 8.1.10 describes this direct limit.)

In particular, the category  $\mathbf{V}$  has small limits, has coequalizers, and has colimits of  $<\gamma_1$ -directed systems, and all of these are the same in  $\mathbf{V}$  as in  $\Omega\text{-Alg}$ .

**Proof.** It is straightforward that if an algebra satisfies an identity, any subalgebra or homomorphic image satisfies the same identity, giving (i) and (iii) above, and that a direct product of algebras satisfying an identity again satisfies that identity. Since arbitrary limits can be constructed using products and equalizers, and in  $\Omega\text{-Alg}$  equalizers are certain subalgebras, we get (ii). To show (iv), let  $L$  be the direct limit in  $\Omega\text{-Alg}$  of a  $<\gamma_1$ -directed system of algebras  $(A_i)_I$  of  $\mathbf{V}$ , let  $(s, t)$  be an identity of  $\mathbf{V}$ , say involving the first  $\alpha < \gamma_1$  variables, which we regard as an identity  $(s', t')$  in an  $\alpha$ -tuple of variables, and let  $v$  be an  $\alpha$ -tuple of elements of  $L$ . By Lemma 8.1.10 (second paragraph)  $|L|$  is the direct limit of the sets  $|A_i|$ , hence by  $<\gamma_1$ -directedness of  $I$ , we can find an  $i \in I$  such that  $A_i$  contains inverse images of all members of  $v$ . The  $\alpha$ -tuple formed from these inverse images will satisfy the relation  $(s', t')$ , hence so does  $v$ , its image. Hence our direct limit object satisfies the identity  $(s', t')$ , and hence the equivalent identity  $(s, t)$ .

The final assertion follows immediately by Lemma 7.6.7.  $\square$

**Corollary 8.4.9.** *Let  $\mathbf{V}$  be a variety of  $\Omega$ -algebras. Then*

(i) *The forgetful functor from  $\mathbf{V}$  to  $\mathbf{Set}$  respects limits, and also colimits over  $<\gamma_1$ -directed partially ordered sets.*

(ii) *The inclusion functor into  $\mathbf{V}$  of any subvariety  $\mathbf{W}$  respects these constructions, and also respects coequalizers.*

(iii) *Direct limits in  $\mathbf{V}$  over  $<\gamma_1$ -directed partially ordered sets respect limits in  $\mathbf{V}$  over categories  $\mathbf{D}$  having  $<\gamma_1$  objects whose morphism-sets are generated by  $<\gamma_1$  morphisms.  $\square$*

**Exercise 8.4:2.** Verify that the above corollary indeed follows from results we have proved.

We saw in Lemma 6.9.3 that if a category  $\mathbf{C}$  is given with a concept of a *subobject* of an object, then one likewise gets a concept of a *subfunctor* of a  $\mathbf{C}$ -valued functor. Let us make, for future reference

**Definition 8.4.10.** *If  $\mathbf{V}$  is a variety of algebras, then unless the contrary is stated, references to subfunctors of  $\mathbf{V}$ -valued functors  $F$  are to be interpreted with “subobject” meaning “subalgebra”.*

*Thus, for any category  $\mathbf{C}$  and functor  $F: \mathbf{C} \rightarrow \mathbf{V}$ , a subfunctor  $G$  of  $F$  is (essentially) a construction associating to every  $X \in \text{Ob}(\mathbf{C})$  a subalgebra  $G(X) \subseteq F(X)$ , in such a way that for every morphism  $f: X \rightarrow Y$  of  $\mathbf{C}$ , the  $\mathbf{V}$ -algebra homomorphism  $F(f)$  carries  $G(X) \subseteq F(X)$  into  $G(Y) \subseteq F(Y)$ .*

The subfunctors of group- and vector-space-valued functors considered in the exercises following Lemma 6.9.3 are examples of this concept. (If you didn't do the last part of Exercise 6.9:12, this might be a good time to look at it again.)

Let us prove for general varieties a pair of facts that we noted earlier in many special cases.

**Proposition 8.4.11.** *A morphism  $f: A \rightarrow B$  in a variety  $\mathbf{V}$  is one-to-one if and only if it is a monomorphism, and surjective if and only if it is a coequalizer map.*

**Proof.** By Exercise 6.8:7, if  $f$  is one-to-one on underlying sets, it is a monomorphism. To get the converse, consider the congruence  $E$  associated to  $f$ . This is the underlying set of a subalgebra  $C$  of  $A \times A$ , hence it is an object of  $\mathbf{V}$ , and the projections of this object onto the two factors are morphisms  $C \rightrightarrows A$  having the same composite with  $f$ . Hence if  $f$  is a monomorphism, these two projections must be equal, which means that  $E$  can contain no nondiagonal elements of  $|A| \times |A|$ , which says that  $f$  is one-to-one. (Observe that this argument is not valid in an arbitrary full subcategory of  $\mathbf{V}$ , since such a subcategory may contain  $A$  and  $B$  without containing  $C$ . For an example, see Exercise 6.7:5.)

We have observed that coequalizers in  $\mathbf{V}$  are coequalizers in  $\Omega\text{-Alg}$ , and that these are surjective. Conversely, if  $f: A \rightarrow B$  is surjective, it is easy to verify that it has the universal property of the coequalizer in  $\mathbf{V}$  of the pair of maps  $C \rightrightarrows A$  defined in the preceding paragraph.  $\square$

The above result does not discuss the relation between onto-ness and being an *epimorphism*, nor between one-one-ness and the condition of being an *equalizer* map. If  $f$  is onto, then by the above proposition it is a coequalizer map, hence by Lemma 7.6.2 it is an epimorphism; likewise, by that lemma an equalizer map is a monomorphism, hence by the above proposition, is one-to-one. Neither of the converse statements is true, but they are tied together in a curious way:

**Exercise 8.4:3.** (i) Show that if a variety  $\mathbf{V}$  of algebras has an epimorphism which is not surjective (cf. Exercise 6.7:6(iii)) then it also has a one-to-one map which is not an equalizer.

(ii) Is the reverse implication true?

**Exercise 8.4:4.** The proof of Proposition 8.4.11 used the facts that  $\mathbf{V}$  is closed in  $\Omega\text{-Alg}$  under products and subalgebras. Which of these two conditions is missing in the example from Exercise 6.7:5 mentioned in that proof? Can you also find an example in which only the other condition is missing?

We turn now to constructions which are not the same in a variety  $\mathbf{V}$  and the larger category  $\Omega\text{-Alg}$ . We will get these via the next lemma. Let us give both the proof of that result based on the “big direct product” idea (Freyd’s Adjoint Functor Theorem), and the one based on “terms modulo consequences of identities”.

**Lemma 8.4.12.** *If  $\mathbf{V}$  is a variety of  $\Omega$ -algebras, the inclusion functor of  $\mathbf{V}$  into  $\Omega\text{-Alg}$  has a left adjoint.*

**First Proof.** We have seen that  $\mathbf{V}$  has small limits and that these are respected by the inclusion functor into  $\Omega\text{-Alg}$ , so by Freyd’s Adjoint Functor Theorem, it suffices to verify the solution-set condition. If  $A \in \text{Ob}(\Omega\text{-Alg})$ , then every  $\Omega$ -algebra homomorphism  $f$  of  $A$  into a  $\mathbf{V}$ -algebra  $B$  factors through the quotient of  $A$  by the congruence  $E$  associated to  $f$ . Since the factor-algebra  $A/E$  is isomorphic to a subalgebra of  $B$ , it belongs to  $\mathbf{V}$ . Hence the set of all factor-algebras of the given  $\Omega$ -algebra  $A$  which belong to  $\mathbf{V}$ , with the canonical morphisms  $A \rightarrow A/E$ , is the desired solution-set.

**Second Proof.** Let  $\mathbf{V}$  be  $\mathbf{V}(J)$ , the variety determined by the set of identities  $J \subseteq |F_{\Omega}(\gamma_0)| \times |F_{\Omega}(\gamma_0)|$ . Given  $A \in \text{Ob}(\Omega\text{-Alg})$ , let  $E \subseteq |A| \times |A|$  be the congruence on  $A$  generated by all pairs  $(f(s), f(t))$  with  $(s, t) \in J$ , and  $f: F_{\Omega}(\gamma_0) \rightarrow A$  a homomorphism. Then it is straightforward to verify that  $A/E$  belongs to  $\mathbf{V}$ , and is universal among homomorphic images of

$A$  belonging to  $\mathbf{V}$ .  $\square$

We shall call the above left adjoint functor the construction of *imposing the identities of  $\mathbf{V}$*  on an  $\Omega$ -algebra  $A$ . Note that if we impose the identities of  $\mathbf{V}$  on an algebra already in  $\mathbf{V}$ , we get the same algebra.

We can now get the rest of the constructions we want:

**Theorem 8.4.13.** *Let  $\mathbf{V}$  be a variety of  $\Omega$ -algebras. Then  $\mathbf{V}$  has small colimits, objects presented by generators and relations, and free objects on all small sets. All of these constructions can be achieved by performing the corresponding constructions in  $\Omega\text{-Alg}$ , and then imposing the identities of  $\mathbf{V}$  on the resulting algebras (i.e., applying the left adjoint obtained in the preceding lemma).*

**Proof.** The existence of these constructions in  $\Omega\text{-Alg}$  was shown in Theorems 8.3.8, 8.3.7 and 8.3.3. That left adjoints respect such constructions was proved in Theorems 7.8.3, 7.7.1, and 7.3.9.  $\square$

We note the

**Corollary 8.4.14** (to proof of Lemma 8.4.12). *If  $\mathbf{V} \subseteq \mathbf{W}$  are varieties of  $\Omega$ -algebras, then the inclusion functor of  $\mathbf{V}$  in  $\mathbf{W}$  has a left adjoint, given by the composite of the inclusion of  $\mathbf{W}$  in  $\Omega\text{-Alg}$  with the left adjoint of the inclusion of  $\mathbf{V}$  in  $\Omega\text{-Alg}$ .*

**Proof.** Given an algebra  $A$  in  $\mathbf{W}$ , the assertion to be proved is that if we regard  $A$  as an object of  $\Omega\text{-Alg}$  and as such impose on it the identities of  $\mathbf{V}$ , the resulting  $\mathbf{V}$ -algebra  $B$  will be universal as a  $\mathbf{V}$ -algebra with a  $\mathbf{W}$ -algebra homomorphism of  $A$  into it. This is immediate because a  $\mathbf{W}$ -algebra homomorphism  $A \rightarrow B$  is the same as an  $\Omega$ -algebra homomorphism  $A \rightarrow B$ .  $\square$

We remark that the above corollary does not follow from Lemma 8.4.12 by Theorem 7.3.9 (on composites of left adjunctions). Rather, it can be regarded as a special case of some important results not yet discussed. One, which will come in the next chapter, gives general conditions for a functor among varieties to have a left adjoint. It can also be gotten from Lemma 8.4.12 using a result on subcategories whose inclusion functors have left adjoints (“reflective subcategories” [17, §IV.3]), a topic I hope to look at in an as-yet-unwritten chapter.

**Exercise 8.4:5.** Give an example of a variety  $\mathbf{V}$  in which the free algebra on one generator  $x$  is also generated *non-freely* by some element  $y$ .

The next exercise, which leads up to an open question in part (iii), is about the left adjoint to the inclusion functor  $\mathbf{Bool}^1 \rightarrow \mathbf{CommRing}^1$ , though its statement does not use the word “functor”.

**Exercise 8.4:6.** Recall that  $\mathbf{Bool}^1$ , the variety of Boolean rings, is the subvariety of  $\mathbf{CommRing}^1$  determined by the one additional identity  $x^2 - x = 0$ .

(i) Show that the following conditions on a commutative ring  $R$  are equivalent: (a)  $R$  admits no homomorphism to the field  $\mathbb{Z}_2$ . (b)  $R$  admits no homomorphism to a nontrivial Boolean ring. (c) The ideal of  $R$  generated by all elements  $r^2 - r$  ( $r \in R$ ) is all of  $R$ . (Hint: from the fact that every commutative ring admits a homomorphism onto a field, deduce that every Boolean ring admits a homomorphism to  $\mathbb{Z}_2$ .)

Let us call a ring  $R$  satisfying the above equivalent conditions “Boolean-trivial”.

(ii) For every positive integer  $n$ , let  $T_n$  denote the commutative ring presented by  $2n$  generators  $a_0, \dots, a_{n-1}, b_0, \dots, b_{n-1}$  and one relation  $\sum a_i(b_i^2 - b_i) = 1$ . Show that a commutative ring  $R$  is Boolean-trivial if and only if for some  $n$  there exists a ring homomorphism  $T_n \rightarrow R$ . Show, moreover, that there exist certain homomorphisms among the rings  $T_n$ , which allow one to deduce that the above family of conditions forms a chain under implication.

Ralph McKenzie (unpublished) has raised the question

(iii) Is the above chain of implications eventually constant?

The above question is equivalent to asking whether beyond some point there exist homomorphisms in the “nonobvious” direction among the  $T_n$ . This seems implausible, but to my knowledge no one has found a way to prove that such homomorphisms do not exist, and the question is open.

For those familiar with the language of logic, what McKenzie actually asked was whether Boolean-triviality was a first-order condition. Since, as noted in the exercise, that condition is the disjunction of a countable chain of first-order conditions, his question is equivalent to asking whether it is given by one member of the chain.

His version of the question also differed from the above in that he on the one hand restricted attention to rings  $R$  of characteristic 2, and on the other hand did not restrict attention to commutative  $R$ . However, the general characteristic case is equivalent to the characteristic 2 case, since given a ring  $R$ , one can translate a first-order sentence about  $R/2R$  into a sentence about  $R$  by replacing relations “ $x = y$ ” with “ $(\exists z) x = y + 2z$ ”. Concerning commutativity, I felt that since the expected answer is negative, and since a negative answer in the commutative case would imply a negative answer in the general case, and since people are more familiar with commutative rings than with general rings, this modification of the question would be for the better.

One can, however, raise the same question about the relation between commutative and noncommutative rings, and it seems equally difficult to answer:

**Exercise 8.4:7.** Prove results analogous to (i)(b) $\Leftrightarrow$ (c) and to (ii) of the preceding exercise with the varieties **CommRing**<sup>1</sup> and **Bool**<sup>1</sup> replaced by **Ring**<sup>1</sup> and **CommRing**<sup>1</sup> respectively, and see whether you can make any progress on the question analogous to (iii) for this case.

We remark that one does not have a result analogous to Exercise 8.4:6(ii) for every pair consisting of a variety and a subvariety; e.g., **Group** and **Ab**. What is special about the varieties of the above two exercises is that triviality of an object is equivalent to a single relation,  $0 = 1$ . (Another variety with this property is **Lattice**<sup>0,1</sup>, the variety of lattices with greatest and least element made into zeroary operations.)

Returning to the general theory of varieties of algebras, let us extend some notation that we had set up for the categories  $\Omega\text{-Alg}$ :

**Definition 8.4.15.** *The free-object functor and the underlying-set functor associated with a variety  $\mathbf{V}$  will be denoted  $F_{\mathbf{V}}: \mathbf{Set} \rightarrow \mathbf{V}$  and  $U_{\mathbf{V}}: \mathbf{V} \rightarrow \mathbf{Set}$ . The  $\mathbf{V}$ -algebra presented by a generating set  $X$  and relation set  $R$  will be denoted  $\langle X | R \rangle_{\mathbf{V}}$ , or  $\langle X | R \rangle$  when there is no danger of confusion.*

In presenting a  $\mathbf{V}$ -algebra, it is often convenient to take a “relation” in an  $X$ -tuple of variables to mean a pair of elements of  $F_{\mathbf{V}}(X)$  rather than of  $F_{\Omega}(X)$ . If we write  $q: F_{\Omega}(X) \rightarrow F_{\mathbf{V}}(X)$  for the canonical homomorphism, it is clear that given  $(s, t) \in |F_{\Omega}(X)| \times |F_{\Omega}(X)|$  and an  $X$ -tuple  $v$  of elements of a  $\mathbf{V}$ -algebra  $A$ , the elements  $s$  and  $t$  will fall together under the

homomorphism  $F_{\Omega}(X) \rightarrow A$  determined by  $v$  if and only if  $q(s)$  and  $q(t)$  fall together under the homomorphism  $F_{\mathbf{V}}(X) \rightarrow A$  determined by  $v$ ; so the same condition is expressed by the original relation  $(s, t) \in |F_{\Omega}(X)| \times |F_{\Omega}(X)|$ , and by the induced pair  $(q(s), q(t)) \in |F_{\mathbf{V}}(X)| \times |F_{\mathbf{V}}(X)|$ . Thus, if  $Y$  is a subset of  $|F_{\mathbf{V}}(X)| \times |F_{\mathbf{V}}(X)|$ , we will often denote by  $\langle X | Y \rangle_{\mathbf{V}}$  the quotient of  $F_{\mathbf{V}}(X)$  by the congruence generated by  $Y$ .

When we considered the concept of *representable functors*  $\mathbf{C} \rightarrow \mathbf{Set}$ , we saw that for  $\mathbf{C} = \mathbf{Group}$ , a presentation of the group representing such a functor yielded a nice concrete description of the functor. One can turn this observation, in the context of arbitrary varieties of algebras, into a characterization of representable functors.

**Lemma 8.4.16.** *Let  $\mathbf{W}$  be a variety of algebras, and  $U: \mathbf{W} \rightarrow \mathbf{Set}$  a functor. Then the following conditions are equivalent:*

- (i)  $U$  is representable; i.e., there exists an object  $R$  of  $\mathbf{W}$  such that  $U$  is isomorphic to the functor  $h_R = \mathbf{W}(R, -)$ .
- (ii) There exists a set  $X$ , and a set of relations in an  $X$ -tuple of variables,  $Y \subseteq |F_{\Omega}(X)| \times |F_{\Omega}(X)|$ , such that  $U$  is isomorphic to the functor associating to every object  $A$  of  $\mathbf{W}$  the set  $\{\xi \in |A|^X \mid (\forall (s, t) \in Y) s_A(\xi) = t_A(\xi)\}$ .

**Proof.** If  $U$  is represented by  $R$ , take a presentation  $R = \langle X | Y \rangle_{\mathbf{W}}$ ; then  $U$  will have the form shown in (ii). Conversely, if  $U$  is as in (ii), it is represented by the algebra with presentation  $\langle X | Y \rangle_{\mathbf{W}}$ .  $\square$

Thus, we immediately see that such functors on  $\mathbf{Group}$  as  $G \mapsto \{x \in |G| \mid x^2 = e\}$  and  $G \mapsto \{(x, y) \in |G|^2 \mid xy = yx\}$  are representable. A less obvious case is the “set of invertible elements” functor on monoids. If we try to use the criterion of the above lemma with  $X$  a singleton, it does not work, because the condition of invertibility is not an equation in  $x$  alone. However, because inverses are *unique* when they exist, we see that this construction is isomorphic to the functor  $S \mapsto \{(x, y) \in |S|^2 \mid xy = e = yx\}$ , which is of the required form.

In condition (ii) of the above lemma,  $X$  and/or  $Y$  may, of course, be empty. If  $Y$  is empty, then  $U$  is the  $X$ th power of the underlying-set functor (Definition 6.8.5), and is represented by  $F_{\mathbf{V}}(X)$ . An example with  $X$  but not  $Y$  empty is the functor  $\mathbf{Ring}^1 \rightarrow \mathbf{Set}$  represented by  $\mathbb{Z}_n$  for an integer  $n$ . We recall that this ring is presented by the empty set of generators, and the one relation  $n = 0$  (where “ $n$ ” as a ring element means the  $n$ -fold sum  $1 + \dots + 1$ ). This ring admits no homomorphism to a ring  $A$  unless  $n = 0$  in  $A$ , while when  $A$  satisfies that equation, there is a unique ring homomorphism  $\mathbb{Z}_n \rightarrow A$  (namely the additive group map taking  $1_{\mathbb{Z}_n}$  to  $1_A$ ). Thus,  $h_{\mathbb{Z}_n}$  takes  $A$  to the empty set if the characteristic of  $A$  does not divide  $n$ , and to a one-element set if it does. In terms of point (ii) of the above lemma, this functor must be described as sending  $A$  to “the set of 0-tuples of elements of  $A$  such that  $n = 0$ ”. This sounds peculiar because the “such that” clause does not refer to anything in the preceding phrase; but it is logically correct: we get the unique 0-tuple if  $n = 0$  in  $A$ , and nothing otherwise.

**Exercise 8.4:8.** Determine which of the following set-valued functors are representable. In each case where the answer is affirmative, give an “ $X$ ” and “ $Y$ ” as in Lemma 8.4.16. In (i)-(v),  $n$  is a fixed integer.

- (i) The functor on  $\mathbf{Ring}^1$  taking  $A$  to a singleton if  $n$  is invertible in  $A$ , and to the empty set otherwise.

- (ii) The functor on **Ring**<sup>1</sup> taking  $A$  to its underlying set if  $n$  is invertible in  $A$ , and to the empty set otherwise.
- (iii) The functor on **Ab** taking  $A$  to the kernel of the endomorphism “multiplication by  $n$ ”.
- (iv) The functor on **Ab** taking  $A$  to the image of this endomorphism.
- (v) The functor on **Ab** taking  $A$  to the cokernel of this endomorphism.
- (vi) The functor on **Lattice** taking  $A$  to the set of pairs  $(x, y)$  such that  $x \leq y$ .
- (vii) For  $P$  a fixed partially ordered set, the functor on **Lattice** taking  $A$  to the set of isotone maps from  $P$  to the “underlying” partially ordered set of  $|A|$ .

We mentioned in §7.12 that it was shown in [42] that contravariant *left* adjunctions among “the kind of categories of algebras we will be studying in this course” were degenerate. We now have the language in which to state this result (which will not be proved here!) precisely: If  $W: \mathbf{V}^{\text{op}} \rightarrow \mathbf{W}$  and  $V: \mathbf{W}^{\text{op}} \rightarrow \mathbf{V}$  are mutually left adjoint contravariant functors between varieties of algebras, then all objects  $W(A)$  are epimorphs of the initial object of  $\mathbf{W}$  (i.e., codomains of epimorphisms with the initial object as domain), and all objects  $V(B)$  are epimorphs of the initial object of  $\mathbf{V}$ . The next exercise shows how to get trivial examples of such adjunctions, and then gives an example which, though rather unnatural, is nontrivial.

**Exercise 8.4:9.** (i) Show that if  $\mathbf{C}$  and  $\mathbf{D}$  are any two categories having initial objects, then the contravariant functors between  $\mathbf{C}$  and  $\mathbf{D}$  each of which takes every object of its domain to the initial object of its codomain, and takes all morphisms to the identity morphism of that object, are mutually left adjoint.

(ii) Let  $n$  be a positive integer, and **CommRing** <sub>$\mathbb{Z}_n$</sub> <sup>1</sup> the variety of commutative  $\mathbb{Z}_n$ -algebras; equivalently, commutative rings satisfying the identity  $n = 0$ . Show how to define a functor  $F: (\mathbf{CommRing}_{\mathbb{Z}_n}^1)^{\text{op}} \rightarrow \mathbf{CommRing}^1$  such that if  $R$  has characteristic  $m \mid n$ , then  $F(R) = \mathbb{Z}[m^{-1}]$ , and verify that  $F$  has a left adjoint.

(iii) Verify that the above functor and its adjoint both take arbitrary objects to epimorphs of the initial object of their codomain categories.

**8.5. Derived operations.** Having identified  $\Omega$ -algebra terms  $s$  with elements of free  $\Omega$ -algebras  $F_{\Omega\text{-Alg}}(X)$ , our viewpoint in “evaluating” these terms has been, “a choice of an  $X$ -tuple  $v$  of elements in an  $\Omega$ -algebra  $A$  induces an evaluation homomorphism  $F_{\Omega\text{-Alg}}(X) \rightarrow A$ ”. But as noted in §1.6, we can modify which variable(s) – the  $X$ -tuple  $v$ , the term  $s$ , or both – we foreground. We do this in the next definition, again replacing  $\Omega\text{-Alg}$  with a general variety  $\mathbf{V}$ .

**Definition 8.5.1.** Let  $\mathbf{V}$  be a variety of algebras,  $X$  a set, and  $(F_{\mathbf{V}}(X), u)$  the free  $\mathbf{V}$ -algebra on  $X$ . For every element  $s \in |F_{\mathbf{V}}(X)|$ , every  $\mathbf{V}$ -algebra  $A$ , and every  $X$ -tuple  $v$  of elements of  $|A|$ , let

$$\text{eval}(s, A, v) \in |A|$$

denote the image of the element  $s$  under the unique homomorphism  $f: F_{\mathbf{V}}(X) \rightarrow A$  such that  $fu = v$ . (Intuitively, the result of substituting into the term  $s$  the  $X$ -tuple  $v$  of elements of  $A$ .)

For fixed  $s$  and  $A$ , let us define

$$s_A: |A|^X \rightarrow |A|.$$

by

$$s_A(v) = \text{eval}(s, A, v).$$

A derived  $X$ -ary operation on  $A$  will mean a map  $|A|^X \rightarrow |A|$  which is equal to  $s_A$  for some  $s \in |F_{\mathbf{V}}(X)|$ .

More generally, given  $s$  and any full subcategory  $\mathbf{C}$  of  $\mathbf{V}$  (e.g., a one-object subcategory, or all of  $\mathbf{V}$ ), if we write  $U_{\mathbf{C}}: \mathbf{C} \rightarrow \mathbf{Set}$  for the restriction to  $\mathbf{C}$  of the underlying-set functor of  $\mathbf{V}$ , and  $U_{\mathbf{C}}^X: \mathbf{C} \rightarrow \mathbf{Set}$  for the functor carrying an object  $A$  to the set  $U_{\mathbf{C}}(A)^X$  (cf. Definition 6.8.5), then  $s_{\mathbf{C}}: U_{\mathbf{C}}^X \rightarrow U_{\mathbf{C}}$  will denote the morphism between these functors  $\mathbf{C} \rightarrow \mathbf{Set}$  which on each object  $A$  of  $\mathbf{C}$  acts by  $s_A$ . A morphism  $U_{\mathbf{C}}^X \rightarrow U_{\mathbf{C}}$  which can be written  $s_{\mathbf{C}}$  for some  $s \in |F_{\mathbf{V}}(X)|$  will be called a derived  $X$ -ary operation of  $\mathbf{C}$ .

Note that the derived operations will in particular include the primitive operations  $s_A: |A|^{\text{ari}(s)} \rightarrow |A|$  (respectively,  $s_{\mathbf{C}}: U_{\mathbf{C}}^{\text{ari}(s)} \rightarrow U_{\mathbf{C}}$ ) induced by the operation symbols  $s \in |\Omega|$ , and the projection operations  $p_{X,x}: |A|^X \rightarrow |A|$  (respectively  $U_{\mathbf{C}}^X \rightarrow U_{\mathbf{C}}$ ), induced by the free generators  $u(x) \in |F_{\mathbf{V}}(X)|$ .

Let us now follow up on some ideas that we toyed with at the end of §2.3. Given any full subcategory  $\mathbf{C}$  of our variety  $\mathbf{V}$ , consider the large set of all “generalized operations on  $\mathbf{C}$  in an  $X$ -tuple of variables”, i.e., functions  $f$  associating to each object  $A$  of  $\mathbf{C}$  a map  $f_A: |A|^X \rightarrow |A|$  in an arbitrary way. If we look at the set of all these generalized operations as a direct product,  $\prod_{A \in \text{Ob}(\mathbf{C})} |A|^{|A|^X}$  (living in the next larger universe), we see that it can be made the underlying set of a large  $\mathbf{V}$ -algebra, namely the product,  $\prod_{A \in \text{Ob}(\mathbf{C})} A^{|A|^X}$ ; let us denote this algebra by  $\text{GenOp}_{\mathbf{C}}(X)$ . We are not interested in this bloated monster for itself, but for the observation that the (still generally large) set of those elements thereof which constitute morphisms of functors  $U_{\mathbf{C}}^X \rightarrow U_{\mathbf{C}}$ , i.e.,  $\mathbf{Set}^{\mathbf{C}}(U_{\mathbf{C}}^X, U_{\mathbf{C}})$ , forms a subalgebra thereof. (A description of the  $\mathbf{V}$ -algebra structure on this hom set might have seemed unnatural without the context of the algebra structure on  $\text{GenOp}(\mathbf{C})$ , which is why we began with the latter. Incidentally, when we first discussed this in §2.3, we were not sure it made sense to talk about large sets. Having adopted the Axiom of Universes, and the associated interpretation of large sets, we can deal with these safely!) We shall call this the algebra of functorial  $X$ -ary operations on  $\mathbf{C}$ . The derived  $X$ -ary operations of  $\mathbf{C}$  form a subalgebra of this subalgebra:

$$(8.5.2) \quad \text{DerOp}_{\mathbf{C}}(X) \subseteq \mathbf{Set}^{\mathbf{C}}(U_{\mathbf{C}}^X, U_{\mathbf{C}}) \subseteq \text{GenOp}_{\mathbf{C}}(X).$$

Note that the algebra of derived operations is quasi-small, i.e., isomorphic to a small algebra, since it is a homomorphic image of  $F_{\mathbf{V}}(X)$ . The image of each generator  $x \in X$  will be the function carrying every  $X$ -tuple to its  $x$ th coordinate, thus, these “coordinate functions” generate  $\text{DerOp}_{\mathbf{C}}(X)$  as an  $\Omega$ -algebra. We can describe the resulting algebra nicely, and, under appropriate hypotheses, the algebra of functorial operations as well:

**Lemma 8.5.3.** *Let  $\mathbf{C}$  be a full subcategory of a variety  $\mathbf{V}$ , and  $X$  a (small) set. Then the (large) algebra of derived  $X$ -ary operations on  $\mathbf{C}$  is isomorphic to the (small) algebra  $F_{\mathbf{V}\text{ar}(\text{Ob}(\mathbf{C}))}(X)$ .*

*Moreover, if  $\mathbf{C}$  contains the free  $\mathbf{V}$ -algebra on  $X$ , then every functorial  $X$ -ary operation on  $\mathbf{C}$  is a derived operation; i.e.,  $\mathbf{Set}^{\mathbf{C}}(U_{\mathbf{C}}^X, U_{\mathbf{C}}) = \text{DerOp}_{\mathbf{C}}(X) \cong F_{\mathbf{V}}(X)$ .*

*In particular, if  $\mathbf{C} = \mathbf{V}$ , the above equality holds for all sets  $X$ .*

**Sketch of Proof.** The first assertion is straightforward, since for two terms  $s$  and  $t$ , we have  $s_{\mathbf{C}} = t_{\mathbf{C}}$  if and only if  $(s, t)$  is an identity of  $\mathbf{C}$ . To prove the second, assume  $\mathbf{C}$  contains the free algebra  $F_{\mathbf{V}}(X)$ , and verify that a functorial  $X$ -ary operation on  $\mathbf{C}$  is determined by its value

on the universal  $X$ -tuple  $u$  of elements of  $F_{\mathbf{V}}(X)$ ; equivalently, apply Yoneda's Lemma to the pair of functors  $U_{\mathbf{V}}^X \cong h_{F_{\mathbf{V}}(X)}$  and  $U_{\mathbf{V}} \cong h_{F_{\mathbf{V}}(1)}$ . The final assertion is clear.  $\square$

**Exercise 8.5:1.** Give the details of the above proof.

**Exercise 8.5:2.** (i) Show that if  $\mathbf{C}$  is the full subcategory of all *finite algebras* in  $\mathbf{V}$ , then the algebra of functorial  $X$ -ary operations on  $\mathbf{C}$  can be described as the inverse limit of all finite factor algebras of  $F_{\mathbf{V}}(X)$ . (Make this statement precise.)

(ii) Show that if  $\mathbf{V} = \mathbf{Group}$  and  $\mathbf{C}$  is as in (i), and  $X = 1$ , then the group of functorial unary operations on  $\mathbf{C}$  is uncountable. Give an explicit example of an operation in this group that is not a derived group-theoretic operation.

(iii) Interpret Exercise 6.9:7, especially part (ii) thereof, in terms of point (i) above, and if you had not yet successfully done that exercise, see whether you can make further progress on it.

In part (ii) above, the map from functorial operations on general groups to functorial operations on finite groups failed to be surjective. There are also situations where such maps fail to be one-to-one:

**Exercise 8.5:3.** (i) Give an example of a variety  $\mathbf{V}$  not generated by its finite algebras. (If possible, get such an example in which the variety is defined by finitely many operation-symbols, all of finite arities, and finitely many identities.)

(ii) Show that the property asked for in the first sentence above is equivalent to saying that the restriction map from functorial operations in finitely many variables on  $\mathbf{V}$  to such operations on the finite objects of  $\mathbf{V}$  is not one-to-one.

Since Exercise 8.5:2(ii) above shows that, though the variety of all groups has only countably many functorial operations of any finite arity, its full subcategory of *finite* groups has uncountably many such operations, one may ask whether, for  $\mathbf{C}$  a full subcategory of a variety  $\mathbf{V}$ , the class of functorial operations of  $\mathbf{C}$  need even be quasi-small!

The answer depends on one's foundational assumptions; I will briefly summarize the situation. Logicians have asked the question,

(8.5.4) Does there exist a proper class (in our language, a non-small set) of (small) models of some first-order theory, none of which is embeddable in another?

The answer to (8.5.4) turns out to depend on one's choice of universe. If  $\mathbb{U}$  is the smallest universe, or is a successor element in the well-ordered set of universes, the answer is yes. The negative answer, on the other hand, is called "Vopěnka's principle"; the *existence* of a universe for which this holds is equivalent to the existence of a cardinal with some special properties (which force it to be enormous) but which are thought likely to be consistent with ZFC.

Now the positive answer to (8.5.4), which, as noted, is true in "most" universes, is known to imply the existence of a non-small set  $C$  of small *algebras* of some finitary type  $\Omega$  such that there are no homomorphisms between distinct members of  $C$ . Given such a  $C$ , let  $\mathbf{C}$  be the full subcategory of  $\Omega\text{-Alg}$  with  $C$  as object-set. Then we see that the definition of a functorial operation  $f$  on  $\mathbf{C}$  involves no conditions relating the behavior of  $f$  on *different* objects. So, for instance, for every subset  $B \subseteq C$ , there is a functorial binary operation on  $\mathbf{C}$  which acts as the first-coordinate function on algebras in  $B$ , and as the second-coordinate function on algebras not in  $B$ . Thus, in "most" universes we have a class of algebras with a non-quasi-small set of functorial binary operations. (Cf. [106], [120].)

Let me end this section with some questions about operations on the real and rational numbers which, so far as I know, are open.

**Exercise 8.5:4.** (Harvey Friedman)

- (i) If we make the set of real numbers an algebra under the single binary operation  $a(x, y) = x^2 + y^3$ , does this algebra satisfy any nontrivial identities?
- (ii) If we make the set of nonnegative real numbers an algebra under the single binary operation  $c(x, y) = x^{1/2} + y^{1/3}$ , does this algebra satisfy any nontrivial identities?
- (iii) Does there exist a derived binary operation on the ring  $\mathbb{Q}$  of rational numbers which is one-to-one as a map  $|\mathbb{Q}| \times |\mathbb{Q}| \rightarrow |\mathbb{Q}|$ ?

If you cannot answer this last question, you might hand in proofs that the answer to the corresponding question for the ring of integers is “yes”, and for the ring of real numbers, “no”.

Another question posed by Friedman along the lines of (i) and (ii) above was whether the group of bijective maps  $\mathbb{R} \rightarrow \mathbb{R}$  generated by the two maps  $p(x) = x+1$  and  $q(x) = x^3$  is free on those two generators. This was answered affirmatively in [127], with 3 replaced by any odd prime. (See [61] for a simplified proof.) The result has subsequently been generalized to show, essentially, that the group of maps generated by exponentiation by all positive rational numbers and addition of all real constants is the coproduct of the two groups generated by these two sorts of maps [60], and, in another direction [33], to show that the group generated by exponentiation by positive rationals with odd numerator and denominator, addition of real algebraic numbers, and *multiplication* by nonzero real algebraic numbers, is the coproduct of the group generated by the above addition and multiplication operations and the group generated by the multiplication and exponentiation operations, with amalgamation of the subgroup of multiplication operations. (For the meaning of a coproduct of groups with amalgamation of a common subgroup, cf. Exercise 6.8:23(i) and paragraph preceding that exercise.)

Another open question that is somewhat similar to the above, though more a question in number theory than general algebra, is

**Exercise 8.5:5.** (B. Poonen) Does there exist a polynomial  $f \in \mathbb{Q}(x, y)$  such that  $f(\mathbb{Z} \times \mathbb{Z}) = \mathbb{N}$ ?

**8.6. Characterizing varieties and equational theories.** We observed at the end of §5.5 that when one obtains a Galois connection from a relation on a pair of sets,  $R \subseteq S \times T$ , the closure  $X^{**}$  or  $Y^{**}$  of a subset  $X \subseteq S$  or  $Y \subseteq T$  is constructed “from above”, namely as the set of members of the set  $S$  or  $T$  that satisfy certain conditions determined by members of the other set; and that a recurring type of mathematical question is how to describe these closures “from below”, as all elements obtainable from members of  $X$  or  $Y$  by iterating some constructions. In the case of the Galois connection between  $\Omega$ -algebras and identities, these questions are: Given a set  $C$  of  $\Omega$ -algebras, how can we construct from these algebras all the algebras in the set  $\text{Ob}(\mathbf{Var}(C)) = C^{**}$  that they generate; and given a set  $J$  of identities, how can we construct from these all members of the equational theory  $J^{**}$  that they generate? Answers to these questions should, in particular, give internal criteria for when a set of algebras is a variety, and for when a set of identities is an equational theory.

I said in §5.5 that a general approach to this kind of question is to look for operations which carry every set  $Y^*$  or  $X^*$  into itself, and having found as many as one can, to try to show that closure under these operations is sufficient, as well as necessary, for a set to be closed.

Now we have shown that a variety of algebras is closed under forming subalgebras, homomorphic images, products, and  $\langle \gamma_1 \rangle$ -directed direct limits. (Closure under general limits need

not be mentioned, since it is implied by closure under products and subalgebras. On the other hand, the existence of free objects, coproducts, etc., cannot be used in such a characterization, since they are only defined relative to the variety we are trying to construct.) The next result shows that three of the above four closure conditions suffice to characterize varieties.

In reading that result, recall that by Convention 8.4.5, sets  $C$  of algebras are not assumed small.

**Theorem 8.6.1** (Birkhoff’s Theorem). *Let  $\Omega$  be a type. Then a set of  $\Omega$ -algebras forms a variety if and only if it is closed under forming homomorphic images, subalgebras, and products (of small families).*

*In fact, if  $C$  is a set of  $\Omega$ -algebras, then any object of  $\mathbf{Var}(C)$  can be written as a homomorphic image of a subalgebra of a product of a small set of members of  $C$ .*

**Proof.** Clearly, it suffices to prove the final assertion. Let  $\mathbf{V} = \mathbf{Var}(C)$ ; then an algebra belonging to  $\mathbf{V}$  can be written as a *homomorphic image* of the free  $\mathbf{V}$ -algebra  $F_{\mathbf{V}}(X)$  for some set  $X$ , hence it suffices to show that  $F_{\mathbf{V}}(X)$  can be obtained as a *subalgebra* of a *product* of objects in  $C$ . To show this, let  $N \subseteq |F_{\Omega}(X)| \times |F_{\Omega}(X)|$  denote the set of all pairs  $(s, t)$  that are *not* identities of  $\mathbf{V}$ ; equivalently, which are not identities of all members of  $C$ . For each  $(s, t) \in N$ , choose an  $X$ -tuple  $v_{(s,t)}$  of elements of an algebra  $A_{(s,t)} \in C$  such that  $v_{(s,t)}$  fails to satisfy the relation  $(s, t)$ . Let  $P$  be the product algebra  $\prod_{(s,t) \in N} A_{(s,t)}$ , and let  $v: X \rightarrow |P|$  be the set map with  $(s, t)$ -component  $v_{(s,t)}$  for each  $(s, t) \in N$ . It follows from its definition that this  $X$ -tuple  $v$  satisfies none of the relations in  $N$ ; on the other hand, since  $P$  belongs to  $\mathbf{V}$ , it must satisfy all relations *not* in  $N$ . It is easily deduced that the subalgebra  $F \subseteq P$  generated by this  $X$ -tuple is isomorphic to the free algebra  $F_{\mathbf{V}}(X)$ .  $\square$

The last sentence of Theorem 8.6.1 is often expressed in operator language:

$$(8.6.2) \quad \mathbf{Var}(C) = \mathbf{HSP}(C).$$

To make this precise, let us fix a type  $\Omega$ , and let  $L_{\Omega}$  denote the large lattice of all subsets  $C \subseteq \mathbf{Ob}(\Omega\text{-Alg})$  which are closed under going to isomorphic algebras (i.e., satisfy  $T \cong S \in C \Rightarrow T \in C$ ). Thus  $L_{\Omega}$  is isomorphic to the lattice of all subsets of the set of isomorphism classes of algebras in  $\Omega\text{-Alg}$ .) For each  $C \in |L_{\Omega}|$ , let us define

$$\mathbf{H}(C) = \{\text{homomorphic images of algebras in } C\},$$

$$\mathbf{S}(C) = \{\text{subalgebras of algebras in } C\},$$

$$\mathbf{P}(C) = \{\text{products of algebras in } C\}.$$

Then (8.6.2) indeed expresses the last sentence of Theorem 8.6.1. (Except that  $\mathbf{Var}(C)$  should, more precisely, be  $\mathbf{Ob}(\mathbf{Var}(C))$ . But we will ignore that distinction in this discussion, to give this statement the form in which it is usually stated.)

(The restriction to classes closed under isomorphism is not assumed in all discussions of this topic, leading to somewhat capricious behavior of the above operators: For  $C$  a class of algebras not necessarily closed under isomorphism,  $\mathbf{H}(C)$  is nevertheless closed under going to isomorphic algebras, by the definition of “homomorphic image”, though it loses this property if the definition of this operator is changed to “quotients of members of  $C$  by congruences”. On the other hand,  $\mathbf{S}(C)$  is not generally closed under going to isomorphic algebras if  $C$  is not, but it acquires that property if one changes the criterion to “algebras embeddable in members of  $C$ ”. Whether  $\mathbf{P}(C)$

is closed under isomorphism depends on whether one defines “product” to mean “any object which can be given a family of ‘projection’ maps having the appropriate universal property”, as we do here, or as the “standard” set-theoretic product. Since these distinctions are irrelevant to the algebraic questions involved, it seems best to eliminate them by restricting attention to isomorphism-closed classes. These are called “abstract classes” by some authors, though I do not favor that term. Incidentally, while discussing this topic, we will, obviously, temporarily set aside our habit of using  $\mathbf{P}$  for “power set”.)

In view of (8.6.2), it is natural to examine the monoid of operators on  $|L_\Omega|$  generated by  $\mathbf{H}$ ,  $\mathbf{S}$  and  $\mathbf{P}$ . We see from that result that the product  $\mathbf{HSP}$  acts as a closure operator, and hence is *idempotent*:  $(\mathbf{HSP})^2 = \mathbf{HSP}$ . From this we can deduce further equalities, e.g.,  $\mathbf{SHSP} = \mathbf{HSP}$ . This deduction is clear when we think of  $\mathbf{H}$ ,  $\mathbf{S}$  and  $\mathbf{P}$  as closure operators; to abstract the argument, let  $Z$  denote the monoid of all operators  $\mathbf{A}: |L_\Omega| \rightarrow |L_\Omega|$  satisfying

- (a)  $(\forall C \in |L_\Omega|) \mathbf{A}(C) \supseteq C$  ( $\mathbf{A}$  is increasing),  
 (b)  $(\forall C, D \in |L_\Omega|) C \supseteq D \Rightarrow \mathbf{A}(C) \supseteq \mathbf{A}(D)$  ( $\mathbf{A}$  is isotone).

This monoid  $Z$  can be partially ordered by writing  $\mathbf{A} \geq \mathbf{B}$  if and only if for all  $C$ ,  $\mathbf{A}(C) \supseteq \mathbf{B}(C)$ . By (a), all elements of  $Z$  are  $\geq$  the identity operator, which we shall denote  $\mathbf{I}$ ; we see from (b) that  $\mathbf{B} \geq \mathbf{C} \Rightarrow \mathbf{AB} \geq \mathbf{AC}$ , and we see by the definition of  $\geq$  that  $\mathbf{B} \geq \mathbf{C} \Rightarrow \mathbf{BA} \geq \mathbf{CA}$ . Hence knowing only that  $\mathbf{H}, \mathbf{S}, \mathbf{P} \in Z$ , we can say that  $(\mathbf{HSP})^2 \geq \mathbf{SHSP} \geq \mathbf{HSP}$ ; hence, as claimed, the equality  $(\mathbf{HSP})^2 = \mathbf{HSP}$  implies  $\mathbf{SHSP} = \mathbf{HSP}$ .

Having illustrated how to calculate with these operators, we invite the reader to combine these methods with considerations of structures of  $\Omega$ -algebras in

**Exercise 8.6:1.** Describe explicitly the partially ordered monoid generated by the operators  $\mathbf{H}$ ,  $\mathbf{S}$  and  $\mathbf{P}$  on classes of  $\Omega$ -algebras for general  $\Omega$ ; i.e., determine the distinct products of these operators, their composition, and the order-relations among them. Are there finitely or infinitely many distinct operators? Which such operators are idempotent?

(When I say “for general  $\Omega$ ”, I mean that a relation  $\leq$  or  $=$  should be considered to hold if and only if it holds for *all*  $\Omega$ . Special cases will be looked at in the next exercise.)

The above is a large task, but an interesting one. To carry it out fully, you need counterexamples showing that each equality or inclusion that you do *not* assert actually fails to hold for some appropriately chosen set of algebras. However, a counterexample for one relation often turns out to be a counterexample for several, so the task is not unreasonably difficult.

There are numerous modifications of this problem. For example.

**Exercise 8.6:2.** Suppose we restrict the operators  $\mathbf{H}$ ,  $\mathbf{S}$ ,  $\mathbf{P}$  to classes of algebras in a particular variety  $\mathbf{V}$ ; then some additional inclusions and equalities may occur among the composites of these restricted operators. Investigate the partially ordered monoids of operators obtained when  $\mathbf{V}$  is **Set**, respectively **Group**, respectively **Ab**. You may add to this list.

One could enlarge the set of operators considered above, introducing, for instance,  $\mathbf{D} = \{\text{equalizers}\}$  (i.e.,  $\mathbf{D}(C) =$  the set of equalizers of pairs of homomorphisms among algebras of  $C$ ; thus,  $\mathbf{D} \leq \mathbf{S}$ ),  $\mathbf{P}_{\text{fin}} = \{\text{products of finite families}\}$ , and  $\mathbf{L} = \{\text{direct limits of directed systems}\}$ . (In considering these last two we should restrict attention to finitary algebras, or else replace “finite families” and “directed systems” by “families of  $< \gamma_1$  objects” and “ $< \gamma_1$ -directed systems” for  $\gamma_1$  as in the preceding section.) Results on the structure of the monoid generated by any subset of  $\{\mathbf{H}, \mathbf{S}, \mathbf{P}, \mathbf{D}, \mathbf{P}_{\text{fin}}, \mathbf{L}\}$ , or any other such family of natural operators, can be turned in as homework, but I will merely pose as an exercise the questions

**Exercise 8.6.3.** Can one in general strengthen (8.6.2) to

- (i)  $\mathbf{Var}(C) = \mathbf{HDP}(C)$ ?
- (ii)  $\mathbf{Var}(C) = \mathbf{HSP}_{\text{fin}}(C)$ ?

The proof of Birkhoff's Theorem leads us to examine the class of  $\Omega$ -algebras that are free in *some* variety.

**Proposition 8.6.3.** *Let  $\Omega$  be a type,  $F$  an  $\Omega$ -algebra,  $X$  a set, and  $u$  an  $X$ -tuple of elements of  $|F|$ . Then the following conditions are equivalent:*

- (i)  $(F, u)$  is a free algebra on the set  $X$  in some variety  $\mathbf{V}$  of  $\Omega$ -algebras.
- (ii)  $(F, u)$  is a free algebra on the set  $X$  in the variety generated by  $F$ .
- (iii)  $F$  is generated by the image of  $X$ , and there exists some full subcategory  $\mathbf{C}$  of  $\Omega\text{-Alg}$  containing  $F$  such that  $(F, u)$  is free in  $\mathbf{C}$  on the set  $X$ .
- (iv)  $F$  is generated by the image of  $X$ , and for every set map  $v: X \rightarrow |F|$ , there exists an endomorphism  $e$  of  $F$  such that  $v = eu$ . (If we assume  $u$  is an inclusion map, this latter condition can be stated, "Every map of  $X$  into  $|F|$  extends to an endomorphism of  $F$ .")
- (v)  $F$  is isomorphic to the quotient of  $F_{\Omega}(X)$  by a congruence  $E$  which is carried into itself by every endomorphism  $f$  of  $F_{\Omega}(X)$  (i.e., which satisfies  $(s, t) \in E \Rightarrow (f(s), f(t)) \in E$ ), and the map  $u$  is the composite of universal maps  $X \rightarrow |F_{\Omega}(X)| \rightarrow |F_{\Omega}(X)/E| \cong |F|$ .

**Proof.** We have (i) $\Rightarrow$ (ii) because a free algebra in a given concrete category is easily seen to remain free in any full subcategory which contains it; (ii) $\Rightarrow$ (iii) is immediate. The universal property of a free object gives (iii) $\Rightarrow$ (iv). To see (iv) $\Rightarrow$ (v), identify  $F$  with the quotient of  $F_{\Omega}(X)$  by the congruence  $E$  consisting of all relations satisfied by the  $X$ -tuple  $u$ . Then if  $f$  is an endomorphism of  $F_{\Omega}(X)$ , (iv) implies that  $f$  induces an endomorphism of  $F = F_{\Omega}(X)/E$ , which can be seen to be equivalent to the condition that  $E$  is carried into itself by  $f$ , which is the assertion of (v).

Finally, given (v) we see that the relations satisfied by  $u$  will be satisfied by every  $X$ -tuple of elements of  $F$ , i.e., will be identities of  $\mathbf{Var}(\{F\})$  in an  $X$ -tuple of variables, and conversely the identities of  $\mathbf{Var}(\{F\})$  are necessarily satisfied by  $u$ . Hence  $F$ , being generated by the image of the  $X$ -tuple  $u$ , which satisfies precisely those relations which are identities of  $\mathbf{Var}(\{F\})$  in an  $X$ -tuple of variables, is the free  $\mathbf{Var}(\{F\})$ -algebra on  $X$ , proving (i).  $\square$

**Exercise 8.6.4.** Suppose  $\mathbf{V} = \mathbf{Monoid}$  and  $\mathbf{C}$  is the class of monoids all of whose elements are invertible.

- (i) Show that  $\mathbf{C}$  has free algebras (i.e., that its underlying-set functor has a left adjoint), but that these are not the free algebras of  $\mathbf{Var}(\mathbf{C})$ .
- (ii) Show using this example that the requirement that  $F$  be generated by the image of  $X$  cannot be removed from condition (iii) or (iv) of Proposition 8.6.3; specifically, that if it is removed, the resulting conditions no longer imply condition (i) of that proposition.

From Proposition 8.6.3 we can deduce the corresponding result with  $\Omega\text{-Alg}$  replaced by an arbitrary variety  $\mathbf{V}$ . In particular, we record

**Proposition 8.6.4.** *Let  $\mathbf{V}$  be a variety. Then*

- (i) *If  $X$  is a set and  $u$  an  $X$ -tuple of elements of an object  $F$  of  $\mathbf{V}$ , then  $(F, u)$  is a free algebra in a subvariety  $\mathbf{W}$  of  $\mathbf{V}$  if and only if it is isomorphic to a quotient of the free*

$\mathbf{V}$ -algebra  $F_{\mathbf{V}}(X)$  by a congruence invariant under all endomorphisms of  $F_{\mathbf{V}}(X)$ .

(ii) If  $\gamma_0$  is an infinite cardinal greater than or equal to the arities of all operations of  $\Omega$ , the subvarieties of  $\mathbf{V}$  are in bijective correspondence with congruences on  $F_{\mathbf{V}}(\gamma_0)$  which are invariant under all endomorphisms of this algebra, each subvariety  $\mathbf{W}$  corresponding to the congruence determined by the natural map  $F_{\mathbf{V}}(\gamma_0) \rightarrow F_{\mathbf{W}}(\gamma_0)$ , and each endomorphism-invariant congruence  $E$  corresponding to the subvariety generated by  $F_{\mathbf{V}}(\gamma_0) / E$ .  $\square$

Point (ii) above, in the case where  $\mathbf{V} = \Omega\text{-Alg}$ , solves the problem of characterizing equational theories:

**Theorem 8.6.5.** Let  $\Omega$  be a type, and  $\gamma_0$  an infinite cardinal greater than or equal to the arities of all operations of  $\mathbf{V}$ . Then a subset  $J \subseteq |F_{\Omega}(\gamma_0)| \times |F_{\Omega}(\gamma_0)|$  is an equational theory if and only if it is a congruence on  $F_{\Omega}(\gamma_0)$ , and is carried into itself by all endomorphisms of  $F_{\Omega}(\gamma_0)$ ; in other words, if and only if it satisfies the following five conditions for all  $s, t, u$ , etc. in  $|F_{\Omega}(\gamma_0)|$  and  $\sigma \in |\Omega|$ . (In (iv) and (v),  $\sigma_{F_{\Omega}(\gamma_0)}$ ,  $s_{F_{\Omega}(\gamma_0)}$  and  $t_{F_{\Omega}(\gamma_0)}$  denote the derived operations on  $F_{\Omega}(\gamma_0)$  induced by  $\sigma$ ,  $s$  and  $t$ .)

- (i)  $(s, s) \in J$ .
- (ii)  $(s, t) \in J \Rightarrow (t, s) \in J$ .
- (iii)  $(s, t) \in J, (t, u) \in J \Rightarrow (s, u) \in J$ .
- (iv)  $(t_i, u_i)_{i \in \text{ari}(\sigma)} \in J^{\text{ari}(\sigma)} \Rightarrow (\sigma_{F_{\Omega}(\gamma_0)}(t_i), \sigma_{F_{\Omega}(\gamma_0)}(u_i)) \in J$ .
- (v)  $(s, t) \in J, (u_i)_{i \in \gamma_0} \in |F_{\Omega}(\gamma_0)|^{\gamma_0} \Rightarrow (s_{F_{\Omega}(\gamma_0)}(u_i), t_{F_{\Omega}(\gamma_0)}(u_i)) \in J$ .  $\square$

There is a standard name for the kind of algebras that have been the key to the above discussion:

**Definition 8.6.6.** A pair  $(F, u)$  with the equivalent properties of Proposition 8.6.3 (in particular, property (i)) is called a relatively free  $\Omega$ -algebra.

By Lemma 8.4.2 an algebra relatively free on  $\gamma_0$  generators uniquely determines the corresponding variety, but a relatively free algebra  $(F, u)$  on an  $\alpha$ -tuple of generators for  $\alpha < \gamma_0$  may be free in more than one variety; the free group on one generator, which is also a free abelian group on one generator, is a familiar example. The variety  $\mathbf{Var}(\{F\})$  used in the proof of Proposition 8.6.3(v) $\Rightarrow$ (i) will necessarily be the *smallest* variety in which  $(F, u)$  is free. The *largest* such variety is the variety defined by the identities in  $\alpha$  variables satisfied by  $F$ ; equivalently, having for identities the relations satisfied by the  $\alpha$ -tuple  $u$  in  $F$ . The details, and some examples, are indicated in

**Exercise 8.6:5.** (i) Let  $\mathbf{V}$  be a variety, and suppose  $F \in \text{Ob}(\mathbf{V})$  is relatively free on an  $\alpha$ -tuple  $u$  of indeterminates. Show that  $(F, u)$  is a free algebra in precisely those subvarieties  $\mathbf{U} \subseteq \mathbf{V}$  which contain the variety  $\mathbf{Var}(F)$  (defined by all identities satisfied in  $F$ ), and are contained in the subvariety of  $\mathbf{V}$  defined by the identities in  $\leq \alpha$  variables holding in  $F$ .

(ii) For  $\mathbf{V} = \mathbf{Group}$ ,  $\alpha = 1$ , and  $(F, u)$  the group  $\mathbb{Z}$ , with  $u$  selecting 1 as free generator, characterize group-theoretically those subvarieties of  $\mathbf{V}$  in which  $(F, u)$  is free.

(iii) Show that if again  $\mathbf{V} = \mathbf{Group}$ , but we now take  $\alpha = 2$ , and for  $(F, u)$  either the free group on 2 generators or the free abelian group on 2 generators, then in each case, the greatest and least subvarieties of  $\mathbf{V}$  in which this group is free coincide.

(iv) Are there any relatively free groups  $(F, u)$  on 2 generators such that the greatest and

least varieties of groups in which  $(F, u)$  is free are distinct?

(v) If  $\Omega$  is the type of groups, and  $(F, u)$  is either the free group on 2 generators or the free abelian group on 2 generators, show that the greatest and the least varieties of  $\Omega$ -algebras in which  $(F, u)$  is free do not coincide, but that if  $(F, u)$  is the free group or free abelian group on 3 generators, they again coincide.

Here are some exercises on subvarieties of familiar varieties.

**Exercise 8.6:6.** (If you do both parts below, give the proof of one in detail, and for the other give details where the proofs differ.)

(i) Let  $G$  be a group, and  $G\text{-Set}$  the variety of all  $G$ -sets. Show that subvarieties of  $G\text{-Set}$  other than the least subvariety (characterized in Remark 8.4.7) are in one-to-one correspondence with the normal subgroups  $N$  of  $G$ , in such a way that the subvariety corresponding to  $N$  is equivalent to the variety  $(G/N)\text{-Set}$ , by an equivalence which respects underlying sets.

(ii) Prove the analogous result for subvarieties of  $R\text{-Mod}$ , where  $R$  is an arbitrary ring. (In that case, the least subvariety is not an exceptional case.)

**Exercise 8.6:7.** (i) Let  $\mathbf{CommRing}^1$  denote the category of commutative rings. Show that if  $\mathbf{V}$  is a proper subvariety of  $\mathbf{CommRing}^1$  generated by an *infinite integral domain*, then  $\mathbf{V}$  is the variety  $\mathbf{V}_p$  determined by the 0-variable identity  $p = 0$  for some prime  $p$ , where the symbol “ $p$ ” in this identity is an abbreviation for  $1+1+\dots+1$  with  $p$  summands.

(ii) Show that the subvariety  $\mathbf{Bool}^1 \subseteq \mathbf{CommRing}^1$  is a *proper* subvariety of the variety  $\mathbf{V}_2$  defined as in (i).

**Exercise 8.6:8.** Let  $F = F\mathbf{Ring}_1(\omega)$ , the free associative (noncommutative) ring on indeterminates  $x_0, x_1, \dots$ . For each positive integer  $n$ , let

$$S_n = \sum_{\pi} (-1)^{\pi} x_{\pi(0)} \dots x_{\pi(n-1)} \in F,$$

where  $\pi$  ranges over the permutations on  $n$  elements, and  $(-1)^{\pi}$  denotes  $+1$  if  $\pi$  is an even permutation,  $-1$  if  $\pi$  is odd.

(i) Show that any ring satisfying  $S_n = 0$  also satisfies  $S_{n'} = 0$  for all  $n' \geq n$ ; i.e., that  $(S_{n'}, 0) \in \{(S_n, 0)\}^{**}$ .

(ii) Show that for every  $d > 0$  there exists  $n > 0$  such that for every commutative ring  $k$ , the ring  $M_d(k)$  of  $d \times d$  matrices over  $k$  satisfies the identity  $S_n = 0$ .

(iii) Show that for every  $n > 0$  there exists  $d > 0$  such that  $M_d(k)$  does not satisfy  $S_n = 0$  for any nontrivial commutative ring  $k$ .

(iv) Deduce that there is an infinite chain of distinct varieties of rings of the form  $\mathbf{V}(\{(S_n, 0)\})$ , and an infinite chain of distinct varieties of rings of the form  $\mathbf{Var}(\{M_d(\mathbb{Z})\})$ . (In these symbols, the expressions in set-brackets denote singletons.)

Note on the above exercise: The *least*  $n$  such that all  $d \times d$  matrix rings  $M_d(k)$  over commutative rings  $k$  satisfy  $S_n = 0$  is  $2d$ . The hard part of this result, namely that  $M_d(k)$  satisfies  $S_{2d} = 0$ , is known as the Amitsur-Levitzki Theorem [35]. All known proofs are either messy (e.g., by graph theory [116]) or tricky (e.g., using exterior algebras [109]). The student is invited to attempt to find a new proof! Part (ii) of the above exercise can be done relatively easily, however, using a larger-than-optimal  $n$ .

The study of varieties of noncommutative rings is called the theory of *rings with polynomial identity*, affectionately known as *PI rings*. See [110, Chapter 6] for an introduction to this subject.

Here is a curious variety closely related to the variety of groups.

**Exercise 8.6:9.** Let  $\Omega$  be the type defined by a single ternary (i.e., “3-ary”) operation-symbol,  $\tau$ . Let  $H: \mathbf{Group} \rightarrow \Omega\text{-Alg}$  be the functor taking a group  $G$  to the  $\Omega$ -algebra with underlying set  $|G|$  and operation

$$(8.6.7) \quad \tau(x, y, z) = xy^{-1}z.$$

(i) Show that the objects  $H(G)$  are the nonempty algebras in a certain subvariety of  $\Omega\text{-Alg}$ , and give a set  $J$  of identities defining this variety.

The algebras (empty and nonempty) in this variety are called *heaps*, so we shall call the variety **Heap**.

(ii) Show that for groups  $G$  and  $G'$ , one has

$$H(G) \cong H(G') \text{ in } \mathbf{Heap} \Leftrightarrow G \cong G' \text{ in } \mathbf{Group}.$$

(iii) Show, however, that not every isomorphism between  $H(G)$  and  $H(G')$  has the form  $H(i)$  for  $i$  an isomorphism between  $G$  and  $G'$ !

(iv) Show that the following categories are equivalent: (a) **Group**, (b) the variety of algebras  $(|A|, \tau, \iota)$  where  $(|A|, \tau)$  is a heap, and  $\iota$  is a zeroary operation, subject to no further identities (intuitively, “heaps with distinguished elements  $\iota$ ”), (c) **Heap**<sup>Pt</sup>, where the construction **C**<sup>Pt</sup> is defined as in Exercise 6.8:3.

(v) Show that if  $X, Y$  are two objects of any category **C**, then the set of isomorphisms  $X \rightarrow Y$  forms a heap under the operation  $\tau(x, y, z) = xy^{-1}z$ . How is the structure of this heap related to those of the groups  $\text{Aut}(X)$  and  $\text{Aut}(Y)$ ?

The concept of heap is not very well known, and many mathematicians have from time to time rediscovered it and given it other names (myself included). Heaps were apparently first studied by Prüfer [107] and Baer [37], under the name *Schar* meaning “crowd” or “flock”, a humorous way of saying “something like a group”. The term was rendered into Russian by Suškevič [115] as *гpyдa*, meaning “heap”, which gave both a loose approximation of the meaning of *Schar*, and a play on the sounds of the Russian words: “group” = *gruppa*, “heap” = *gruda*. Since the concept and its generalizations have gotten most attention in Russian-language works, it has come back into Western European languages via translations of this Russian term rather than of the original German. (Incidentally, there is an unrelated notion with the name “heap” in the theory of data structures [126, p.72].)

Part (ii) of the above exercise shows that there is no need for a separate theory of the *structure* of heaps; this is essentially contained in that of groups. However, the variety of heaps is both a taking-off point for various generalizations (“semiheaps” etc.), and a source of examples in general algebra and category theory.

Point (iv) of the preceding exercise suggests

**Exercise 8.6:10.** (i) For what varieties **V** is it true that the category **V**<sup>Pt</sup> can be identified with the variety gotten by adding to **V** one zeroary operation, and no additional identities?

(ii) What varieties **V** satisfy the conditions (a)-(d) of Exercise 6.8:2? (Note that for varieties these conditions are all equivalent, by the last part of that exercise.)

Let us remark that in stretching the concept of “variety” from its classical definition as a class of algebras defined by identities to our present category-theoretic use, we have pulled it over a lot of ground, so that care is needed in using the term. For example, when should we think of two varieties as being “essentially the same”? If they are precisely equal? If we can establish a bijection between their types such that they are defined by the corresponding identities? If they are equivalent as categories? If there is a category-theoretic equivalence which also respects the underlying-set functors of the varieties?

There is no right answer, but these four conditions are all inequivalent.

**Exercise 8.6:11.** What implications exist among the above four conditions on a pair of varieties? Give examples showing that no two of those conditions are equivalent.

**8.7. Lie algebras.** Let me digress here to introduce a variety important in algebra, geometry, and differential equations, that of *Lie algebras*. I have referred to these in previous chapters in a few comments “for the reader familiar with the concept”. The reader who prefers to remain unfamiliar with it for the time being may skip this section, and perhaps come back later on. In subsequent sections, Lie algebras will be again be referred to only in occasional exercises and remarks.

To motivate the definition, consider an associative algebra  $A$  over a field (or more generally, over a commutative ring)  $k$ , and suppose we look at the underlying set of  $A$  together with its *operations of  $k$ -vector-space* (or  $k$ -module), and the *commutator bracket operation*,

$$(8.7.1) \quad [x, y] = xy - yx.$$

These operations obviously satisfy the identities saying

$$(8.7.2) \quad +, -, 0 \text{ and the scalar multiplications by members of } k \text{ make } |A| \text{ a } k\text{-module, and } [-, -] \text{ is a } k\text{-bilinear operation with respect to this } k\text{-module structure.}$$

There is a further obvious identity satisfied by  $[-, -]$ , and another that, though not so obvious, is straightforward to verify:

$$(8.7.3) \quad [x, x] = 0 \quad (\text{alternating identity}),$$

$$(8.7.4) \quad [x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0 \quad (\text{Jacobi identity}).$$

Note that in the presence of (8.7.2), (8.7.3) implies (and if 2 is invertible in  $k$ , is equivalent to)

$$(8.7.5) \quad [x, y] + [y, x] = 0 \quad (\text{anticommutativity}).$$

The expansion of (8.7.4) in terms of the multiplication of  $A$  involves 12 terms, and writing these out it is not hard to check that the identity holds, but the following slightly simpler verification gives some useful insight. Recall that a *derivation* on a  $k$ -algebra (associative or not) means a  $k$ -linear map  $D$  satisfying the identity

$$(8.7.6) \quad D(yz) = D(y)z + yD(z).$$

Now it is easy to check that if  $x$  is any element of an associative  $k$ -algebra  $A$ , then the operation  $[x, -]$  is a derivation on  $A$ ; that is, for all  $y$  and  $z$ ,

$$(8.7.7) \quad [x, yz] = [x, y]z + y[x, z].$$

A map that is a derivation with respect to a given multiplication is also a derivation with respect to the opposite multiplication,  $y*z = zy$ . Subtracting the opposite multiplication from the original multiplication gives the commutator map (8.7.1); so by (8.7.7)  $[x, -]$  also acts as a derivation with respect to that map:

$$(8.7.8) \quad [x, [y, z]] = [[x, y], z] + [y, [x, z]].$$

We can use anticommutativity to rearrange this identity so that the bracket arrangement of the second term, like that of the other two, becomes  $[-, [-, -]]$ , and so that the last term has the same

cyclic order of  $x$ ,  $y$  and  $z$  as the other two terms. Bringing all three terms to the same side, we see that the above formula becomes precisely (8.7.4). Thus, the Jacobi identity (8.7.4) is equivalent to the condition (8.7.8), saying that the commutator bracket operation with one variable fixed is a derivation with respect to the commutator bracket operation as a function of two variables!

One now makes

**Definition 8.7.9.** *Let  $k$  be a commutative ring (often assumed a field). Then a Lie algebra over  $k$  means a  $k$ -module given with a  $k$ -bilinear operation  $[-, -]$  which is alternating and satisfies the Jacobi identity; in other words, a set  $|A|$  with operations  $+$ ,  $-$  and  $0$ , a “scalar multiplication” operation corresponding to each element of  $k$ , and a binary operation  $[-, -]$ , satisfying (8.7.2)-(8.7.4).*

*The variety of Lie algebras over  $k$  will be denoted  $\mathbf{Lie}_k$ .*

*For an element  $x$  of a Lie algebra  $L$ , the map  $[x, -]: |L| \rightarrow |L|$  is often denoted  $\text{ad}_x$ . (This stands for “adjoint”, but for obvious reasons we will not call it by that name here.)*

In view of the way we motivated (8.7.2)-(8.7.4), we see that if we write  $\mathbf{Ring}_k^1$  for the category of associative  $k$ -algebras, we have a functor

$$B: \mathbf{Ring}_k^1 \rightarrow \mathbf{Lie}_k$$

taking each associative  $k$ -algebra  $A$  to the Lie algebra with the same underlying  $k$ -module, and with bracket operation given by the commutator bracket (8.7.1). It is not hard to do

**Exercise 8.7:1.** Show that  $B$  has a left adjoint

$$E: \mathbf{Lie}_k \rightarrow \mathbf{Ring}_k^1.$$

This is called the *universal enveloping algebra* construction.

In a future chapter on normal forms, I hope to prove the Poincaré-Birkhoff-Witt Theorem. That theorem gives a normal form for  $E(L)$  when  $L$  is free as a  $k$ -module (as is automatic if  $k$  is a field), and in particular, shows that the maps giving the unit of the above adjunction,  $\eta(L): L \rightarrow B(E(L))$ , are then one-to-one. Thus, every Lie algebra over a field can be “embedded in” an associative algebra. An important consequence is

**Exercise 8.7:2.** Suppose  $k$  is a field.

- (i) Assuming, as asserted above, that for all Lie algebras  $L$ , the map  $\eta(L)$  is one-to-one, show that the Lie algebras of the form  $B(L)$  generate the variety  $\mathbf{Lie}_k$ .
- (ii) Deduce that every identity satisfied by the  $k$ -module structure and the derived operation  $[-, -]$  in all associative  $k$ -algebras  $A$  is a consequence of the identities (8.7.2)-(8.7.4).
- (iii) Describe how one can use the normal form for free associative  $k$ -algebras found in §3.12 to test whether two terms in the Lie operations and an  $X$ -tuple of generator-symbols represent the same element of the free Lie algebra  $F_{\mathbf{Lie}_k}(X)$ .

(This is not quite the same as having a normal form in  $F_{\mathbf{Lie}_k}(X)$ , but it is useful in many of the same ways. Normal forms for  $F_{\mathbf{Lie}_k}(X)$  have been found, but they are messy.)

If  $R$  is any  $k$ -algebra (which for the moment need not even be associative), and  $S$  is the associative  $k$ -algebra of all  $k$ -linear maps (i.e.,  $k$ -module homomorphisms)  $R \rightarrow R$ , then it easy to verify that if  $s, t \in S$  are both derivations (i.e., satisfy (8.7.6)), then  $[s, t] \in S$  is also a derivation. Thus, the  $k$ -derivations on  $R$  form a Lie subalgebra  $\text{Der}_k(R) \subseteq B(S)$ ; we will just write this  $\text{Der}(R)$  when there is no danger of ambiguity. For  $R$  a Lie algebra or an associative algebra, a derivation of the form  $\text{ad}_x = [x, -]$  is called an *inner derivation*.

**Exercise 8.7:3.** Let  $R$  be a not necessarily associative  $k$ -algebra, with multiplication denoted  $*$ , and for  $x \in |R|$  let  $\text{Ad}_x: |R| \rightarrow |R|$  denote the map  $y \mapsto x*y - y*x$ . (Thus if  $R$  is associative,  $\text{Ad}_x$  coincides with the operation  $\text{ad}_x$  of the Lie algebra  $B(R)$ , while if  $R$  is a Lie algebra, so that  $*$  denotes  $[-, -]$ ,  $\text{Ad}_x$  will be  $2 \text{ad}_x$ .)

Write down the identity that  $R$  must satisfy for all of the maps  $\text{Ad}_x$  to be derivations. Show that if  $R$  is anticommutative (satisfies  $x*y + y*x = 0$ ) and 2 is invertible in  $k$ , this identity is equivalent to the Jacobi identity, but that in general (in particular, if  $R$  is associative) it is not.

In terms of the “ad” notation, we can get yet another interpretation of the Jacobi identity. It is not hard to check that (8.7.8) is equivalent to

$$(8.7.10) \quad \text{ad}_{[y, z]} = \text{ad}_y \text{ad}_z - \text{ad}_z \text{ad}_y.$$

Thus the Jacobi identity also tells us that  $\text{ad}: L \rightarrow \text{Der}(L)$  is a homomorphism of Lie algebras.

If  $R$  is a commutative algebra, we see that the Lie algebra  $B(R)$  has trivial bracket operation. However, even for such  $R$ , the associative  $k$ -algebra of  $k$ -module endomorphisms of  $R$  is in general noncommutative, hence the Lie algebra  $\text{Der}(R)$ , a sub-Lie-algebra thereof, can have nonzero bracket operation; and indeed, such Lie algebras  $\text{Der}(R)$  are important in commutative ring theory and differential geometry.

For example, let us take for  $k$  the field  $\mathbb{R}$  of real numbers, and for  $R$  the commutative  $\mathbb{R}$ -algebra of  $C^\infty$  (i.e., infinitely differentiable) real-valued functions on  $\mathbb{R}^n$  for some positive integer  $n$ . Then the derivations on  $R$  are precisely the left  $R$ -linear combinations of the  $n$  derivations  $\partial/\partial x_1, \dots, \partial/\partial x_n$  (see next exercise). Geometers identify the derivation  $D = \sum a_i(x) \partial/\partial x_i$  ( $a_i(x) \in R$ ) with the  $C^\infty$  vector field  $a(x) = (a_1(x), \dots, a_n(x))$ , because for  $f \in R$ ,  $D(f)$  gives, at each point  $p$ , the rate of change of  $f$  that would be seen by a particle at  $p$  moving with velocity  $a(p)$ . In this way they make such vector fields into a Lie algebra; more generally, they do this with the vector fields on any  $C^\infty$  manifold.

**Exercise 8.7:4.** Let  $R$  be the ring of  $C^\infty$  functions on  $\mathbb{R}^n$ , and  $D: R \rightarrow R$  any  $\mathbb{R}$ -linear derivation. For  $i = 0, \dots, n-1$ , let  $a_i(x) = D(x_i)$ , where  $x_i$  denotes the  $i$ th projection map  $\mathbb{R}^n \rightarrow \mathbb{R}$ . You will show below that

$$D = \sum a_i(x) \partial/\partial x_i,$$

in other words, that for all  $f \in R$  and  $p \in \mathbb{R}^n$ ,  $D(f)(p) = \sum a_i(p) (\partial/\partial x_i f)(p)$ .

The plan of the proof is to verify that near  $p$ , the function  $f(x)$  looks like  $f(p) + \sum (\partial/\partial x_i f)(p) \cdot (x_i - p_i)$  plus a second-order remainder term. If one can show that  $D$  takes this remainder to a function that is 0 at  $p$ , then by applying  $D$  to the above expression and evaluating at  $p$ , one will get the desired value for  $D(f)$  at that point.

We begin with two general facts about derivations that we will need:

(i) Show that a  $k$ -linear derivation on any  $k$ -algebra  $R$  with multiplicative neutral element 1 has  $k$  (i.e.,  $k \cdot 1$ ) in its kernel.

(ii) Show that if  $I$  is an ideal of a commutative ring  $R$ , and we write  $I^2$  for the ideal spanned by  $\{gh \mid g, h \in I\}$ , then for any derivation  $D: R \rightarrow R$ , we have  $D(I^2) \subseteq I$ .

Now let  $R$ ,  $D$ ,  $f$  and  $p$  be as in the first paragraph above, and let  $I_p = \{a \in R \mid a(p) = 0\}$ , an ideal of  $R$ . In the next step you will show that

$$f(x) = f(p) + \sum (\partial/\partial x_i f)(p)(x_i - p_i) + \text{term in } I_p^2.$$

(iii) For any point  $x \in \mathbb{R}^n$ , evaluate  $f(x) - f(p)$  by the Fundamental Theorem of Calculus, applied along the line-segment from  $p$  to  $x$  parametrized by  $t \in [0, 1]$ . From the summand of this integral involving each operator  $\partial/\partial x_i$ , extract a factor  $x_i - p_i$ . Show that the integral remaining as the coefficient of  $x_i - p_i$  is, as a function of  $x$ , a member of  $R$  whose value at

$p$  is  $(\partial/\partial x_i f)(p)$ ; i.e., that the integral equals this constant plus a member of  $I_p$ . (In showing that these functions are  $C^\infty$ , you can use the fact that if  $g(x_0, \dots, x_{n-1}, t)$  is a  $C^\infty$  function of  $n+1$  variables, then  $\int_0^1 g(x_0, \dots, x_{n-1}, t) dt$  is a  $C^\infty$  function of  $n$  variables.)

(iv) Complete the proof of the desired formula for  $D$ .

An analogous purely algebraic construction is to start with any commutative ring  $k$ , regard the polynomial ring  $R = k[x_1, \dots, x_n]$  as “functions on affine  $n$ -space over  $k$ ”, and regard its Lie algebra of derivations as “polynomial vector fields”. These are easily shown (e.g., by calculation on monomials) to have the form  $\sum a_i(x) \partial/\partial x_i$ , where the derivations  $\partial/\partial x_i$  are this time the operations of formal partial differentiation, and again  $a_i(x) \in R$ .

It turns out that Lie algebras of vector fields on  $n$ -dimensional space, in either the geometric or algebraic sense, satisfy some additional identities, beyond those satisfied by all Lie algebras. The case  $n = 1$  is examined in the next exercise.

**Exercise 8.7:5.** (i) Let  $C^\infty(\mathbb{R}^1)$  denote the ring of  $C^\infty$  functions on the real line  $\mathbb{R}^1$ , and consider the Lie algebra of vector fields  $L(\mathbb{R}^1) = \{f d/dx \mid f \in (C^\infty(\mathbb{R}^1))\}$ . Verify that the Lie bracket operation on  $L(\mathbb{R}^1)$  is given by the formula

$$[f d/dx, g d/dx] = (fg' - gf') d/dx.$$

For notational convenience, let us regard this as a Lie algebra structure on  $|C^\infty(\mathbb{R}^1)|$ :

$$[f, g] = fg' - gf'.$$

We shall continue to denote this Lie algebra  $L(\mathbb{R}^1)$ .

(ii) Show that  $L(\mathbb{R}^1)$  does not generate  $\mathbf{Lie}_{\mathbb{R}}$ . (You will want to find an identity satisfied in  $L(\mathbb{R}^1)$ , but not in all Lie algebras. To see how to prove the latter property, cf. Exercise 8.7:2(iii).)

The above result requires some computational dirty-work. On the other hand, even if you do not do that part, a little ingenuity will allow you to do the remaining parts.

(iii) Show that for every positive integer  $n$ ,  $L(\mathbb{R}^1)$  contains a subalgebra which is free on  $n$  generators in  $\mathbf{Var}(L(\mathbb{R}^1))$ .

(iv) Show that  $\mathbf{Var}(\text{Der}(\mathbb{R}[x])) = \mathbf{Var}(L(\mathbb{R}^1))$ , i.e., that polynomial vector fields satisfy no identities not satisfied by  $C^\infty$  vector fields. However, show that in contrast to (iii),  $\text{Der}(\mathbb{R}[x])$  does *not* contain a subalgebra which is free on more than one generator in this variety.

You can carry the idea of part (ii) farther if you are interested, letting  $n$  be a positive integer and looking at the variety generated by the Lie algebra  $L(\mathbb{R}^n)$  of  $C^\infty$  vector fields on  $\mathbb{R}^n$ , and possibly various subalgebras, such as the subalgebra of vector fields with zero divergence (in the sense defined in multivariable calculus).

One of the most important interpretations of Lie algebras lies outside the scope of this course, and I will only sketch it: the connection with Lie groups.

A *Lie group* is a topological group  $G$  whose underlying topological space is a manifold. Typical examples are the rotation group of real 3-space, which is a 3-dimensional compact Lie group, and the group of motions of 3-space generated by rotations and translations, which is 6-dimensional and noncompact. Some degenerate but important examples are the real line, which is 1-dimensional, its compact homomorphic image the circle group  $\mathbb{R}/\mathbb{Z}$ , and finally, the discrete groups, which are the zero-dimensional Lie groups. It is known that every Lie group admits a unique  $C^\infty$  structure respected by the group operations.

If  $G$  is a Lie group,  $e$  its identity element, and  $T_e$  the tangent space to  $G$  at  $e$ , then every tangent vector  $t \in T_e$  extends by left translation to a left-translation-invariant vector field on  $G$ . Hence the space of left-invariant vector fields may be identified in a natural manner with  $T_e$ .

The commutator bracket of two left-invariant vector fields is left-invariant, so such vector fields form a Lie algebra; hence the above identification gives us a Lie algebra structure on  $T_e$ .

Here is another way of arriving at the same Lie algebra. Let us think of the *additive* structure of  $T_e$  as the “first order approximation to the group structure of  $G$  in the neighborhood of  $e$ ”. This approximation is abelian, which corresponds to the fact that the commutator of two elements of  $G$  both of which are close to  $e$  deviates from  $e$  only “to second order”. To measure the second-order *noncommutativity* of  $G$  near the identity, let us identify a neighborhood of  $0 \in T_e$  with a neighborhood of  $e \in G$  in a  $C^\infty$  manner, and on this identified neighborhood use vector-space notation for the operations of  $T_e$ , and  $\circ$  for the multiplication of  $G$ . Then for  $x, y \in T_e$  and real variables  $s$  and  $t$ , that second-order noncommutativity is measured by the limit

$$\lim_{s, t \rightarrow 0} \frac{(sx)^\circ(ty) - (ty)^\circ(sx)}{st}.$$

Let us call this limit  $[x, y] \in T_e$ . This turns out to coincide with the operation on  $T_e$  constructed above using left invariant vector fields; but one can discover the identities (8.7.2)-(8.7.4) (with  $\mathbb{R}$  for  $k$ ) directly by examining the properties of the above limit; this gives another standard motivation of the concept of Lie algebra. (In proving (8.7.4), the group identity of Exercise 2.4:2(i) can be useful.) Hence elements of this Lie algebra are often discussed heuristically as “infinitesimal” elements of the Lie group  $G$ .

For a familiar case, let  $G$  be the rotation group of Euclidean 3-space, so that elements of  $G$  represent rotations of space through various angles about various axes. Then elements of its Lie algebra  $L$  represent *angular velocities* about various axes. As a vector space,  $L$  may be identified with  $\mathbb{R}^3$ , each  $x \in L$  being described by a vector pointing along the axis of rotation, with magnitude equal to the angular velocity. The Lie bracket on  $L$  is an operation on  $\mathbb{R}^3$  known to every math or engineering student: the “cross product” of vectors.

In general, it turns out that the “local” structure of a Lie group  $G$  is determined by its Lie algebra: two Lie groups with isomorphic Lie algebras have neighborhoods of the identity which are isomorphic under the restrictions of the group operations to partial operations on that set.

**Exercise 8.7:6.** The ideas of Exercise 2.3:2 and the discussion preceding it showed that in the variety generated by a *finite* algebra, a free object on finitely many generators is finite. Is it similarly true that in the variety  $\mathbf{Var}(A)$  generated by any finite-dimensional associative or Lie algebra  $A$  over a field  $k$ , a free object on finitely many generators is finite-dimensional? If not, can you prove some related condition (e.g., of small “growth-rate” in the sense of Exercises 4.2:2-4.2:9)?

Can you at least show that such a variety  $\mathbf{Var}(A)$  must be distinct from the whole variety  $\mathbf{Ring}_k^1$ , respectively  $\mathbf{Lie}_k$ ? In the Lie case, if  $k = \mathbb{R}$ , can you show it distinct from the subvariety  $\mathbf{Var}(L(\mathbb{R}^1))$  of Exercise 8.7:5(ii)?

If we combine the observation that the derivations on a  $k$ -algebra form a Lie algebra over  $k$  with the intuition that elements of the Lie algebra associated with a Lie group represent “infinitesimal” elements of that group, we get the heuristic principle that a derivation on an algebra may be regarded as an “infinitesimal” algebra automorphism. This suggests that such a derivation should be determined by what it does on a generating set, and, in the case of a free algebra, that it should be possible to specify it in an arbitrary way on the free generators. The next exercise obtains results of these sorts.

**Exercise 8.7:7.** Let  $A$  be a not necessarily associative algebra over a commutative ring  $k$ .

(i) Show that the *kernel* of any derivation  $d: A \rightarrow A$  is a subalgebra of  $A$ . (This is analogous to the *fixed subalgebra* of an automorphism.) Deduce that two derivations which

agree on a generating set for  $A$  are equal.

The other result we want, about derivations on free algebras, requires a trick to turn derivations into something to which we can apply the universal property of free algebras.

(ii) Let  $A'$  denote the  $k$ -algebra whose  $k$ -module structure is that of  $A \times A$ , and whose multiplication is given by  $(a, x)(b, y) = (ab, ay + xb)$ . Verify that this  $k$ -algebra is associative, respectively associative and commutative, respectively Lie, if and only if  $A$  has the same property.

(iii) Show that a map  $d: A \rightarrow A$  is a derivation if and only if the map  $a \mapsto (a, d(a))$  is a homomorphism  $A \rightarrow A'$  as  $k$ -algebras. Deduce that if  $A$  is the free nonassociative  $k$ -algebra, the free associative  $k$ -algebra, the free associative commutative  $k$ -algebra, or the free Lie algebra over  $k$  on a set  $X$ , then every set-map  $X \rightarrow |A|$  extends uniquely to a derivation  $A \rightarrow A$ .

(iv) Show that if  $A$  is a field or a division ring, and  $X$  a subset generating  $A$  as a field or division ring, then any derivation  $A \rightarrow A$  is determined by its restriction to  $X$ . Can you generalize this result?

We remark that the concept of a derivation from a  $k$ -algebra  $A$  into itself is a case of the more general concept of a derivation  $A \rightarrow B$ , where  $A$  is a  $k$ -algebra and  $B$  is an  $A$ -module (if  $A$  is commutative and associative, or Lie) or an  $A$ -bimodule (in the general associative or nonassociative case); but we will not go into the details of these concepts here.

Some general references for the theory of Lie algebras are [78], [80], [112]. I will end this section by briefly mentioning two concepts related to that of Lie algebra.

Our observation that for  $A$  a  $k$ -algebra, the set of derivations  $A \rightarrow A$  is closed under  $k$ -module operations and commutator brackets, and thus forms a Lie algebra, makes the concept of Lie algebra a useful tool for studying derivations when  $k$  is a field of characteristic 0. But when  $k$  has characteristic  $p$ , one finds that the set of derivations is also closed under the operation of taking  $p$ th powers, and this fact needs to be taken into account in studying them. This leads to an extension of the concept of a Lie algebra, called a  $p$ -Lie algebra; see [80].

Finally, our motivation of the definition of Lie algebra starting from (8.7.1) suggests the analogous question of what identities will be satisfied by the operation

$$(x, y) = xy + yx$$

on an associative algebra. This is the starting-point of the theory of *Jordan algebras*, though the subject is not as neat as that of Lie algebras. Jordan algebras are defined using the identities of degree  $\leq 4$  satisfied by the above operation, but that operation also satisfies identities of higher degrees not implied by the Jordan identities; Jordan algebras satisfying these are called “semispecial”. No analog of the connection between Lie groups and Lie algebras appears to exist for Jordan algebras. A standard reference for the theory of Jordan algebras is [81].

Let us now return to general algebras.

### 8.8. Some necessary trivialities.

**Definition 8.8.1.** If  $g: S^X \rightarrow S$  is an  $X$ -ary operation on a set  $S$ , and  $a: X \rightarrow Y$  is a set map, then by the  $Y$ -ary operation on  $S$  induced by  $g$  via the map  $a$  of arity-sets, we shall mean the map  $f: S^Y \rightarrow S$  defined by

$$f((c_y)_{y \in Y}) = g((c_{a(x)})_{x \in X}).$$

The covariance of this construction in the arity-set is actually the result of two contravariations:

$a: X \rightarrow Y$  induces a map  $S^Y \rightarrow S^X$ , then this gives a map  $S^{(S^X)} \rightarrow S^{(S^Y)}$ .

If in the above definition we take for  $g$  an operation (primitive or derived) of an algebra structure on  $S$ , say corresponding to an element  $s \in |F_\Omega(X)|$ , then  $f$  corresponds to the image of  $s$  under the homomorphism  $F_\Omega(a): F_\Omega(X) \rightarrow F_\Omega(Y)$ . (In terms of this description, the covariance is straightforward.)

**Definition 8.8.2.** If  $f: S^Y \rightarrow S$  is an operation on a set, and  $X$  is a subset of the index set  $Y$ , we shall say that  $f$  depends only on the indices in  $X$  if  $f$  takes on the same value at any two  $Y$ -tuples that (regarded as functions on  $Y$ ) have the same restriction to  $X$ .

**Lemma 8.8.3.** If in the context of Definition 8.8.2 either  $S$  or  $X$  is nonempty, then  $f$  depends only on the indices in  $X$  if and only if  $f$  is induced by an  $X$ -ary operation  $g$  on  $S$ , via the inclusion of  $X$  in  $Y$ .  $\square$

**Exercise 8.8:1.** Prove Lemma 8.8.3. Your proof should show why the condition “ $S$  or  $X$  is nonempty” is needed.

That finishes the material from this section that we will need in what follows! But there are several interesting related points, explored in the exercises below.

**Exercise 8.8:2.** Let  $S$  and  $Y$  be sets and  $f: S^Y \rightarrow S$  a  $Y$ -ary operation on  $S$ .

- (i) Suppose  $W, X \subseteq Y$  are sets such that  $f$  depends only on the indices in  $W$ , and  $f$  also depends only on the indices in  $X$ . Show that  $f$  depends only on the indices in  $W \cap X$ .
- (ii) On the other hand, show that given an infinite family of subsets  $X_i \subseteq Y$  such that for each  $i$ ,  $f$  depends only on the indices in  $X_i$ , it may not be true that  $f$  depends only on the indices in  $\bigcap X_i$ . (Suggestion: Let  $S = [0, 1] \subseteq \mathbb{R}$ ,  $Y = \omega$ , and  $f$  be the operation  $\lim \sup$ .)
- (iii) In general, given a  $Y$ -ary operation  $f$  on  $S$ , what properties must the set

$$D_f = \{X \subseteq Y \mid f \text{ depends only on indices in } X\}$$

have? Specifically, try to find conditions on a family  $U$  of subsets of  $Y$  which are necessary and sufficient for there to exist a set  $S$  and a function  $f: S^Y \rightarrow S$  such that  $U = D_f$ .

In some works on general algebra, there is a confusion between *zeroary* derived operations, and *constant* derived operations of *nonzero* arities. The next two exercises show some of the basis of this confusion:

**Exercise 8.8:3.** (Like Exercise 8.8:1, but for derived operations.)

- (i) Show that if a derived  $Y$ -ary operation  $s$  of an algebra  $A$  depends only on indices in a subset  $X \subseteq Y$ , and  $X$  is *nonempty*, then  $s$  is in fact induced by an  $X$ -ary derived operation of  $A$ .
- (ii) On the other hand, suppose the derived  $Y$ -ary operation  $s$  of  $A$  depends only on the empty set of indices in  $Y$ , i.e., is constant. If  $A$  has zeroary operations, show that, as (i) would suggest, but for a different reason,  $s$  is induced by a zeroary derived operation of  $A$ . Show, however, that if  $A$  has no zeroary operations, derived operations depending on the empty set of indices can still exist, but will not be induced by derived zeroary operations.

Thus, for  $m \leq n$ , derived  $m$ -ary operations correspond to derived  $n$ -ary operations depending only on the first  $m$  variables, *except* for the  $m = 0$  case, where this is not true unless the algebra has zeroary primitive operations.

One is still more susceptible to the confusion referred to above if one excludes empty algebras,

as is shown by

**Exercise 8.8:4.** We have seen that the  $X$ -ary derived operations of a variety  $\mathbf{V}$  can be characterized as the morphisms  $U_{\mathbf{V}}^X \rightarrow U_{\mathbf{V}}$  where  $U_{\mathbf{V}}$  is the underlying-set functor of  $\mathbf{V}$ .

Suppose now that  $\mathbf{V}$  is a variety *without* zeroary operations, hence having an empty algebra  $I$ . Let  $\mathbf{V}-\{I\}$  denote the full subcategory of  $\mathbf{V}$  consisting of all nonempty algebras, and let  $U_{\mathbf{V}-\{I\}}$  denote the restriction of  $U_{\mathbf{V}}$  to this subcategory.

(i) Show that morphisms  $(U_{\mathbf{V}-\{I\}})^X \rightarrow U_{\mathbf{V}-\{I\}}$  correspond to derived  $X$ -ary operations of  $\mathbf{V}$  *except* in the case  $X = \emptyset$ , in which case they can be put in natural correspondence with the constant derived unary operations.

(ii) Show that if  $\mathbf{V}$  has constant derived unary operations, then  $\mathbf{V}-\{I\}$  is isomorphic in a natural way to a variety of algebras (of a different type) having zeroary operations.

As an example, suppose that (as has been proposed from time to time) one sets up a variant of the concept of ‘‘group’’, based only on the two operations of composition and inverse, axiomatizing these by the associative law for composition, and the following identities, which hold in ordinary groups as consequences of the inverse and neutral element laws:

$$x = xy^{-1} = y^{-1}yx.$$

(iii) Let  $\mathbf{V}$  be the variety so defined. Show that the category  $\mathbf{V}-\{I\}$  is isomorphic to **Group**.

**8.9. Clones and clonal theories.** Given a family of *unary* operations on a set  $S$ , i.e., maps  $S \rightarrow S$ , the composites of these (together with the ‘‘empty composite’’, the identity map) form a *monoid* of maps of  $S$  into itself. In this section we will look at the structure of the set of derived operations of a family of *not necessarily unary* operations, under the operations analogous to composition of unary operations.

We will limit ourselves to finitary operations. (There is no problem with the infinitary case, but I thought the concepts would come across more clearly in the familiar finitary context. The reader interested in the infinitary case can easily make the appropriate generalizations, replacing ‘‘finite’’ by ‘‘ $< \gamma_0$ ’’, for  $\gamma_0$  any regular infinite cardinal.) We will also, in this presentation, make our arities natural numbers (for the infinitary case read ‘‘cardinals  $< \gamma_0$ ’’) rather than arbitrary finite sets, since allowing all finite sets as arities would mean that every algebra would have a *large* set of formally distinct operations.

**Definition 8.9.1.** *Let  $S$  be a set. Then a clone of operations on  $S$  will mean a set  $C$  of operations on  $S$ , of natural-number arities, which is closed under formation of derived operations. Concretely, this says that*

(i) *For every natural number  $n$ ,  $C$  contains the  $n$  projection maps  $p_{n,i}: S^n \rightarrow S$  ( $i \in n$ ), defined by*

$$(8.9.2) \quad p_{n,i}(\xi_0, \dots, \xi_{n-1}) = \xi_i,$$

*and*

(ii) *Given natural numbers  $m, n \in \omega$ , an  $m$ -ary operation  $s \in C$ , and  $m$   $n$ -ary operations  $t_0, \dots, t_{m-1} \in C$ , the set  $C$  also contains the  $n$ -ary operation*

$$(8.9.3) \quad (\xi_0, \dots, \xi_{n-1}) \mapsto s(t_0(\xi_0, \dots, \xi_{n-1}), \dots, t_{m-1}(\xi_0, \dots, \xi_{n-1}))$$

*i.e., the composite*

$$S^n \xrightarrow{(t_0, \dots, t_{m-1})} S^m \xrightarrow{s} S.$$

The least clone on  $S$  containing a given set of operations will be called the clone generated by that set. Thus, for any finitary type  $\Omega$  and any  $\Omega$ -algebra  $A$ , the set of derived operations of  $A$  of natural-number arities constitutes the clone generated by the primitive operations of  $A$ .

Let us look at an example of how this procedure of generation works. Given a binary operation  $f$  and a ternary operation  $g$  on a set  $S$ , how do we express in terms of the constructions (8.9.2) and (8.9.3) the 6-ary operation

$$(\xi_0, \dots, \xi_5) \mapsto f(g(\xi_0, \xi_1, \xi_2), g(\xi_3, \xi_4, \xi_5))?$$

It should clearly arise as an instance of (8.9.3) with  $f$  for  $s$ , but we cannot, as we might first think, take  $g$  for  $t_0$  and  $t_1$ . That would give the ternary operation  $(\xi_0, \xi_1, \xi_2) \mapsto f(g(\xi_0, \xi_1, \xi_2), g(\xi_0, \xi_1, \xi_2))$ . We need, rather, to use as  $t_0$  and  $t_1$  the two 6-ary operations  $(\xi_0, \dots, \xi_5) \mapsto g(\xi_0, \xi_1, \xi_2)$  and  $(\xi_0, \dots, \xi_5) \mapsto g(\xi_3, \xi_4, \xi_5)$ . We get these, in turn, as instances of (8.9.3) with  $g$  in the role of  $s$ , and projection maps (8.9.2) in the role of the  $t_i$ 's. Namely, taking for  $t_0, t_1, t_2$  the projection maps  $p_{6,0}, p_{6,1}, p_{6,2}$ , we get the first of the above 6-ary operations, and using the remaining three 6-ary projection maps, we get the other. We can then, as noted, apply (8.9.3) to  $f$  and these two 6-ary operations to get the 6-ary operation first asked for. (In this example, each of our variables happened to appear exactly once in the final expression, and the occurrences were in ascending order of subscripts, but obviously, by different choices of projection maps, we can get expressions in which variables appear more than once, and in arbitrary orders.)

Above, we got a “new” operation by inserting into the ternary operation  $g$  the 6-ary projection maps  $p_{6,0}, p_{6,1}, p_{6,2}$ . It is clear that if we had, instead, inserted the ternary projections  $p_{3,0}, p_{3,1}, p_{3,2}$  (in that order) we would have gotten back precisely the operation  $g$ . Note also that if we substitute any operation  $f$  into the unary projection map  $p_{1,0}$ , we get the operation  $f$  back. These phenomena are analogs of the *neutral element laws* in a monoid.

One also has an analog of the associative law: If  $m, n$  and  $p$  are nonnegative integers, then given an  $m$ -ary operation  $s$ , any  $m$   $n$ -ary operations  $t_i$  ( $i \in m$ ), and any  $n$   $p$ -ary operations  $u_j$  ( $j \in n$ ), one can either substitute the  $t$ 's into  $s$ , and the  $u$ 's into the resulting operation, or first substitute the  $u$ 's into the  $t$ 's, and then the resulting operations into  $s$ . In each case one gets the  $p$ -ary operation which is the composite of the set maps

$$S^p \xrightarrow{(u_0, \dots, u_{n-1})} S^n \xrightarrow{(t_0, \dots, t_{m-1})} S^m \xrightarrow{s} S.$$

Hence the results of these two substitution procedures are equal.

It looks as though we ought to abstract these properties, and use them as the definition of a new sort of algebraic object, which we might call a “formal substitution algebra” or a “clonal algebra”. We would then have a new way of looking at varieties of algebras: Given a type  $\Omega$  and a family  $J$  of identities, we would construct a “clonal algebra”  $\langle \Omega \mid J \rangle$  presented by these operation-symbols and identities. We could then define a “representation” of this clonal algebra on a set  $|A|$  to mean a homomorphism of  $\langle \Omega \mid J \rangle$  into the clone of *all* finitary operations on that set. Such representations could be identified with  $\Omega$ -algebras satisfying the identities of  $J$ ; thus, each variety of algebras could be looked at as the category of representations of a clonal algebra.

Unfortunately, these “clonal algebras” would not be algebras as we have so far defined the term. Our algebras  $A$  have an underlying set  $|A|$ ; but a “clonal algebra” would have an

underlying *family* of sets, one set for each arity of the operations symbolized, with composition operations associated to appropriate combinations of these.

Now there is, in fact, a concept of *many-sorted algebra* (algebra having different “sorts” of elements), and in an as-yet-unwritten chapter, I hope to show that our general theory of algebras can be adapted to that context in a straightforward way. If I were going to develop the ideas sketched above using many-sorted algebras, I would wait for that chapter.

But in fact, we don’t need a new kind of mathematical object to do what we have been discussing. After all, we introduced the concept of a *category* to formalize the properties of composition of maps, which is what we are dealing with here. The apparent difficulty with looking at the members of a clone of operations as morphisms in a category is that an  $n$ -ary operation in a clone is composed on the right, not with a single operation, but with a family of  $n$  operations. The solution is to define our category so that a morphism therein is not a single  $n$ -ary operation  $|A|^n \rightarrow |A|$ , but an  $m$ -tuple of  $n$ -ary operations, corresponding to a map  $|A|^n \rightarrow |A|^m$ .

Now everything falls into place! The category should have objects  $X_n$  in one-to-one correspondence with the natural numbers  $n$ , and a morphism between  $X_n$  and  $X_m$  should correspond to an  $m$ -tuple of  $n$ -ary operations in our clone.

In saying “a morphism between  $X_n$  and  $X_m$ ”, I have skirted the question of which of these objects is to be the domain, and which the codomain. This is a notational choice: whether we want to encode our structure as a certain category, or as its opposite. The development we have just seen suggests that the morphisms corresponding to  $m$ -tuples of  $n$ -ary operations should go from  $X_n$  to  $X_m$ , since an  $m$ -tuple of  $n$ -ary operations of an algebra  $A$  gives a set map  $|A|^n \rightarrow |A|^m$ . More globally, an  $m$ -tuple of derived  $n$ -ary operations of a variety  $\mathbf{V}$  is equivalent to a morphism  $U_{\mathbf{V}}^n \rightarrow U_{\mathbf{V}}^m$ , so the “clone of derived operations” of  $\mathbf{V}$  is encoded by the full subcategory of  $\mathbf{Set}^{\mathbf{V}}$  having the functors  $U_{\mathbf{V}}^n$  as objects.

But there is also motivation for the opposite choice. Recall that the derived  $n$ -ary operations of a variety  $\mathbf{V}$  correspond to the elements of the free algebra  $F_{\mathbf{V}}(n)$ . An  $m$ -tuple of such elements is picked out by a homomorphism  $F_{\mathbf{V}}(m) \rightarrow F_{\mathbf{V}}(n)$ ; so the full subcategory of  $\mathbf{V}$  consisting of the free objects  $F_{\mathbf{V}}(n)$  also embodies the structure of the operations of  $\mathbf{V}$ , in the manner opposite to way it is embodied in morphisms  $U_{\mathbf{V}}^n \rightarrow U_{\mathbf{V}}^m$ . This is, of course, a case of the contravariance of the Yoneda equivalence between the covariant functors  $U_{\mathbf{V}}^n$  and their representing objects  $F_{\mathbf{V}}(n)$ .

Postponing the above question for a moment, let us note that, whichever choice we make, we will want to know *which* categories with object-set of the form  $\{X_n \mid n \in \omega\}$  correspond in this way to clones of operations. Clearly, such a category should be given with a distinguished family of  $n$  morphisms  $p_{n,i}$  ( $i \in n$ ) between  $X_1$  and  $X_n$  for each  $n$  (corresponding, in one description to the  $n$  projection maps  $|A|^n \rightarrow |A|$ , and in the other to the  $n$  obvious morphisms  $F_{\mathbf{V}}(1) \rightarrow F_{\mathbf{V}}(n)$ ). It must also have the property that the morphisms between  $X_n$  and  $X_m$  (in the appropriate direction) correspond, via composition with the  $p_{m,i}$ , to the  $m$ -tuples of morphisms between  $X_n$  and  $X_1$ .

These conditions together say that in the category, each object  $X_n$  is the *product* (or *coproduct*) of  $n$  copies of  $X_1$ , with the given morphisms  $p_{n,i}$  as (co)projection maps. As to the choice of direction of the morphisms, F.W. Lawvere, in his doctoral thesis [14] where he introduced this approach, made  $X_n$  a *product* of copies of  $X_1$ , but in later published work switched to the definition under which it would be a coproduct, in other words, under which the category would look like the category of free algebras  $F_{\mathbf{V}}(n)$ . An attractive feature of the latter choice for Lawvere is that the category having *only* the maps  $p_{n,i}$  for morphisms  $X_1 \rightarrow X_n$

(corresponding to the variety with no primitive operations) is the full subcategory  $\mathbf{N} \subseteq \mathbf{Set}$  having the set  $\mathbb{N} = \omega$  of natural numbers for object-set; hence the category corresponding to a general variety can be characterized as a certain kind of extension of  $\mathbf{N}$ . This fits with his project of creating a category-theoretic foundation for set theory and for mathematics, with the category  $\mathbf{N}$  as a basic building-block. I prefer the other choice of variance because it leads to a covariant relationship between this category of formal operations, and the actual operations in the variety. I will include both versions in the definition below, calling them the “contravariant” and “covariant” versions, but from that point on, we will generally work with the covariant formulation.

Lawvere calls the category of formal operations of a variety  $\mathbf{V}$  the “theory of  $\mathbf{V}$ ”, and any category of this form an “algebraic theory”. For us this would be awkward, for though these categories carry approximately the same information as equational theories (Definition 8.4.6), the two concepts are different enough that we cannot identify them. So let us introduce a different term.

**Definition 8.9.4.** A covariant clonal category will mean a category  $\mathbf{X}$  given with a bijective indexing of its object-set by the natural numbers,

$$\text{Ob}(\mathbf{X}) = \{X_n \mid n \in \omega\},$$

and given with morphisms

$$p_{n,i}: X_n \rightarrow X_1 \quad (i \in n)$$

which make each  $X_n$  the product of  $n$  copies of  $X_1$ , and such that  $p_{1,0}$  is the identity map of  $X_1$ . (Equivalently, letting  $\mathbf{N}$  denote the full subcategory of  $\mathbf{Set}$  whose objects are the natural numbers, this means a category  $\mathbf{X}$  given with a functor  $\mathbf{N}^{\text{op}} \rightarrow \mathbf{X}$  which is bijective on object-sets, and turns finite coproducts in  $\mathbf{N}$  to products in  $\mathbf{X}$ .)

A contravariant clonal category will mean a category  $\mathbf{X}$  given with the dual sort of structure, equivalently, given with a covariant clonal category structure on  $\mathbf{X}^{\text{op}}$ , equivalently, given with a functor  $\mathbf{N} \rightarrow \mathbf{X}$  which is bijective on object-sets and respects finite coproducts.

(More generally, for any infinite regular cardinal  $\gamma_0$ , one may define concepts of covariant and contravariant  $< \gamma_0$ -clonal category, using in place of  $\mathbf{N}$  the full subcategory of  $\mathbf{Set}$  having for object-set the cardinal  $\gamma_0$ , and in place of finite (co)products, (co)products of  $< \gamma_0$  factors.)

**Exercise 8.9:1.** Establish the equivalence of structures noted parenthetically in the first paragraph of the above definition.

A clonal category is itself a mathematical object, so we make

**Definition 8.9.5.** By **Clone** we shall denote the category whose objects are the covariant clonal categories, and where a morphism  $\mathbf{X} \rightarrow \mathbf{Y}$  is a functor which carries  $X_n$  to  $Y_n$  for each  $n$ , and respects the morphisms  $p_{n,i}$ . In other words, **Clone** will denote the full subcategory of the comma category  $(\mathbf{N}^{\text{op}} \downarrow \mathbf{Cat})$  (Exercise 6.8:26(iii)) whose objects are the clonal categories.

Incidentally, when one forms the category of *contravariant* clonal categories, this is isomorphic to our present category **Clone**, *not* opposite thereto, since the direction of morphisms within clonal categories does not affect the direction of functors among them.

We now wish to establish the relation between clonal categories and varieties of algebras. First,

given a variety  $\mathbf{V}$ , how shall we define the associated clonal category? The most convenient choice from the formal point of view would be to use  $n$ -ary derived operations of  $\mathbf{V}$  as the morphisms from  $X_n$  to  $X_1$ . (This construction was sketched for  $\mathbf{V} = \mathbf{Group}$  when we were noting “nonprototypical” ways categories could arise, in the paragraph containing (6.2.1).) Unfortunately, these derived operations are not small as sets (though the set of them is quasi-small). So let us use in their stead the corresponding elements of the free  $\mathbf{V}$ -algebra  $F_{\mathbf{V}}(n)$ . Of course, we will define the composition operation of the clone so as to correspond to composition of derived operations.

**Definition 8.9.6.** *If  $\mathbf{V}$  is a variety of finitary algebras, the covariant clonal theory of  $\mathbf{V}$  will mean the clonal category  $\mathbf{Cl}(\mathbf{V})$ , with objects denoted  $Cl_n(\mathbf{V})$ , in which a morphism from  $Cl_n(\mathbf{V})$  to  $Cl_m(\mathbf{V})$  means an  $m$ -tuple of elements of  $|F_{\mathbf{V}}(n)|$ , composition of such morphisms*

$$Cl_p(\mathbf{V}) \xrightarrow{|F_{\mathbf{V}}(p)|^n} Cl_n(\mathbf{V}) \xrightarrow{|F_{\mathbf{V}}(n)|^m} Cl_m(\mathbf{V})$$

*is defined by substitution of  $n$ -tuples of expressions in  $p$  indeterminates into expressions in  $n$  indeterminates, and each morphism  $p_{n,i}$  is given by  $i$ th member of the universal  $n$ -tuple of generators of  $F_{\mathbf{V}}(n)$ . We note that this is equivalent (via a natural isomorphism) to the full subcategory of the large category  $\mathbf{Set}^{\mathbf{V}}$  having for objects the functors  $U_{\mathbf{V}}^n$  ( $n \in \omega$ ), and also to the opposite of the small full subcategory of  $\mathbf{V}$  having for objects the free  $\mathbf{V}$ -algebras  $F_{\mathbf{V}}(n)$ .*

*Given a clonal category  $\mathbf{X}$ , an  $\mathbf{X}$ -algebra will mean a functor  $\mathbf{X} \rightarrow \mathbf{Set}$  respecting the product structures defined on the objects  $X_n$  by the projection maps  $p_{n,i}$ . The category of all  $\mathbf{X}$ -algebras will be written  $\mathbf{X}\text{-Alg}$ . The functor  $\mathbf{X}\text{-Alg} \rightarrow \mathbf{Set}$  taking each  $\mathbf{X}$ -algebra  $A$  to the set  $A(X_1)$  will be written  $U_{\mathbf{X}\text{-Alg}}$ , or  $U$  when there is no danger of ambiguity, and called the “underlying-set functor” of  $\mathbf{X}\text{-Alg}$ .*

*(The analogs of the categories and objects named in this and the preceding definition with arities taken from an arbitrary regular infinite cardinal  $\gamma_0$  rather than the natural numbers, may be written  $\mathbf{Clone}^{(\gamma_0)}$ ,  $\mathbf{Cl}^{(\gamma_0)}(\mathbf{V})$ ,  $Cl_{\alpha}^{(\gamma_0)}(\mathbf{V})$ , etc..)*

Note that by our general conventions, unless the contrary is stated, a clonal category  $\mathbf{X}$  is legitimate, hence, as it has by definition a small set of objects, it is small. Thus, the corresponding category  $\mathbf{X}\text{-Alg}$  is legitimate.

We have designed these concepts so that the categories  $\mathbf{X}\text{-Alg}$  are essentially the same as classical varieties of algebras. Let us state this property as

**Lemma 8.9.7.** *If  $\mathbf{X}$  is a clonal category, then  $\mathbf{X}\text{-Alg}$  (second paragraph of Definition 8.9.6) is equivalent to a variety  $\mathbf{V}$  of finitary algebras, by an equivalence respecting underlying-set functors. For  $\mathbf{V}$  so constructed,  $\mathbf{Cl}(\mathbf{V})$  (first paragraph of Definition 8.9.6) is naturally isomorphic in  $\mathbf{Clone}$  to  $\mathbf{X}$ .*

*Inversely, if  $\mathbf{V}$  is any variety of finitary algebras, then  $\mathbf{Cl}(\mathbf{V})\text{-Alg}$  is equivalent to  $\mathbf{V}$ .  $\square$*

**Exercise 8.9:2.** Prove Lemma 8.9.7.

For  $\mathbf{X}$  a clonal category, an  $\mathbf{X}$ -algebra can be thought of as a “representation” of the clone  $\mathbf{X}$  by sets and set maps. This suggests the following more general definition, which we will find useful in the next chapter:

**Definition 8.9.8.** If  $\mathbf{X}$  is a clonal category and  $\mathbf{C}$  any category with finite products, a representation of  $\mathbf{X}$  in  $\mathbf{C}$  will mean a covariant functor  $A: \mathbf{X} \rightarrow \mathbf{C}$  respecting the product structures defined on the objects  $X_n$  by the projection maps  $p_{n,i}$ .

We remark that the information given by a clonal category is not quite the same as that given by a variety, in that the clonal category does not distinguish between *primitive* and *derived* operations, while under our definition, a variety does.

Lawvere defines a variety of algebras (in his language, an “algebraic category”) to mean a category of the form  $\mathbf{X}\text{-Alg}$ , where  $\mathbf{X}$  is what we call a clonal category (and he calls a theory). This is a reasonable and elegant definition, but since we began with the classical concepts of variety and theory, and it is pedagogically desirable to hold to one definition, we shall keep to our previous definition of variety, and study the categories  $\mathbf{X}\text{-Alg}$  as a closely related concept.

**Exercise 8.9:3.** Let  $2\mathbf{N}$  be the full subcategory of  $\mathbf{Set}$  having for objects the nonnegative *even* integers. For each integer  $n$ , the object  $2n$  of  $2\mathbf{N}$  is a coproduct of  $n$  copies of the object  $2$ , hence the opposite category  $(2\mathbf{N})^{\text{op}}$  can be made a covariant clonal category by an appropriate choice of maps  $p_{n,i}$ . Write down such a system of maps  $p_{n,i}$ , and obtain an explicit description of  $(2\mathbf{N})^{\text{op}}\text{-Alg}$  as a variety  $\mathbf{V}$  determined by finitely many operations and finitely many identities. Your answer should show what it means to put a  $(2\mathbf{N})^{\text{op}}$ -algebra structure on a set.

**Exercise 8.9:4.** In defining an  $\mathbf{X}$ -algebra as a certain kind of functor in Definition 8.9.6, we required that this functor respect the given structures of the objects  $X_n$  as  $n$ -fold products of  $X_1$ .

- (i) Show that under the conditions of that definition, to respect these distinguished products is equivalent to respecting *all* finite products that exist in  $\mathbf{X}$ .
- (ii) Show on the other hand that an  $\mathbf{X}$ -algebra may fail to respect infinite products in  $\mathbf{X}$ . (To do this, you must start by finding a clonal category  $\mathbf{X}$  having a nontrivial infinite product of objects!)

**Exercise 8.9:5.** Show that, up to isomorphism, there are just two clonal categories  $\mathbf{X}$  such that the functor  $\mathbf{N}^{\text{op}} \rightarrow \mathbf{X}$  is not faithful. What are the corresponding varieties?

The next exercise does not involve the concept of clonal category, and could have come immediately after the definition of a clone of operations on a set, but I didn’t want to break the flow of the discussion. It requires familiarity with a bit of elementary electronics.

**Exercise 8.9:6.** (Inspired by a question of F. E. J. Linton.)

If  $n$  is a positive integer, let us understand an “ $n$ -labeled circuit graph” to mean a finite connected graph  $\Gamma$  (which may have more than one edge between two given vertices), with two distinguished vertices  $v_0$  and  $v_1$ , and given with a function sending the edges of  $\Gamma$  to the set  $n = \{0, \dots, n-1\}$ . To each such graph let us associate the  $n$ -ary operation on the nonnegative real numbers that takes each  $n$ -tuple  $(r_0, \dots, r_{n-1})$  of such numbers to the *resistance* that would be measured between  $v_0$  and  $v_1$  if each edge of  $\Gamma$  labeled  $i$  were a resistor with resistance  $r_i$ .

- (i) Explain (briefly) why the set of operations on nonnegative real numbers arising in this way from labeled circuit graphs forms a *clone*.
- (ii) Let  $s$  denote the binary operation in this clone corresponding to putting two resistors in *series*,  $p$  the binary operation corresponding to putting two resistors in *parallel*, and  $w$  the 5-ary operation corresponding to a *Wheatstone Bridge*; i.e., determined by the graph  $\diamond$ , with  $v_0$  and  $v_1$  the top and bottom vertices, and distinct labels on all five edges. Show that none of these three operations is in the subclone generated by the other two. (Suggestion: Look at the behavior of these three operations with respect to the order relation on the nonnegative reals.)

A much more difficult question is

(iii) Is the clone of (i) generated by the three operations listed in (ii)?

I do not know answers to the next two questions.

(iv) Can one *characterize* the set of operations belonging to the clone of part (i), i.e., describe some test that can be applied to an  $n$ -ary operation on positive real numbers to determine whether it belongs to the clone?

(v) Can one find a generating set for the identities satisfied by the two binary operations  $s$  and  $p$ ? (This was the question of Fred Linton's which inspired this exercise.)

Generating sets for identities of other families of operations in this clone would, of course, likewise be of interest.

(vi) Suppose one is interested in more general electrical circuits; e.g., circuits containing resistors, capacitors and inductances, and possibly other elements. Can one somehow extend the "clonal" viewpoint to such circuits? (If we allow circuit components such as rectifiers, which do not behave symmetrically, we must work with directed rather than undirected graphs.)

(vii) If you succeed in extending the clonal approach to circuits composed of resistors, capacitors and inductances, is the clone you get isomorphic to the clone of part (i) (the clone one obtains assuming all elements are resistors)?

You might also try to answer this question for other sets of circuit elements.

Recall that the morphisms between clonal categories are the functors respecting the indexing of the object-set and the morphisms  $p_{n,i}$ . What do such functors mean from the viewpoint of the corresponding varieties of algebras? If  $\mathbf{V}$  is a variety of  $\Omega$ -algebras and  $\mathbf{W}$  is a variety of  $\Omega'$ -algebras, we see that to specify a morphism  $f \in \mathbf{Clone}(\mathbf{Cl}(\mathbf{V}), \mathbf{Cl}(\mathbf{W}))$  one must associate to every primitive operation  $s$  of  $\mathbf{V}$  a derived operation  $f(s)$  of  $\mathbf{W}$  of the same arity, so that the defining identities for  $\mathbf{V}$  in those primitive operations are satisfied by the derived operations  $f(s)$  in  $\mathbf{W}$ . We find that such a morphism  $f$  determines a functor in the opposite direction,  $\mathbf{W} \rightarrow \mathbf{V}$ ; namely, given a  $\mathbf{W}$ -algebra  $A$ , we get a  $\mathbf{V}$ -algebra  $A_f$  with the same underlying set by using for each primitive  $\mathbf{V}$ -operation  $s_{A_f}$  the derived operation  $f(s)_A$  of the  $\mathbf{W}$ -structure on  $|A|$ . In fact we have

**Lemma 8.9.9.** (Lawvere) *Functors between varieties of algebras which preserve underlying sets correspond bijectively to morphisms in the opposite direction between the clonal theories of these varieties, via the construction described above.  $\square$*

**Exercise 8.9:7.** Prove Lemma 8.9.9.

Easy examples of such functors among varieties are the *forgetful* functors  $\mathbf{Group} \rightarrow \mathbf{Monoid}$ ,  $\mathbf{Ring}^1 \rightarrow \mathbf{Monoid}$ ,  $\mathbf{Ring}^1 \rightarrow \mathbf{Ab}$ ,  $\mathbf{Lattice} \rightarrow \vee\text{-Semilattice}$ , and similar constructions, including the underlying-set functor of every variety, and the inclusion functor of any subvariety in a larger variety, e.g.,  $\mathbf{Ab} \rightarrow \mathbf{Group}$ . In the above list of cases, each primitive operation of the codomain variety happens to be mapped to a primitive operation of the domain variety. Some examples in which primitive operations are mapped to non-primitive operations are the functor  $\mathbf{Bool}^1 \rightarrow \vee\text{-Semilattice}$  under which the semilattice operation  $x \vee y$  is mapped to (i.e., given by) the Boolean ring operation  $(x, y) \mapsto x + y + xy$ ; the functor  $H: \mathbf{Group} \rightarrow \mathbf{Heap}$  of Exercise 8.6:9, under which the ternary heap operation  $\tau$  is mapped to the group operation  $xy^{-1}z$ , and the functor  $B: \mathbf{Ring}_k^1 \rightarrow \mathbf{Lie}_k$  of §8.7, under which, though the primitive  $k$ -module operations of  $\mathbf{Lie}_k$  are mapped to the corresponding primitive operations of  $\mathbf{Ring}_k^1$ , the *Lie bracket* is mapped to the commutator operation  $xy - yx$ .

We have seen most of the above constructions before as examples of functors having left adjoints. In fact, one can prove that any functor between varieties induced by a morphism of their clonal theories – in other words, every functor between varieties that preserves underlying sets – has a left adjoint! We will not stop to do this here, because it will be an immediate consequence of a necessary and sufficient criterion for a functor between varieties to have a left adjoint that we will obtain in the next chapter. But you can, if you wish, do this case now as an exercise:

**Exercise 8.9:8.** (Lawvere) Show that any functor between varieties of finitary algebras which preserves underlying sets has a left adjoint.

You may drop the “finitary” condition if you wish, either using generalized versions of the results of this section, or proving the result without relying on the ideas of this section.

**Exercise 8.9:9.** Since morphisms  $\mathbf{X} \rightarrow \mathbf{Y}$  in **Clone** are defined to be certain functors, we can also look at *morphisms between two such functors*, giving a concept of morphisms between morphisms in **Clone**. Interpret this concept in terms of varieties of algebras. That is, given two varieties of algebra  $\mathbf{V}$  and  $\mathbf{W}$ , and two underlying-set-preserving functors  $F, G: \mathbf{V} \rightarrow \mathbf{W}$ , corresponding to functors  $f, g: \mathbf{Cl}(\mathbf{W}) \rightarrow \mathbf{Cl}(\mathbf{V})$ , what data relating  $F$  and  $G$  corresponds to a morphism  $f \rightarrow g$ ?

In particular, describe one or more underlying-set preserving functors  $\mathbf{Boole}^1 \rightarrow \mathbf{Semilat}$  and/or  $\mathbf{Group} \rightarrow \mathbf{Monoid}$ , describe the corresponding functors between clonal categories, and then find examples of morphisms between two of those functors, or nonidentity morphisms from such a functor to itself, and interpret these in terms of the given varieties of algebras.

Here are some exercises on particular underlying-set-preserving functors and their adjoints:

**Exercise 8.9:10.** Let  $U: \mathbf{Group} \rightarrow \mathbf{Monoid}$  denote the forgetful functor, and  $F: \mathbf{Monoid} \rightarrow \mathbf{Group}$  its left adjoint (called in §3.11 the “universal enveloping group” construction).

(i) Show that there exist proper subvarieties  $\mathbf{V} \subseteq \mathbf{Group}$  such that  $U(\mathbf{V})$  does not lie in a proper subvariety of  $\mathbf{Monoid}$ .

A much harder problem is

(ii) If  $\mathbf{V}$  is a proper subvariety of  $\mathbf{Monoid}$ , must  $F(\mathbf{V})$  be contained in a proper subvariety of  $\mathbf{Group}$ ? Must one in fact have  $UF(\mathbf{V}) \subseteq \mathbf{V}$ ?

**Exercise 8.9:11.** Let  $H: \mathbf{Group} \rightarrow \mathbf{Heap}$  be the functor described by (8.6.7) in Exercise 8.6:9, and  $F: \mathbf{Heap} \rightarrow \mathbf{Group}$  its left adjoint. Let  $A$  be a nonempty heap. We recall that  $A \cong H(G)$  for some group  $G$ .

(i) Describe the group  $F(A)$  as explicitly as possible in terms of  $G$ .

(ii) It follows from Exercise 8.6:9(iii) that in general,  $A = H(G)$  has automorphisms not arising from automorphisms of the group  $G$ . Take an example of such an automorphism  $i$  (or better, obtain a complete characterization of automorphisms of any nonempty heap  $A = H(G)$  and let  $i$  be a general automorphism of this form), and describe the induced automorphism  $F(i)$  of the group  $F(H(G))$ .

**Exercise 8.9:12.** Show that there exist exactly two underlying-set-preserving functors  $\mathbf{Set} \rightarrow \mathbf{Semigroup}$ . (Hint: What derived operations does  $\mathbf{Set}$  have?) Find their left adjoints.

The next exercise looks at clonal categories as mathematical objects:

**Exercise 8.9:13.** Show that the category **Clone** has small limits and colimits.

The approach of the paragraph preceding Lemma 8.9.9 also shows that for any clonal category  $\mathbf{X}$  and any type  $\Omega$ , to give a morphism  $\mathbf{Cl}(\Omega\text{-Alg}) \rightarrow \mathbf{X}$  is simply to pick for each  $s \in |\Omega|$  an appropriate morphism in  $\mathbf{X}$ ; and gives a similar characterization of the morphisms from clonal categories  $\mathbf{Cl}(\mathbf{V})$  to  $\mathbf{X}$ . We record these observations as

**Lemma 8.9.10.** *Let  $\Omega = (|\Omega|, \text{ari})$  be any type. Then the functor  $\mathbf{Clone} \rightarrow \mathbf{Set}$  associating to each clonal category  $\mathbf{X}$  the set of maps*

$$(8.9.11) \quad \{ f: |\Omega| \rightarrow \bigsqcup_n \mathbf{X}(X_n, X_1) \mid \forall s \in |\Omega|, f(s) \in \mathbf{X}(X_{\text{ari}(s)}, X_1) \}$$

*is representable, with representing object  $\mathbf{Cl}(\Omega\text{-Alg})$ . Thus,  $\mathbf{Cl}(\Omega\text{-Alg})$  may be regarded as a “free clonal category on an  $|\Omega|$ -tuple of formal operations of arities given by the function  $\text{ari}_\Omega$ ”.*

*Suppose further that  $J$  is a set of identities for  $\Omega$ -algebras, which we will here express, not as pairs of elements of  $|F_\Omega(\omega)|$ , but as pairs of elements of  $|F_\Omega(n)|$  for various  $n \in \omega$ ; and let us identify these sets  $|F_\Omega(n)|$  with the sets  $\mathbf{Cl}(\Omega\text{-Alg})(Cl_n(\Omega\text{-Alg}), Cl_1(\Omega\text{-Alg}))$ . Let*

$$A_{\Omega, J}: \mathbf{Clone} \rightarrow \mathbf{Set}$$

*denote the functor associating to each clonal category  $\mathbf{X}$  the subset of (8.9.11) consisting of maps that satisfy the additional condition:*

*For each  $(s, t) \in J$ , the induced map  $\mathbf{Cl}(\Omega\text{-Alg}) \rightarrow \mathbf{X}$  corresponding to  $f$  carries  $s$  and  $t$  to the same element.*

*Then  $A_{\Omega, J}$  is representable, with representing object  $\mathbf{Cl}(\mathbf{V}(J))$ . Thus,  $\mathbf{Cl}(\mathbf{V}(J))$  may be written  $\langle \Omega \mid J \rangle_{\mathbf{Clone}}$ , i.e., may be regarded as the clonal category “presented by the family  $\Omega$  of formal operations, and the family  $J$  of relations in this family”.  $\square$*

Let me end this section by mentioning a few related concepts on which there is considerable literature, though we will not study them further here.

One is often interested in properties of a variety  $\mathbf{V}$  of algebras that do not depend on which operations are considered primitive. These can be expressed as statements about the clonal category  $\mathbf{Cl}(\mathbf{V})$ . The formally simplest such statements are universally or existentially quantified equations, in families of operations of specified arities. Universally quantified equations of this sort are called *hyperidentities* [118]. An example, and its interpretation in terms of ordinary identities, is noted in

**Exercise 8.9:14.** Show that the following conditions on a variety  $\mathbf{V}$  are equivalent: (a)  $\mathbf{V}$  satisfies the hyperidentity saying that all derived *unary* operations are equal. (b) All *primitive* operations  $s$  of  $\mathbf{V}$  (of all arities) satisfy the identity of idempotence:  $s(x, \dots, x) = x$ . (c) For every  $A \in \text{Ob}(\mathbf{V})$ , every one-element subset of  $|A|$  is the underlying set of a subalgebra of  $A$ .

The above hyperidentity is satisfied, for instance, by the varieties of lattices, semilattices, and heaps. On the other hand, there are many varieties that satisfy no nontrivial hyperidentities; e.g., it is shown in [118] that this is true of the variety of commutative rings. A class of varieties determined by a family of hyperidentities is called a *hypervariety*. (However, the term “hyperidentity” is used by some authors, e.g., in [102], with the similar but different meaning of an identity holding for all families of *primitive* operations of given arities in a variety.)

**Exercise 8.9:15.** (i) Show that for every monoid identity  $s = t$  there is a hyperidentity  $s' = t'$  such that for  $S$  a monoid, the variety  $S\text{-Set}$  satisfies  $s' = t'$  if and only if the monoid  $S$  satisfies  $s = t$ .

(ii) Is the inverse statement true, that for every hyperidentity there exists a monoid identity such that  $S\text{-Set}$  satisfies the hyperidentity if and only if  $S$  satisfies the monoid identity? If not, is there a modified version of this statement that is correct?

(iii) Is the analog of the result of (i), and of whatever answer you got for (ii), true for ring identities and hyperidentities of varieties  $R\text{-Mod}$ ? If not, how much can be said about the

relation between hyperidentities satisfied by varieties  $R\text{-Mod}$  and identities or other conditions satisfied by  $R$ ?

Because hyperidentities involve both universal quantification over derived operations and universal quantification over the algebra-elements to which these operations are applied, they tend to be very strong, and hence somewhat “crude” conditions, as illustrated by the fact that the variety of commutative rings satisfies no hyperidentities. *Existentially* quantified equations in derived operations, on the other hand, which translate to certain “ $\exists \forall$ ” conditions on operations and elements, have proved a more versatile tool in General Algebra. An example of this sort of condition on a variety  $\mathbf{V}$  is the statement that there exists a derived ternary operation  $M$  of  $\mathbf{V}$  satisfying the identities

$$M(x, x, y) = M(x, y, x) = M(y, x, x) = x.$$

This is satisfied, for instance, by the variety of lattices, where one can take  $M(x, y, z) = (x \vee y) \wedge (y \vee z) \wedge (z \vee x)$ . Another example is the condition of the same form, but with “ $= y$ ” on the right in place of “ $= x$ ”; this is satisfied by the variety of abelian groups of exponent 2, with  $M(x, y, z) = x + y + z$ . Many important technical conditions on a variety  $\mathbf{V}$  (for instance, the condition that for any two congruences  $E$  and  $E'$  on an object  $A$  of  $\mathbf{V}$ , one has  $E \circ E' = E' \circ E$  under composition of binary relations on  $|A|$ ; or the condition that each subalgebra  $B$  of a direct product algebra  $A_1 \times \dots \times A_n$  in  $\mathbf{V}$  is determined by its images in the pairwise products  $A_i \times A_j$ .) turn out to be equivalent to the statement that  $\mathbf{V}$  belongs to the union of some chain of classes, each determined by an existentially quantified equation in derived operations. The condition that a variety belong to such a union is called a *Mal'cev condition*; see [5, §II.12] and [11, §60] for examples and applications.

Finally, let me sketch the idea of another sort of structure, called an *operad*, similar to a clonal category but designed to apply to a wider class of situations. To motivate this, suppose that we wish to think of an algebra over a field  $k$ , not as a set with operations  $+$ ,  $0$ ,  $-$ ,  $\cdot$ , etc., but as a  $k$ -vector-space with a single additional  $k$ -bilinear operation “ $\cdot$ ”, and that we want to look at this in the context of other systems consisting of  $k$ -vector-spaces  $V$  given with  $k$ -multilinear operations satisfying various multilinear identities. To study such entities, we would like to set up an abstract model, analogous to a clonal category, but modeling, not an unstructured set with set-theoretic operations, but a vector space with multilinear operations. As in the situation that motivated clonal categories, one can form *derived* multilinear operations from given multilinear operations; but there are things one can do in a clone of set-theoretic operations but not in this context: The projection maps  $V^n \rightarrow V$  are not multilinear, so they will not appear in our structure, nor, for the same reason, will any derived operations in which some variable occurs more than once. On the other hand, there is also structure in this multilinear context which one does not have for ordinary clones, namely a  $k$ -vector-space structure on the set of multilinear operations of each arity, under which composition of multilinear operations is given by multilinear maps. The analog of a clonal category that one gets on taking these features into account is called a  *$k$ -linear operad*.

Now let the role that was held by direct products of sets in our development of the concept of clonal category, and implicitly by tensor products of vector spaces in that of  $k$ -linear operad (since a  $k$ -multilinear map  $V^n \rightarrow V$  is equivalent to a vector space map  $V \otimes_k \dots \otimes_k V \rightarrow V$ ) be filled by a general bifunctor “ $\square$ ” on some category  $\mathbf{C}$ , satisfying appropriate associativity conditions. One can write down a description of the sort of composition of operations that is possible without any more specific assumptions on  $\square$ . The structure one obtains in this way is called an *operad*.

For more details, see [69].

**8.10. Structure and Semantics.** The results of this section will not be essential to what follows, and our presentation will be sketchy. They give, however, a useful perspective on what we have been doing, and in the next chapter we will often cite them in formulating alternative descriptions of some concept.

Let us look back at the way we associated a clonal theory to a variety  $\mathbf{V}$  in Definition 8.9.6. I claim that the various equivalent forms of that construction all reduce to an observation that is applicable in much broader contexts, namely

**Lemma 8.10.1.** *Let  $\mathbf{C}$  be a category and  $A$  an object of  $\mathbf{C}$  such that all finite products  $A \times \dots \times A$  exist in  $\mathbf{C}$ , and suppose such a product  $A^n = \prod_{i \in n} A$  ( $n \in \omega$ ) is chosen for each  $n$ , so that the objects  $A^n$  are distinct in  $\text{Ob}(\mathbf{C})$ . Then the full subcategory of  $\mathbf{C}$  whose objects are the  $A^n$ , given with the projection maps  $p_{n,i}: A^n \rightarrow A$ , is a clonal category.  $\square$*

To see that this was essentially what we were using in Definition 8.9.6, note on the one hand that each free object  $F_{\mathbf{V}}(n)$  is a coproduct of  $n$  copies of  $F_{\mathbf{V}}(1)$ , hence in  $\mathbf{C}^{\text{op}}$ , the corresponding objects are products of  $n$  copies of one object, and the full subcategory of  $\mathbf{C}^{\text{op}}$  with these objects is one of our descriptions of the clonal theory of  $\mathbf{V}$ . The description based on looking at the products  $U_{\mathbf{V}}^n$  of copies of the functor  $U_{\mathbf{V}}$  used the same idea in the large category  $\text{Set}^{\mathbf{V}}$ .

We may generalize this latter example by considering any category  $\mathbf{C}$  given with a functor  $U: \mathbf{C} \rightarrow \text{Set}$ . The full subcategory of  $\text{Set}^{\mathbf{C}}$  having for objects the functors  $U^n$  will in general be large; however, in many cases it will be *quasi-small*, i.e., isomorphic to a small category  $\mathbf{X}$ . (This is true whenever the  $U^n$  are representable, or more generally, if the solution-set condition for representability holds, even though the other conditions may not.) To formalize this class of examples, let us make

**Definition 8.10.2.** *For the remainder of this section,  $\mathbf{Conc}$  will denote the large category having for objects all pairs  $(\mathbf{C}, U)$ , where  $\mathbf{C}$  is a category, and  $U$  a functor  $\mathbf{C} \rightarrow \text{Set}$ , such that for every integer  $n$ ,  $\text{Set}^{\mathbf{C}}(U^n, U)$  is quasi-small, and where a morphism  $(\mathbf{C}, U) \rightarrow (\mathbf{D}, V)$  means a functor  $F: \mathbf{C} \rightarrow \mathbf{D}$  such that  $VF = U$ .*

(I've chosen the symbol  $\mathbf{Conc}$  as an abbreviation for “concrete”, though that term is only an approximation, since we are not assuming that the functors to  $\text{Set}$  are faithful, while we are assuming a quasi-smallness hypothesis not in the definition of “concrete category”. The point of this terminology is to make us think of  $U$  (at least at the beginning) as like an “underlying-set functor”, so that we can picture the morphisms of  $\mathbf{Conc}$  as the underlying-set-preserving functors.)

If we associate to each object of  $\mathbf{Conc}$  the clonal category having for object-set the powers of  $U$  (Definition 6.8.5), this gives a contravariant construction (because of the way morphisms are defined in  $\mathbf{Conc}$ ) of clonal categories from these objects. Unfortunately, this cannot be regarded as a functor to  $\mathbf{Clone}$  because the values assumed, though quasi-small, are not in general small. Hence, for each  $(\mathbf{C}, U) \in \text{Ob}(\mathbf{Conc})$  let us choose a small category isomorphic to the category of natural-number powers of  $U$ , and regard this as an object of  $\mathbf{Clone}$ . (We did this in the preceding section, for the particular case where  $(\mathbf{C}, U)$  had the form  $(\mathbf{V}, U_{\mathbf{V}})$ , and we used the opposite of the category of free  $\mathbf{V}$ -algebras on the natural numbers in this way.) Thus we get a

functor  $\mathbf{Conc}^{\text{op}} \rightarrow \mathbf{Clone}$ . Since the morphisms  $X_n \rightarrow X_1$  in the category constructed in this way from  $(\mathbf{C}, U)$  correspond to the  $n$ -ary operations we can put on the sets  $U(C)$  ( $C \in \text{Ob}(\mathbf{C})$ ) in a functorial manner, the category can be thought of as describing the *algebraic structure* that can be put on the values the functor  $U$ ; hence Lawvere has named this functor ‘‘Structure’’.

**Exercise 8.10:1.** Describe precisely how to make Structure a functor. (Cf. Lemma 7.2.8.)

On the other hand, Lawvere calls the construction taking a clonal category  $\mathbf{X}$  to the variety  $\mathbf{X}\text{-Alg}$  given with its underlying set functor, i.e., the concrete category  $(\mathbf{X}\text{-Alg}, U_{\mathbf{X}\text{-Alg}})$  (which we have seen is also a contravariant construction) ‘‘Semantics’’, because it takes a category of *symbolic* operations, and *interprets* these in all possible ways as operations on sets.

Consider now an arbitrary  $(\mathbf{C}, U) \in \text{Ob}(\mathbf{Conc})$ , and let  $\mathbf{X} = \text{Structure}(\mathbf{C}, U)$ . By construction of  $\mathbf{X}$ , the sets  $U(C)$  ( $C \in \text{Ob}(\mathbf{C})$ ) have structures of  $\mathbf{X}$ -algebra, and these are functorial, in the sense that for  $f$  a morphism of  $\mathbf{C}$ , the set-map  $U(f)$  is a homomorphism of  $\mathbf{X}$ -algebras. This is equivalent to saying that we have an underlying-set-preserving functor  $(\mathbf{C}, U) \rightarrow (\mathbf{X}\text{-Alg}, U_{\mathbf{X}\text{-Alg}})$ . Of course, there are other clonal categories  $\mathbf{Y}$  for which one can put functorial  $\mathbf{Y}$ -algebra structures on these sets (e.g., clonal subcategories of  $\mathbf{X}$ ), but it is not hard to verify that  $\mathbf{X}$  is universal for this property, i.e., that every functorial  $\mathbf{Y}$ -algebra structure arises from a morphism of clones,  $\mathbf{Y} \rightarrow \mathbf{X}$ . This universal property is expressed in Lawvere’s celebrated slogan, ‘‘Structure is adjoint to Semantics’’.

Since in the universal property, the general clonal category  $\mathbf{Y}$  is mapped to the universal clonal category  $\mathbf{X}$ , the latter is *right universal*. So the precise statement is:

**Theorem 8.10.3** (Lawvere). *The functors*

$$\text{Structure: } \mathbf{Conc}^{\text{op}} \rightarrow \mathbf{Clone} \quad \text{and} \quad \text{Semantics: } \mathbf{Clone}^{\text{op}} \rightarrow \mathbf{Conc}$$

*are mutually right adjoint contravariant functors.*  $\square$

**Exercise 8.10:2.** Prove the above theorem.

As with any adjunction, we have a pair of *universal morphisms* connecting the two *composites* of these functors with the identity functors of the given categories. In the more familiar case of a *covariant* adjunction, one of these morphisms, the unit, goes from the identity functor to the composite (e.g., the map from each set  $X$  to the underlying set of the free group on  $X$ ), and the other, the counit, from the composite to the identity (e.g., from the free group on the underlying set of a group  $G$  to  $G$  itself). But in the case of a *contravariant* adjunction, they both go in the same direction; in the right-adjoint case, which we have here, from the identity functor to the composite functor. In the present example, one of these universal maps, namely

$$(8.10.4) \quad \text{Id}_{\mathbf{Clone}} \rightarrow \text{Structure} \circ \text{Semantics}$$

is an isomorphism; this is essentially the last assertion of Lemma 8.5.3. Looking at the other composite,

$$(8.10.5) \quad \text{Id}_{\mathbf{Conc}} \rightarrow \text{Semantics} \circ \text{Structure},$$

it is not hard to see that it will give an equivalence when applied to an object of  $\mathbf{Conc}$  if and only if that object is (up to equivalence) of the form  $(\mathbf{V}, U_{\mathbf{V}})$  where  $\mathbf{V}$  is a variety and  $U_{\mathbf{V}}$  its underlying-set functor. When we apply  $\text{Semantics} \circ \text{Structure}$  to a general object  $(\mathbf{C}, U)$  of

**Conc**, it can be thought of as giving a best approximation of that category by a variety and its underlying-set functor. Thus, for every given pair  $(\mathbf{C}, U)$ , (8.10.5) gives a “comparison functor”

$$(\mathbf{C}, U) \rightarrow \text{Semantics} \circ \text{Structure}(\mathbf{C}, U),$$

between the given object of **Conc** and that “approximation”.

**Exercise 8.10:3.** Describe  $\text{Structure}(\mathbf{C}, U)$  in each of the following cases (e.g., by choosing a set of “primitive operations” and identities), and determine whether the comparison functor is an equivalence.

- (i)  $\mathbf{C} = \mathbf{Set}$ ,  $U(X) = X \times X$ .
- (ii)  $\mathbf{C} = \mathbf{Set} \times \mathbf{Set}$ ,  $U(X, Y) = X \times Y$ .
- (iii)  $\mathbf{C} = \mathbf{Ab}$ ,  $U(X) = U_{\mathbf{Ab}}(X \times X)$ .
- (iv)  $\mathbf{C} = \mathbf{Ab} \times \mathbf{Ab}$ ,  $U(X, Y) = U_{\mathbf{Ab}}(X \times Y)$ .
- (v)  $\mathbf{C} = \mathbf{POSet}$ ,  $U =$  the underlying-set functor.

In cases (iii) and (iv), show that the clonal category  $\text{Structure}(\mathbf{C}, U)$  can be naturally identified with the clonal theory of modules over a certain ring.

**Exercise 8.10:4.** (i) Same task as in the above exercise, for  $\mathbf{C} = \mathbf{Set}^{\text{op}}$ , and  $U$  the power-set functor  $\mathbf{Set}^{\text{op}} \rightarrow \mathbf{Set}$ .

- (ii) If you are comfortable generalizing the concepts of this and the preceding section to algebras with operations of possibly infinite arities, getting in particular a functor  $\text{Structure}^{(\gamma)}: \mathbf{Conc}^{\text{op}} \rightarrow \mathbf{Clone}^{(\gamma)}$  for  $\gamma$  an infinite regular cardinal, investigate  $\text{Structure}^{(\gamma)}(\mathbf{C}, U)$  for the case of part (i).

**Exercise 8.10:5.** Let **CpLattice** denote the category of complete lattices, and **CpSemilattice**<sup>0</sup> the category of complete upper semilattices with least element (regarded as a zeroary operation). We recall that the objects of these two categories are essentially the same, but the morphisms are not (cf. Proposition 5.2.3).

- (i) Show that the underlying-set functor on one of these categories satisfies the smallness condition in the definition of **Conc**, but that of the other does not.
- (ii) In the case that does give an object  $(\mathbf{C}, U)$  of **Conc**, describe the variety  $\text{Semantics} \circ \text{Structure}(\mathbf{C}, U)$ . (Note that in contrast to part (ii) of the preceding exercise, we are here talking about finitary “Structure”.)

Let us end this section with a few observations on the question, “Given a category, how can one tell whether it is equivalent to a variety of algebras?” (Birkhoff’s Theorem tells us which full subcategories of a category  $\Omega\text{-Alg}$  are varieties, but the above question, about abstract categories and equivalence, is of a different sort.) By our preceding observations, a necessary and sufficient condition is that there exist a functor  $U: \mathbf{C} \rightarrow \mathbf{Set}$  such that  $(\mathbf{C}, U)$  lies in **Conc**, and the comparison functor

$$(\mathbf{C}, U) \rightarrow \text{Semantics} \circ \text{Structure}(\mathbf{C}, U)$$

is an equivalence. Note also that the underlying-set functor of any variety is *representable* (by the free object on one generator), so if the above condition holds,  $U$  can be taken to have the form  $h_G$  for some object  $G$  of  $\mathbf{C}$ . In this situation (since by our general convention,  $\mathbf{C}$  is assumed legitimate), the quasi-smallness condition on the powers of  $U$  automatically holds by Yoneda’s Lemma. In summary:

**Lemma 8.10.6.** *A category  $\mathbf{C}$  is equivalent to a variety of finitary algebras if and only if there exists some  $G \in \text{Ob}(\mathbf{C})$  such that the comparison map*

$$(8.10.7) \quad (\mathbf{C}, h_G) \rightarrow \text{Semantics} \circ \text{Structure}(\mathbf{C}, h_G)$$

*is an isomorphism in **Conc**.*

*(The analogous result holds with “finitary” replaced by “having all operations of arity  $\leq \gamma$ ” for any fixed regular infinite cardinal  $\gamma$ , if we use corresponding modified functors  $\text{Structure}^{(\gamma)}$  and  $\text{Semantics}^{(\gamma)}$ .)  $\square$*

Though this does not say very much, it gives a useful heuristic pointer: If we want to determine whether a category  $\mathbf{C}$  is equivalent to a variety of algebras, we should look at possible candidates for the free object on one generator. The next exercise gives several cases where you can show that no such object exists. I do not advise trying to use the above lemma in this and the next two exercises, but only the “heuristic pointer”.

**Exercise 8.10:6.** Show that none of the following categories are equivalent to varieties of algebras, even if we allow the latter to have infinitary operations (though, as always, we assume the set of all operations to be small).

- (i) **POSet**. (Suggestion: For each of the situations (a)  $\mathbf{C}$  a variety of algebras, and  $A$  a free algebra in  $\mathbf{C}$  on a nonempty set, (b)  $\mathbf{C} = \mathbf{POSet}$ , and  $A$  a discrete partially ordered set, and (c)  $\mathbf{C} = \mathbf{POSet}$ , and  $A$  a nondiscrete partially ordered set, investigate the relationship between the set of coequalizer maps in  $\mathbf{C}$ , and the set of morphisms in  $\mathbf{C}$  that  $h_A$  takes to surjective set maps.)
- (ii) **Compact**, the category of compact Hausdorff spaces and continuous maps. (Suggestion: If  $\mathbf{V}$  is a variety with all operations having arities  $< \gamma$ , what does this imply about the closure operator “subalgebra generated by  $-$ ” on the underlying sets of algebras in  $\mathbf{V}$ ? (Cf. Definition 5.3.7 for the case  $\gamma = \omega$ .) Translate this into a statement involving the free object on one generator in  $\mathbf{V}$ , and show that no object has this property in **Compact**.)
- (iii) The full subcategory of **Ab** whose objects are the torsion-free abelian groups.
- (iv) The full subcategory of **Ab** whose objects are the divisible abelian groups (groups such that for every group element  $x$  and nonzero integer  $n$ , the equation  $ny = x$  has a solution  $y$  in  $A$ .)

**Exercise 8.10:7.** In contrast to the last two cases above, show that the full subcategory of **Ab** whose objects are the divisible torsion-free abelian groups *is* equivalent to a variety of algebras.

**Exercise 8.10:8.** Show that **Clone** is not equivalent to any variety of finitary algebras. (Suggestion: Show that (a) an object corresponding to a free object on one generator would have to be a finitely generated clonal category, (b) if it were generated by elements of arities  $\leq n$ , this would be true of all clonal categories, and (c) this is not the case.)

Can you prove that it is or is not equivalent to a variety of possibly infinitary algebras?

In contrast to Exercise 8.10:6(ii), it is proved in [98] that **Compact** *can* be identified with a “variety” if we generalize that concept to allow a *large* set of operations – as we would also have to do, for instance, to speak of the “variety” of complete lattices or semilattices. Under this construction of **Compact**, the operations of each cardinality  $\alpha$  correspond to the points of the Stone-Čech compactification of the discrete space  $\alpha$ . Note that this means that, in contrast to the case of complete lattices (but as for complete upper or lower semilattices, cf. Exercise 8.10:5), for each  $\alpha$ , the set of  $\alpha$ -ary operations is small; i.e., the corresponding generalized clonal category, though not generated by a small set, is legitimate. A consequence is that compact Hausdorff spaces actually behave more like ordinary algebras than do complete lattices! In particular, there is a

“free compact Hausdorff space” on every small set  $X$ , namely, the Stone-Čech compactification of  $X$  as a discrete space.

The difference between the cases of complete *semilattices* and *lattices* noted in the above paragraph has the curious consequence that though complete semilattices behave “well”, the category of sets with *two* complete semilattice operations  $\vee_1$  and  $\vee_2$  does not, since if it behaved like a variety, then complete lattices would behave like a subvariety. As a still more striking example of this sort, though compact Hausdorff spaces are well behaved, the category of sets with a compact Hausdorff topology and a single unary operation not assumed continuous in that topology will not have a free object on one generator. Indeed any non-limit ordinal  $\alpha$  can be given a compact Hausdorff topology in which each nonzero limit ordinal  $\beta < \alpha$  is a topological limit of the lower ordinals; and using this topology and the unary successor operation, the whole algebra  $\alpha$  will be generated by  $\{0\}$ .

Lemma 8.10.6(ii) does *not* say that an object  $G$  with the indicated properties is unique up to isomorphism if it exists. Let us examine the extent to which we can vary  $G$  in a couple of familiar varieties.

**Exercise 8.10:9.** (i) When  $\mathbf{C} = \mathbf{Ab}$ , determine for what objects  $G$  the functor (8.10.7) is an equivalence. Show that for every such  $G$ ,  $\text{Structure}(\mathbf{Ab}, h_G)$  can be identified with the theory of modules over some ring  $R$ .

(ii) Similarly, for  $\mathbf{C} = \mathbf{Set}$  determine what objects make (8.10.7) an equivalence, and try to describe in these cases the theory  $\text{Structure}(\mathbf{Set}, h_G)$ .

The answer to (i) shows that  $\mathbf{Ab}$  is equivalent to several different varieties  $R\text{-Mod}$ , and in (ii) we similarly discover that  $\mathbf{Set}$  is equivalent to several varieties of algebras.

Lawvere gives in his thesis [14, §III.2] a version of Lemma 8.10.6 which is less trivial than ours, but also more complicated to formulate; we will not present it here.

Despite the technical meaning given the word “structure” in this section, we will also continue to use it as a non-specific meta-term in our mathematical discussions.