

Part III. More on adjunctions.

Chapter 9 (the only chapter of this part yet written) represents the culmination of the course. In it we obtain Freyd's beautiful characterization of functors among varieties of algebras that have left adjoints, and study several classes of examples and related results.

Chapter 9. Algebra and coalgebra objects in categories, and functors having adjoints.

One of our long-range goals, since we took our “Cook’s tour” of universal constructions in Chapter 3, has been to obtain general results on when algebras with given universal properties exist. We have gotten several existence results holding in any variety \mathbf{V} , namely, for free objects, limits and colimits, and objects presented by generators and relations. The result for free objects can be restated as the existence of a left adjoint to the forgetful functor $\mathbf{V} \rightarrow \mathbf{Set}$, and we have also shown that the inclusion $\mathbf{V} \rightarrow \Omega\text{-Alg}$ has a left adjoint, where Ω is the type of \mathbf{V} . In the first four sections of this chapter, we shall develop a result of a much more sweeping sort: a characterization of *all* functors between varieties of algebras \mathbf{V} and \mathbf{W} which have left adjoints.

To get an idea what this characterization should be, we should look at some representative examples. Most of the functors with left adjoints among varieties of algebras that we have seen so far have been cut from a fairly uniform mold: underlying-set-preserving constructions that forget some of the operations, and things close to these. We shall begin by looking at an example of a different sort, which will give us some perspective on the features that make the construction of the adjoint possible. We will then formalize these features, arriving at a pair of concepts (those of algebra and coalgebra objects in a general category) of great beauty in their own right, in terms of which we shall establish the desired condition in §9.4. In the remaining sections of this chapter we will work out in detail several general cases, and note various related results.

9.1. An example: $\mathbf{SL}(n)$. Let n be a positive integer. Then for any commutative ring A , the $n \times n$ matrices over A having determinant 1 form a group, called the *special linear* group $\mathbf{SL}(n, A)$. (Recall from §3.12 that rings are assumed to be associative and have 1 unless the contrary is stated.) Clearly, $\mathbf{SL}(n, -)$ is a functor $\mathbf{CommRing}^1 \rightarrow \mathbf{Group}$. Let us simplify our name for this functor to $\mathbf{SL}(n)$, but continue to write its value at A as $\mathbf{SL}(n, A)$.

Does $\mathbf{SL}(n)$ have a left adjoint? In concrete terms, this asks: Given a group G , can we find a universal example of a commutative ring A_G with a homomorphism $G \rightarrow \mathbf{SL}(n, A_G)$?

Let us approach this question in our standard way (first noted in comment 2.2.10), namely, by considering an arbitrary commutative ring A with a homomorphism

$$h: G \rightarrow \mathbf{SL}(n, A),$$

and asking what elements of A , and what relations among these, are determined by this situation.

Clearly, we can get n^2 elements of A from each element g of G , to wit, the components of the matrix $h(g)$:

$$(9.1.1) \quad h(g)_{ij} \quad (g \in |G|, \quad i, j = 1, \dots, n).$$

By definition of $\mathbf{SL}(n, A)$, these satisfy the relation saying that the determinant of the matrix they form is 1:

$$(9.1.2) \quad \det(h(g)_{ij}) = 1 \quad (g \in |G|).$$

The condition that h be a group homomorphism says that for every two elements $g, g' \in |G|$, the matrix $(h(gg')_{ij})$ is the product of the matrices $(h(g)_{ij})$ and $(h(g')_{ij})$. Each such matrix equation is equivalent to n^2 equations in the ring A :

$$(9.1.3) \quad h(gg')_{ik} = \sum_j h(g)_{ij} h(g')_{jk} \quad (g, g' \in |G|, \quad i, k = 1, \dots, n).$$

Clearly, a system of elements (9.1.1) satisfying (9.1.2) and (9.1.3) is equivalent to a homomorphism $G \rightarrow SL(n, A)$. Hence, if we let A_G be the object of **CommRing**¹ presented by *generators* (9.1.1) and *relations* (9.1.2) and (9.1.3), and denote by $h: G \rightarrow SL(n, A_G)$ the resulting group homomorphism, then the pair (A_G, h) will be initial among commutative rings A given with such homomorphisms, and the construction $G \mapsto A_G$ will be the desired left adjoint to $SL(n)$.

What properties of the functor $SL(n)$ have we used here? First, the fact that for every commutative ring A , the elements of $SL(n, A)$ could be described as all families of elements of A indexed by a certain fixed set (in this case the set $n \times n$) which satisfied certain equations (in this case, the single equation saying that the matrix they formed had determinant 1). It was this that allowed us to write down the generators (9.1.1) and relations (9.1.2) in the definition of A_G . Secondly, we used the fact that the multiplication of the group $SL(n, A)$ takes a pair of matrices s, t to a matrix st whose entries are given by certain fixed polynomials (i.e., derived operations) in the $2n^2$ entries of the two given matrices. This allowed us to express the condition that h be a homomorphism by the equations (9.1.3).

We also used, implicitly, a fact special to the variety of groups, namely that for a map of underlying sets to be a homomorphism, it suffices that it respect multiplication. If we want to put this example into a form that generalizes to arbitrary varieties, we should note that the unary “inverse” operation and the zeroary “neutral element” operation of $SL(n, A)$ also have the property that their entries are given by polynomials in the entries of their arguments: The inverse of a matrix of determinant 1 is a matrix of determinants of minors (with certain \pm signs); the identity matrix consists of 0’s and 1’s in certain positions, and these 0’s and 1’s can be regarded as polynomials in the empty set of variables. Hence if we do not wish to call on the special property of group homomorphisms mentioned, we can still guarantee the universal property of A_G , by supplementing (9.1.3) with relations saying that for all $g \in |G|$, the entries of $h(g^{-1})$ are given by the appropriate signed minors in the entries of $h(g)$, and that the (i, j) entry of $h(e)$ has the value δ_{ij} .

To abstract the conditions noted above, let us now consider arbitrary varieties \mathbf{V} and \mathbf{W} (in general of different types), and a functor

$$V: \mathbf{W} \rightarrow \mathbf{V}$$

for which we hope to find a left adjoint. The analog of the first property noted for $SL(n)$ above should be that for $A \in \text{Ob}(\mathbf{W})$, the underlying set $|V(A)|$ is describable as the set of X -tuples of elements of $|A|$, for some fixed set X , which satisfy a fixed set Y of relations. We recall from Lemma 8.4.16 that this is equivalent to saying that the set-valued functor $A \mapsto |V(A)|$, i.e., the functor $U_{\mathbf{V}}V$ (where $U_{\mathbf{V}}: \mathbf{W} \rightarrow \mathbf{Set}$ is the underlying-set functor of \mathbf{W}) is *representable*, with representing object the \mathbf{W} -algebra defined using X and Y as generators and relations:

$$(9.1.4) \quad R = \langle X \mid Y \rangle_{\mathbf{W}}.$$

The object (9.1.4) thus “encodes” the functor V at the set level! Is there a way to extend these observations so as to encode also the \mathbf{V} -algebra structures on the sets $|V(A)|$?

Let us look at this question in the case $V = SL(n)$. We see that the functor $U_{\mathbf{Group}} \circ SL(n)$ is represented by the commutative ring R presented by n^2 generators r_{ij} and one relation $\det(r_{ij}) = 1$; in other words, the commutative ring having a universal $n \times n$ matrix r of determinant 1. Can we find a universal instance of *multiplication* of such matrices? Since multiplication is a binary operation, we should multiply a universal *pair* of matrices of

determinant 1. The ring with such a universal pair is the coproduct of two copies of R . If we denote these two matrices $r_0, r_1 \in |\mathrm{SL}(n, R \amalg R)|$, then the n^2 entries of the product matrix $r_0 r_1 \in |\mathrm{SL}(n, R \amalg R)|$ can, like any elements of $R \amalg R$, be expressed as polynomials in our generators for that ring, the entries of r_0 and r_1 . Using the universality of $r_0, r_1 \in |\mathrm{SL}(n, R \amalg R)|$, it is not hard to show that those same polynomials, when applied to the entries of two *arbitrary* elements of $\mathrm{SL}(n, A)$ for an *arbitrary* commutative ring A , must also give the entries of their product. So it appears that $r_0 r_1$ does in some sense encode the multiplication operation of $\mathrm{SL}(n)$.

There is a more abstract way of looking at this encoding. By the universal property of R , the element $r_0 r_1 \in |\mathrm{SL}(n, R \amalg R)|$ corresponds to some morphism

$$(9.1.5) \quad \mathbf{m}: R \rightarrow R \amalg R$$

(the unique morphism taking the entries of r to those of $r_0 r_1$). Now given a commutative ring A , any two elements $x, y \in |\mathrm{SL}(n, A)|$ arise as images of the universal element $r \in |\mathrm{SL}(n, R)|$ under unique homomorphisms $f, g: R \rightarrow A$. Such a pair of morphisms corresponds, by the universal property of the coproduct, to a single morphism $(f, g): R \amalg R \rightarrow A$ (the morphism carrying the entries of r_0 to those of x and the entries of r_1 to those of y). Composing with (9.1.5), we get a morphism

$$(9.1.6) \quad R \xrightarrow{\mathbf{m}} R \amalg R \xrightarrow{(f, g)} A,$$

which corresponds to an element of $\mathrm{SL}(n, A)$. From the facts that \mathbf{m} corresponds to (i.e., sends r to) the *product* of r_0 and r_1 , and that $\mathrm{SL}(n)$, applied to the map (f, g) gives a *group homomorphism* $\mathrm{SL}(n, R \amalg R) \rightarrow \mathrm{SL}(n, A)$, we can deduce that the matrix given by (9.1.6) (i.e., the result of applying the ring-homomorphism (9.1.6) entrywise to r) is the product of x and y . So the ring homomorphism \mathbf{m} indeed “encodes” our multiplication.

We note similarly that $r^{-1} \in |\mathrm{SL}(n, R)|$ will be the image of the universal element r under a certain morphism

$$(9.1.7) \quad \mathbf{i}: R \rightarrow R$$

and we find that this morphism \mathbf{i} encodes the *inverse* operation on $\mathrm{SL}(n)$.

If we are going to treat the zeroary neutral-element operation similarly, it should correspond to a morphism from R to the coproduct of zero copies of itself. This vacuous coproduct is the *initial object* of $\mathbf{CommRing}^1$, namely the ring \mathbb{Z} of integers. And indeed, if we let

$$(9.1.8) \quad \mathbf{e}: R \rightarrow \mathbb{Z}$$

be the map sending the universal element $r \in |\mathrm{SL}(n, R)|$ to the identity matrix in $\mathrm{SL}(n, \mathbb{Z})$, we find that for every commutative ring A , the composite of (9.1.8) with the unique homomorphism $\mathbb{Z} \rightarrow A$ is the morphism $R \rightarrow A$ that specifies the identity matrix of A .

The structure $(R, \mathbf{m}, \mathbf{i}, \mathbf{e})$ sketched above is, as we shall soon see, what is called a *cogroup* in the category $\mathbf{CommRing}^1$. The maps (9.1.5), (9.1.7), (9.1.8) are called its *comultiplication*, its *coinverse*, and its *co-neutral-element*, and the cogroup $(R, \mathbf{m}, \mathbf{i}, \mathbf{e})$ is said to *represent* the functor $\mathrm{SL}(n): \mathbf{CommRing}^1 \rightarrow \mathbf{Group}$, just as R alone is said to represent the functor $U_{\mathbf{Group}} \circ \mathrm{SL}(n): \mathbf{CommRing}^1 \rightarrow \mathbf{Set}$.

In the next three sections we shall develop general definitions and results of which the case sketched above is an example. We shall see that given a functor $V: \mathbf{W} \rightarrow \mathbf{V}$, if the first of the two properties we called on above holds, namely that the set-valued functor $U_{\mathbf{V}} V$ is

representable, then the other condition, that the operations of the algebras $V(A)$ arise from a co- \mathbf{V} -structure on the representing object, follows automatically. (Indeed, our development of (9.1.5) above did not use our knowledge that the group operations of $\mathrm{SL}(n)$ had this form, but deduced that fact from their functoriality.)

This does not mean that we will ignore the co- \mathbf{V} structure, however! Rather, since it encodes the \mathbf{V} -algebra structure of our otherwise merely set-valued functors, it will be the key to the investigation of these constructions.

9.2. Algebra objects in a category. For pedagogic reasons, let us approach the concept of a coalgebra object in a category \mathbf{C} by starting with the dual concept, that of an algebra object. Let us make:

Convention 9.2.1. *Throughout this section, γ will be a regular cardinal, \mathbf{C} will be a category admitting products indexed by all families of cardinality $< \gamma$ (which we will abbreviate to “ $< \gamma$ -fold products”), and Ω will be a type all of whose operations have arities $< \gamma$.*

(If you are most comfortable with finitary algebras, you may assume $\gamma = \omega$ without missing any of the ideas of this chapter.)

Definition 9.2.2. *For $\beta < \gamma$, a β -ary operation on an object R of \mathbf{C} will mean a morphism $s = s_R: R^\beta \rightarrow R$.*

By Yoneda’s Lemma, such operations correspond bijectively to morphisms of the induced contravariant hom-functors, $h^{R^\beta} \rightarrow h^R$; and by the universal property of the product object R^β , we can identify h^{R^β} with $(h^R)^\beta$, so such a map corresponds to a morphism $(h^R)^\beta \rightarrow h^R$, i.e., a β -ary operation on h^R . In concrete terms, if s_R is a β -ary operation of R , then given an object A of \mathbf{C} and a β -tuple of elements $(\xi_\alpha)_{\alpha < \beta} \in \mathbf{C}(A, R)^\beta$, we first combine these into a single element of $\mathbf{C}(A, R^\beta)$, then compose this with $s_R: R^\beta \rightarrow R$ to get an element of $\mathbf{C}(A, R)$, which we may denote $s_{\mathbf{C}(A, R)}((\xi_\alpha)_{\alpha < \beta})$. This is the category-theoretic abstraction of the familiar technique of making the set of functions from a space A to an algebra R an algebra under *pointwise* application of the operations of R . These observations are summarized in the next lemma (in which the equivalence of (ii) and (iii) holds by the definition of morphism of functors).

Lemma 9.2.3. *Let β be a cardinal $< \gamma$, and R an object of \mathbf{C} . Then the following data are equivalent (via the construction just described):*

- (i) A β -ary operation $s_R: R^\beta \rightarrow R$.
- (ii) A morphism $s_{\mathbf{C}(-, R)}: \mathbf{C}(-, R)^\beta \rightarrow \mathbf{C}(-, R)$ as functors $\mathbf{C}^{\mathrm{op}} \rightarrow \mathbf{Set}$, i.e., as contravariant set-valued functors on \mathbf{C} .
- (iii) A way of defining on each set $\mathbf{C}(A, R)$ ($A \in \mathrm{Ob}(\mathbf{C})$) a β -ary operation $s_{\mathbf{C}(A, R)}: \mathbf{C}(A, R)^\beta \rightarrow \mathbf{C}(A, R)$, so that for every morphism $f \in \mathbf{C}(A, B)$, the induced map $\mathbf{C}(B, R) \rightarrow \mathbf{C}(A, R)$ respects these operations. \square

Recalling that Ω denotes a type all of whose operation-symbols have arities $< \gamma$, we now make

Definition 9.2.4. An Ω -algebra object R in the category \mathbf{C} (or a \mathbf{C} -based Ω -algebra) will mean a pair $(|R|, (s_R)_{s \in |\Omega|})$, where $|R| \in \text{Ob}(\mathbf{C})$, and each s_R is an operation

$$s_R: |R|^{\text{ari}(s)} \rightarrow |R| \quad (s \in |\Omega|).$$

A morphism between Ω -algebra objects of \mathbf{C} will mean a morphism between their underlying \mathbf{C} -objects which forms commuting squares with these operations.

If R is an Ω -algebra object of \mathbf{C} , and A any object of \mathbf{C} , then $\mathbf{C}(A, R)$ will denote the ordinary (i.e., set-based) Ω -algebra with underlying set $\mathbf{C}(A, |R|)$, and operations induced by those of R as in Lemma 9.2.3.

Below, the word ‘‘algebra’’ will continue to mean ‘‘set-based algebra’’ except when the contrary is indicated by writing ‘‘algebra object’’, ‘‘ \mathbf{C} -based algebra’’, etc.. When referring to set-based algebras, I may occasionally add the words ‘‘set-based’’ for emphasis.

Observe that the $| \cdot |$ -notation introduced above is relative. E.g., if \mathbf{C} is itself a category of algebras, and R a \mathbf{C} -based algebra, then $|R|$ denotes the *underlying \mathbf{C} -object* of R , and if S is this \mathbf{C} -object, then $|S| = ||R||$ denotes its *underlying set*. I shall, in fact, sometimes, as in the above definition, use the letter R and its alphabetical neighbors for algebra-objects in categories \mathbf{C} , and other times, as in Lemma 9.2.3, use it for the underlying \mathbf{C} -objects of such objects. Of course, in any given statement I shall be consistent about which meaning I am giving a symbol.

Finally, the reader should note the new use of the symbol $\mathbf{C}(A, R)$ introduced in the above definition: Though A denotes an object of \mathbf{C} , R does not; rather, it is a *\mathbf{C} -based Ω -algebra*, and the whole symbol denotes, not a set, but a (set-based) Ω -algebra. Of course, a \mathbf{C} -based Ω -algebra is intuitively ‘‘an object of \mathbf{C} with additional structure’’, and an Ω -algebra is likewise a set with additional structure; and modulo this additional structure, we have the old meaning of $\mathbf{C}(A, R)$. So this extended notation is ‘‘reasonable’’. But we need to remember when discussing algebra objects of categories that to know what is meant by a symbol $\mathbf{C}(A, R)$, we have to check whether R is assumed to be an object of \mathbf{C} , or is a \mathbf{C} -based Ω -algebra for some Ω .

The above definition also introduced the concept of a *morphism* of \mathbf{C} -based Ω -algebras. Combining this with Yoneda’s Lemma, we easily get

Lemma 9.2.5. Let R and S be Ω -algebra objects in \mathbf{C} . Then the following data are equivalent:

- (i) A morphism of \mathbf{C} -based algebras $R \rightarrow S$.
- (ii) A morphism $f \in \mathbf{C}(|R|, |S|)$ such that for every object A of \mathbf{C} , the induced set map $\mathbf{C}(A, |R|) \rightarrow \mathbf{C}(A, |S|)$ is a homomorphism of Ω -algebras $\mathbf{C}(A, R) \rightarrow \mathbf{C}(A, S)$.
- (iii) A morphism $\mathbf{C}(-, R) \rightarrow \mathbf{C}(-, S)$ of functors $\mathbf{C} \rightarrow \Omega\text{-Alg}$. \square

We next want to define, for an Ω -algebra object R of a category \mathbf{C} , the *derived operations* of R corresponding to the various derived operations of set-based Ω -algebras. This will allow us to say what it means for such an object to satisfy a given *identity*; namely, that the derived operations specified by the two sides of the identity are equal.

One cannot, of course, describe a derived operation of R by giving a formula for its value on a tuple of ‘‘elements of $|R|$ ’’ when \mathbf{C} is a general category. An approach that is often used is to express operations and identities by diagrams. For example, observe that if m is a binary operation on a *set* $|R|$, the condition that m be associative can be expressed as the condition that the diagram

$$(9.2.6) \quad \begin{array}{ccc} |R| \times |R| \times |R| & \xrightarrow{m \times \text{id}_{|R|}} & |R| \times |R| \\ \downarrow \text{id}_{|R|} \times m & & \downarrow m \\ |R| \times |R| & \xrightarrow{m} & |R| \end{array}$$

commute, since the path that goes through the upper right-hand corner gives the ternary derived operation $(x, y, z) \mapsto m(m(x, y), z)$, and the one through the lower left-hand corner gives $(x, y, z) \mapsto m(x, m(y, z))$. Analogously, for any object $|R|$ of a general category \mathbf{C} and any binary operation $m: |R| \times |R| \rightarrow |R|$, the same diagram can be used to define two ternary “derived operations” on $|R|$, and their equality (the commutativity of the diagram) can be made the definition of associativity of the \mathbf{C} -based algebra $R = (|R|, m)$.

The above approach is nice in simple cases, but has the disadvantage of requiring us to figure out the diagram appropriate to every derived operation we want to consider. Another approach, which is equivalent to the above but avoids this dependence on diagrams, is based on considering the algebra $\mathbf{C}(A, R)$ for an appropriate *universal choice* of A . If we want to consider derived operations in β variables, let us look at $\mathbf{C}(|R|^\beta, R)$. Since this is a set-based algebra, we know how to construct its derived β -ary operations from its primitive operations. Applying such a derived operation u to the β projections $p_\alpha: |R|^\beta \rightarrow |R|$ ($\alpha \in \beta$), we get an element $u((p_\alpha)_{\alpha \in \beta}) \in \mathbf{C}(|R|^\beta, |R|)$ which we *define* to be the derived operation u_R of the \mathbf{C} -based algebra R . Identities are then defined as equalities among such derived operations.

Incidentally, although in §8.4 we found it convenient to reduce all identities for Ω -algebras to identities (pairs of terms) in a fixed γ -tuple of variables, we shall here revert to expressing them as identities in β -tuples of variables for various ordinals $\beta < \gamma$. (So, for instance, the diagram (9.2.6) expresses associativity using three variables, rather than countably many.) The advantage will be that we only need to assume that \mathbf{C} has these β -fold products, rather than making the unnecessary stronger assumption that it has γ -fold products.

It is easy to see (cf. Lemma 8.10.1) that the operations of arity $< \gamma$ on $|R|$ (equivalently, on $h^{|R|}$) yield a γ -clonal category; and that a \mathbf{C} -based Ω -algebra structure on $|R|$ as defined above is equivalent to a representation of the γ -clonal category $\mathbf{Cl}^{(\gamma)}(\Omega\text{-Alg})$ in \mathbf{C} which takes $X_1 \in \text{Ob}(\mathbf{Cl}^{(\gamma)}(\Omega\text{-Alg}))$ to $|R| \in \text{Ob}(\mathbf{C})$. The condition that this \mathbf{C} -based algebra R satisfy the identities of a given variety \mathbf{V} is equivalent to saying that this representation of $\mathbf{Cl}^{(\gamma)}(\Omega\text{-Alg})$ arises from (i.e., factors through) a representation of $\mathbf{Cl}^{(\gamma)}(\mathbf{V})$:

$$(9.2.7) \quad \mathbf{Cl}^{(\gamma)}(\Omega\text{-Alg}) \rightarrow \mathbf{Cl}^{(\gamma)}(\mathbf{V}) \rightarrow \mathbf{C},$$

where the first arrow is induced by the given indexing of the operations of \mathbf{V} by Ω .

In the next lemma and definition we set down the observations of the preceding paragraphs, and prove the one nontrivial implication.

Lemma 9.2.8. *Let $R = (|R|, (s_R)_{s \in |\Omega|})$ be an Ω -algebra object of \mathbf{C} , and let u, v be two derived β -ary operations ($\beta < \gamma$) for ordinary (i.e., set-based) algebras of type Ω . Then the following conditions are equivalent:*

- (i) *For all $A \in \text{Ob}(\mathbf{C})$, the algebra $\mathbf{C}(A, R)$ satisfies the identity $u = v$.*
- (ii) *In the algebra $\mathbf{C}(|R|^\beta, R)$, one has $u((p_\alpha)_{\alpha \in \beta}) = v((p_\alpha)_{\alpha \in \beta})$, where the p_α ($\alpha \in \beta$) are the projection maps.*
- (iii) *The morphisms $u, v: Cl_\beta(\mathbf{V}) \rightrightarrows Cl_1(\mathbf{V})$ in the γ -clonal category $\mathbf{Cl}^{(\gamma)}(\Omega\text{-Alg})$ fall*

together under the functor from $\mathbf{Cl}^{(\gamma)}(\Omega\text{-Alg})$ to the γ -clonal theory of $|R|$ induced by the $|\Omega|$ -tuple of operations (s_R) . (See Lemma 8.9.10 for the universal property of $\mathbf{Cl}^{(\gamma)}(\Omega\text{-Alg})$ which allows one to define this morphism.)

(iv) The algebra object R satisfies the “diagrammatic translation” of the identity $u = v$.

Proof. (ii)-(iv) are simply different ways of stating the same condition. Clearly, (i) \Rightarrow (ii). The converse can be gotten by Yoneda’s Lemma; to see it directly, consider any object A of \mathbf{C} and any β -tuple $(\xi_\alpha)_{\alpha \in \beta}$ of elements of $\mathbf{C}(A, |R|)$. By the universal property of the product object $|R|^\beta$, these morphisms correspond to a single morphism $\xi: A \rightarrow |R|^\beta$, and applying $\mathbf{C}(-, R)$ we get an Ω -algebra homomorphism $\mathbf{C}(|R|^\beta, R) \rightarrow \mathbf{C}(A, R)$ carrying each p_α to ξ_α . Hence, any equation satisfied by the former β -tuple is also satisfied by the latter. \square

Definition 9.2.9. If the equivalent conditions of Lemma 9.2.8 hold, the Ω -algebra object R of \mathbf{C} will be said to satisfy the identity $u = v$.

If \mathbf{V} is a variety of Ω -algebras, defined by a family J of identities, then a \mathbf{V} -object of \mathbf{C} will mean an Ω -algebra object R of \mathbf{C} satisfying the identities in J in this sense; equivalently, such that the induced functor $\mathbf{C}(-, R)$ carries \mathbf{C} into \mathbf{V} ; equivalently, such that the corresponding representation of $\mathbf{Cl}^{(\gamma)}(\Omega\text{-Alg})$ in \mathbf{C} arises as in (9.2.7) from a representation of $\mathbf{Cl}^{(\gamma)}(\mathbf{V})$ in \mathbf{C} .

Of course, since the same subvariety $\mathbf{V} \subseteq \Omega\text{-Alg}$ can be determined by more than one set of identities J , we need to be sure that the condition of being a \mathbf{V} -object of \mathbf{C} is independent of our choice of defining identities. The formulation “ $\mathbf{C}(-, R)$ carries \mathbf{C} into \mathbf{V} ” in the above definition makes this clear.

We have been discussing how to put operations on representable functors $\mathbf{C}(-, |R|): \mathbf{C}^{\text{op}} \rightarrow \mathbf{Set}$ ($|R| \in \text{Ob}(\mathbf{C})$), and when such operations will satisfy the identities of a variety \mathbf{V} . Note now that the concept of “a representable set-valued functor given with operations that make it \mathbf{V} -valued” can also be looked at as “a \mathbf{V} -valued functor whose composite with the forgetful functor $\mathbf{V} \rightarrow \mathbf{Set}$ is a representable set-valued functor”. This yields the equivalence of the two formulations of the next definition, in which we extend the term “representable functor” to include algebra-valued constructions.

Definition 9.2.10. If \mathbf{V} is a variety of Ω -algebras, a functor $\mathbf{C}^{\text{op}} \rightarrow \mathbf{V}$ will be called representable if it is isomorphic to a functor of the form $\mathbf{C}(-, R)$, for R a \mathbf{V} -object of \mathbf{C} , equivalently, if its composite with the underlying-set functor $\mathbf{V} \rightarrow \mathbf{Set}$ is representable in the sense already defined for set-valued functors (Definition 7.2.3).

9.3. Coalgebra objects in a category. In the next few sections we shall study *coalgebra* objects, and the functors these represent. A \mathbf{V} -coalgebra object in a category \mathbf{C} will be defined simply as a \mathbf{V} -algebra object in \mathbf{C}^{op} . But pedagogically, the relationship between these two concepts is tricky. The definition of algebra object is easier to think about (to begin with) because it generalizes the familiar concept of a set-based algebra. But in varieties of algebras, *coalgebra* objects and the covariant functors they represent will turn out to be more diverse and interesting than algebra objects and their associated contravariant representable functors, and, as suggested by our example of $\text{SL}(n)$, they will be the main object of study in this chapter. Hence our flip-flop approach of using the algebra concept to introduce the definitions, but then moving immediately to coalgebras. However, in §§9.12-9.13 we will return briefly to algebra objects, and note some

examples and results on these.

In this section we continue to assume that γ is a regular cardinal, and Ω a type all of whose operations have arity $< \gamma$. However, we drop here the assumption of the preceding section that \mathbf{C} has $< \gamma$ -fold products; what we will want is the dual hypothesis, and we will state that explicitly when it is needed, as in the following definition.

Definition 9.3.1. *Let \mathbf{C} be a category having coproducts of all families of $< \gamma$ objects. Then for $\beta < \gamma$, a β -ary co-operation on an object $|R|$ of \mathbf{C} will mean a morphism of $|R|$ into the coproduct of β copies of $|R|$; in other words, a β -ary operation on $|R|$ in \mathbf{C}^{op} . A pair $R = (|R|, (\mathbf{s}_R)_{s \in |\Omega|})$ such that $|R| \in \text{Ob}(\mathbf{C})$, and for each $s \in |\Omega|$, \mathbf{s}_R is an $\text{ari}(s)$ -ary co-operation on $|R|$, will be called an Ω -coalgebra object in \mathbf{C} . A morphism of Ω -coalgebra objects of \mathbf{C} will mean a morphism of underlying \mathbf{C} -objects which respects co-operations.*

For any Ω -coalgebra object R and object A of \mathbf{C} , we shall write $\mathbf{C}(R, A)$ for the set-based algebra whose underlying set is $\mathbf{C}(|R|, A)$, and whose operations are induced by the co-operations of R under the dual of the construction of the preceding section. Explicitly, for $s \in |\Omega|$, the operation $s_{\mathbf{C}(R, A)}$ induced by \mathbf{s}_R on $\mathbf{C}(|R|, A)$ is defined to take each $\text{ari}(s)$ -tuple $(\xi_\alpha) \in \mathbf{C}(|R|, A)^{\text{ari}(s)}$ to the composite morphism

$$|R| \xrightarrow{\mathbf{s}_R} \coprod_{\text{ari}(s)} |R| \xrightarrow{(\xi_\alpha)_{\alpha \in \text{ari}(s)}} A,$$

where the second arrow denotes the map whose composite with the α th coprojection $|R| \rightarrow \coprod_{\text{ari}(s)} |R|$ is ξ_α for each $\alpha \in \text{ari}(s)$.

I will in general, as above, use lower-case boldface letters \mathbf{s} etc. to denote co-operations corresponding to operations denoted by the corresponding lower-case italic letters, s etc..

Note that (as in the parallel definition in the preceding section), the R in the above definition of $\mathbf{C}(R, A)$ is not an object of \mathbf{C} ; here it is a \mathbf{C} -based coalgebra with underlying \mathbf{C} -object denoted $|R|$, and $\mathbf{C}(R, A)$ is likewise not a set, but an algebra with underlying set $\mathbf{C}(|R|, A)$.

Let us recall from Lemma 8.4.16 what the general covariant representable set-valued functor $\mathbf{C}(|R|, -)$ “looks like” in the important case where its domain category \mathbf{C} is a variety \mathbf{W} of algebras. Taking a presentation $|R| = \langle X \mid Y \rangle_{\mathbf{W}}$ for the representing object, the functor can be described as carrying each object A to the set of all X -tuples of elements of A that satisfy the family of relations Y . Let us now examine the form that a β -ary operation s on such a functor takes.

We know that s will be induced by a co-operation $\mathbf{s}_R: |R| \rightarrow \coprod_{\beta} |R|$ of the representing object $|R| = \langle X \mid Y \rangle_{\mathbf{W}}$. The homomorphism \mathbf{s}_R will correspond to some X -tuple of elements of $\coprod_{\beta} |R|$ which satisfies the relations Y . For each $x \in X$, the x th entry of this X -tuple, being an element of $\coprod_{\beta} |R|$, may be expressed in terms of the β images of X generating that coproduct algebra, using some derived operation, which we may name

$$(9.3.2) \quad s_x \in |F_{\mathbf{W}}(\beta \times X)|.$$

Now using the universality of $\coprod_{\beta} |R|$ as a \mathbf{W} -algebra with a β -tuple of elements of $\mathbf{W}(|R|, -)$, we can deduce that if A is an arbitrary \mathbf{W} -algebra, and we regard elements of $\mathbf{W}(|R|, A)$ as X -tuples ξ of elements of A which satisfy the relations Y , then for each β -tuple $(\xi_\alpha)_{\alpha \in \beta}$ of such X -tuples, the x th coordinate of the element $s_{\mathbf{W}(R, A)}(\xi_\alpha)_{\alpha \in \beta} \in \mathbf{W}(|R|, A)$ will be expressed in terms of the coordinates of the β X -tuples ξ_α by the same derived operation (9.3.2). In summary:

Lemma 9.3.3. *Let \mathbf{W} be a variety of algebras, $|R|$ an object of \mathbf{W} , and $\langle X | Y \rangle_{\mathbf{W}}$ a presentation of $|R|$ by generators and relations. For any \mathbf{W} -algebra A , any element $\xi \in \mathbf{W}(|R|, A)$, and any $x \in X$, let us call the image in A of the generator x of $|R|$ under ξ “the x th coordinate of ξ ”.*

Let $\mathbf{s} : |R| \rightarrow \coprod_{\beta} |R|$ be a β -ary co-operation on $|R|$, and for any object A of \mathbf{W} , let us write s for the operation on the set $\mathbf{W}(|R|, A)$ induced by this co-operation on $|R|$. Then there exists an X -tuple of derived $\beta \times X$ -ary operations $(s_x)_{x \in X}$ of \mathbf{W} , such that for every such A , for every β -tuple $(\xi_{\alpha})_{\alpha \in \beta}$ of elements of $\mathbf{W}(|R|, A)$, and for every $x \in X$, the x th coordinate of $s(\xi_{\alpha})$ is computed from the coordinates of the given elements ξ_{α} by the derived operation s_x .

Conversely, given an X -tuple of derived $\beta \times X$ -ary operations s_x of \mathbf{W} ($x \in X$), if the identities of \mathbf{W} imply that, when applied to any β X -tuples all of which satisfy the relations Y , the s_x give (as x ranges over X) an X -tuple of elements which also satisfies Y , then $(s_x)_{x \in X}$ determines a morphism of functors $s : \mathbf{W}(|R|, -)^{\beta} \rightarrow \mathbf{W}(|R|, -)$, equivalently, a β -ary co-operation $\mathbf{s} : |R| \rightarrow \coprod_{\beta} |R|$. \square

So, for instance, if \mathbf{W} is the variety of commutative rings, and $|R|$ the commutative ring with a universal $n \times n$ matrix of determinant 1, we can take for X a family of n^2 symbols $(x_{ij})_{i,j \leq n}$, and for Y the set consisting of the single relation $\det(x_{ij}) = 1$. To describe from the above point of view the comultiplication \mathbf{m} on $|R|$ sketched in §9.1, take $\beta = 2$ and for each $i, j \leq n$ let m_{ij} be the polynomial in $2n^2$ indeterminates by which one computes the (i, j) th entry of the product of two matrices. The multiplicativity of the determinant function implies that these operations, when applied to the entries of two matrices of determinant 1, give the entries of a third matrix of determinant 1, so the condition of the last sentence of the above lemma is satisfied. Thus, these n^2 derived operations yield a binary co-operation on $|R|$, which induces, in a manner described abstractly in Definition 9.3.1 and concretely in Lemma 9.3.3, a binary operation on the sets $\mathbf{CommRing}^1(|R|, A) = |\mathbf{SL}(n, A)|$, namely, multiplication of matrices of determinant 1.

Back, now, to dualizing the concepts and results of the preceding section for a general category \mathbf{C} (not necessarily a variety of algebras). Dualizing Definitions 9.2.9 and 9.2.10 respectively, we get

Definition 9.3.4. *Let \mathbf{C} be a category with $\leq \gamma$ -fold coproducts, and \mathbf{V} a variety of Ω -algebras defined by a set J of identities. Then a co- \mathbf{V} object of \mathbf{C} (or \mathbf{V} -coalgebra in \mathbf{C}) will mean an Ω -coalgebra R in \mathbf{C} satisfying the following equivalent conditions:*

- (i) *For all objects A of \mathbf{C} , the algebra $\mathbf{C}(R, A)$ (Definition 9.3.1) lies in \mathbf{V} .*
- (ii) *For each identity $(u, v) \in J$, say in β variables, if we form the β -fold coproduct $\coprod_{\beta} |R|$ with its canonical coprojections q_{α} ($\alpha \in \beta$), then in the algebra $\mathbf{C}(R, \coprod_{\beta} |R|)$, one has $u(q_{\alpha}) = v(q_{\alpha})$. (This equality of morphisms $|R| \rightarrow \coprod_{\beta} |R|$ may be called the “coidentity” corresponding to the identity $u = v$.)*
- (iii) *Writing $\mathbf{Cl}^{(\gamma)}(|R|^{\text{op}})$ for the clone of all co-operations of arities $\leq \gamma$ on $|R|$ (i.e., operations on $|R|$ in \mathbf{C}^{op}), the morphism of clones $\mathbf{Cl}^{(\gamma)}(\Omega\text{-Alg}) \rightarrow \mathbf{Cl}^{(\gamma)}(|R|^{\text{op}})$ induced by the Ω -coalgebra structure of $|R|$ factors through the canonical map $\mathbf{Cl}^{(\gamma)}(\Omega\text{-Alg}) \rightarrow \mathbf{Cl}^{(\gamma)}(\mathbf{V})$,*

$$\mathbf{Cl}^{(\gamma)}(\Omega\text{-Alg}) \rightarrow \mathbf{Cl}^{(\gamma)}(\mathbf{V}) \rightarrow \mathbf{Cl}^{(\gamma)}(|R|^{\text{op}}).$$

- (iv) R satisfies the dual of the diagrammatic condition corresponding to each identity in J .
- (v) Regarded as an Ω -algebra object of \mathbf{C}^{op} , R is a \mathbf{V} -object.

Definition 9.3.5. Let \mathbf{C} be a category with $<\gamma$ -fold coproducts, and \mathbf{V} a variety of Ω -algebras. Then a covariant functor $\mathbf{C} \rightarrow \mathbf{V}$ will be called representable if (i) it is isomorphic to a functor of the form $\mathbf{C}(R, -)$, for R a co- \mathbf{V} object of \mathbf{C} ; equivalently, if (ii) its composite with the forgetful functor $\mathbf{V} \rightarrow \mathbf{Set}$ is representable in the sense of Definition 7.2.3; equivalently, in the case where \mathbf{C} is a variety \mathbf{W} of algebras, if (iii) there is some set Y of relations in a family X of variables such that this composite is the functor associating to every object A of \mathbf{C} the set of all X -tuples of elements of A satisfying Y .

The full subcategory of $\mathbf{V}^{\mathbf{C}}$ consisting of the representable covariant functors will be denoted $\mathbf{Rep}(\mathbf{C}, \mathbf{V})$.

The equivalence of (i) and (ii) above follows from the equivalence of the corresponding conditions of Definition 9.2.10, which, we recall, followed from Lemmas 9.2.3 and 9.2.8; the equivalence of (ii) and (iii) follows, as already noted, from Lemma 8.4.16. As an example of the last sentence of the definition, $\text{SL}(n)$ is an object of $\mathbf{Rep}(\mathbf{CommRing}^1, \mathbf{Group})$. The next few sections will study further classes of representable functors among varieties of algebras. For students with some knowledge of topology, I insert here a nonalgebraic example.

Exercise 9.3:1. Let $\mathbf{HtpTop}^{(\text{pt})}$ be the category whose objects are Hausdorff topological spaces with basepoint, and whose morphisms are homotopy classes of basepoint-preserving maps.

- (i) Show that $\mathbf{HtpTop}^{(\text{pt})}$ has finite coproducts.
- (ii) We noted at the end of §6.5 that the functor $\mathbf{HtpTop}^{(\text{pt})} \rightarrow \mathbf{Set}$ taking an object (X, x_0) to $|\pi_1(X, x_0)|$ (the underlying set of its fundamental group) was representable, with representing object $(S^1, 0)$. By the above results, the structure of group on these sets must be induced by a *cogroup* structure on $(S^1, 0)$. Describe the co-operations, and verify the cogroup identities.
- (iii) Describe likewise the structure of *group* object on $(S^1, 0)$ which represents the contravariant first *cohomotopy group* functor π^1 .

Note that \mathbf{W} -algebra objects of a category \mathbf{C} represent *contravariant* functors $\mathbf{C}^{\text{op}} \rightarrow \mathbf{W}$, while *covariant* functors $\mathbf{C} \rightarrow \mathbf{W}$ are represented by *coalgebra* objects. This is a consequence of the behavior of the covariant and contravariant Yoneda embeddings, discussed in Remark 7.2.7. For the same reason, *morphisms* among covariant representable functors correspond *contravariantly* to morphisms among their representing coalgebras:

Corollary 9.3.6 (to Lemma 9.2.5). *If \mathbf{C} is a category with $<\gamma$ -fold coproducts, and \mathbf{V} a variety of Ω -algebras, then the category $\mathbf{Rep}(\mathbf{C}, \mathbf{V})$ of covariant representable functors $\mathbf{C} \rightarrow \mathbf{V}$ is equivalent to the opposite of the category of co- \mathbf{V} objects of \mathbf{C} . \square*

We are now ready to relate representability and the existence of adjoints!

9.4. Freyd's criterion for the existence of left adjoints. Recall that in Chapter 7 we obtained some curiously similar results about the classes of covariant representable (\mathbf{Set} -valued) functors and right adjoint functors (functors having left adjoints): both sorts of functors respected limits, and in both cases, all examples of functors respecting limits that did not have the property in question arose from the failure of a "solution-set" condition. The big difference was that the former sort of functors were by definition \mathbf{Set} -valued, while the latter could have values in any category; but

Exercise 7.2:5 showed that assuming that the domain category had small coproducts, if the codomain category was **Set**, the two classes coincided.

We have just defined a much more general concept of “representable functor”, and we can now prove that these coincide with the functors having left adjoints in this broader context.

One direction remains easy: If a functor $V: \mathbf{C} \rightarrow \mathbf{V}$, where \mathbf{C} is any category and \mathbf{V} is a variety of algebras, has a left adjoint G , then since the forgetful functor $U_{\mathbf{V}}: \mathbf{V} \rightarrow \mathbf{Set}$ also has a left adjoint, so does their composite. Hence that composite is representable, namely, by the image under that left adjoint of a 1-element set. I.e., we have

$$U_{\mathbf{V}} V(-) \cong \mathbf{Set}(1, U_{\mathbf{V}} V(-)) \cong \mathbf{V}(F_{\mathbf{V}}(1), V(-)) \cong \mathbf{C}(GF_{\mathbf{V}}(1), -).$$

But we have seen that if \mathbf{C} has finite coproducts, representability of the set-valued functor $U_{\mathbf{V}} V$ is equivalent to representability of the algebra-valued functor V .

When we were considering only **Set**-valued functors, the other direction was also easy: If a functor had representing object R , then its left adjoint G could be constructed as taking each set Z to the coproduct of a Z -tuple of copies of R . To adapt this construction to the case where **Set** is replaced by a general variety \mathbf{V} , we will, in the proof of the next theorem, take, for an arbitrary algebra A in \mathbf{V} , a presentation by *generators and relations*,

$$(9.4.1) \quad A = \langle Z \mid S \rangle_{\mathbf{V}}.$$

We will again take the coproduct of a Z -tuple of copies of R , but we will now need a second colimit construction, essentially a coequalizer construction applied to this coproduct, to “impose the relations in S ”, and give us an object $G(A)$ of \mathbf{C} representing the functor $\mathbf{V}(A, V(-))$. (We use symbols Z and S here rather than X and Y so that if one considers the case where \mathbf{C} is a variety \mathbf{W} of algebras, there will be no confusion between this presentation for A in \mathbf{V} , and the presentation in \mathbf{W} for the representing object $|R|$, which was written $\langle X \mid Y \rangle_{\mathbf{W}}$ in Lemma 9.3.3.)

This is in fact essentially the construction used at the beginning of §9.1 to get a left adjoint for $SL(n)$. However, there we could give an explicit generators-and-relations description of the universal ring, while here, where \mathbf{C} is not assumed a variety of algebras, we abstract the method used as a colimit construction.

For completeness the statement of the theorem below shows (as conditions (ii) and (iii)) both versions of the concept of representability, whose equivalence was noted in Definition 9.3.5.

Theorem 9.4.2 (after Freyd [10]). *Let \mathbf{C} be a category with small colimits, \mathbf{V} a variety of Ω -algebras, and*

$$V: \mathbf{C} \rightarrow \mathbf{V}$$

a (covariant) functor. Then the following conditions are equivalent:

- (i) V has a left adjoint $G: \mathbf{V} \rightarrow \mathbf{C}$.
- (ii) V is representable, i.e., is isomorphic to the \mathbf{V} -valued functor represented by a co- \mathbf{V} object R of \mathbf{C} (Definition 9.3.4).
- (iii) The composite $U_{\mathbf{V}} V$ of V with the underlying set functor $U_{\mathbf{V}}: \mathbf{V} \rightarrow \mathbf{Set}$ is representable, i.e., is isomorphic to the set-valued functor $h_{|R|}$ represented by an object $|R|$ of \mathbf{C} .

Proof. We already know that (ii) \Leftrightarrow (iii), and the straightforward proof of (i) \Rightarrow (iii) was given

above. We shall complete the proof by showing (ii) \Rightarrow (i).

Given $A \in \text{Ob}(\mathbf{V})$, we want a $G(A) \in \text{Ob}(\mathbf{C})$ such that $\mathbf{C}(G(A), -) \cong \mathbf{V}(A, V(-))$ (Theorem 7.3.7(ii)). Let us take a presentation (9.4.1) of A in \mathbf{V} . Thus, $\mathbf{V}(A, V(-))$ can be described as associating to each $B \in \text{Ob}(\mathbf{C})$ the set of all Z -tuples of elements of the \mathbf{V} -algebra $V(B)$ that satisfy the relations given by S .

Let us form a coproduct $\coprod_{z \in Z} |R|^{(z)} \in \text{Ob}(\mathbf{C})$ of a Z -tuple of copies, $|R|^{(z)}$ ($z \in Z$), of the underlying \mathbf{C} -object $|R|$ of our representing coalgebra. Then for any object B of \mathbf{C} , the set $\mathbf{C}(\coprod_Z |R|^{(z)}, B)$ can be naturally identified with $\mathbf{C}(|R|, B)^Z \cong |V(B)|^Z$, the set of all Z -tuples of elements of $V(B)$. To get the subset of Z -tuples satisfying the relations of S , we want to formally “impose” these relations on $\coprod_Z |R|^{(z)}$. Hence, for each relation $(s, t) \in S$ let us form the two morphisms $|R| \rightrightarrows \coprod_Z |R|^{(z)}$ corresponding to s and t , namely $s(q_z)$ and $t(q_z)$, where $(q_z)_{z \in Z}$ is the Z -tuple of coprojection morphisms $|R| \rightarrow \coprod_Z |R|^{(z)}$, and let $G(A)$ be the colimit of the diagram formed out of all these pairs of arrows (one pair for each element of S):

$$\begin{array}{c}
 : \\
 |R| \rightrightarrows \\
 |R| \rightrightarrows \\
 |R| \rightrightarrows \\
 :
 \end{array}
 \coprod_Z |R|^{(z)} \longrightarrow G(A).$$

It follows from the universal property of this colimit that $G(A)$ has the desired property $\mathbf{C}(G(A), -) \cong \mathbf{V}(A, V(-))$. \square

(Note that the above theorem required that \mathbf{C} have arbitrary small colimits, so that we could construct $G(A)$ as just described for arbitrary \mathbf{V} -algebras A . This requirement subsumes the condition of having $< \gamma$ -fold coproducts assumed earlier.)

Exercise 9.4:1. Verify the equivalence of the universal properties of $G(A)$ asserted in the last line of the above proof.

Exercise 9.4:2. Describe the construction used in proving (ii) \Rightarrow (i) above in the particular case $\mathbf{C} = \mathbf{CommRing}^1$, $\mathbf{V} = \mathbf{Group}$, $V = \text{SL}(n)$, $A = \mathbb{Z}_2$. (You are not asked to find a normal form for the ring obtained; simply show the generators-and-relations description that the construction gives in this case.) Show directly from your description that the result is a ring with a universal determinant-1 $n \times n$ matrix of exponent 2.

An alternative way to complete the proof of the above theorem, by showing (iii) \Rightarrow (i) rather than (ii) \Rightarrow (i), is indicated in

Exercise 9.4:3. Assuming condition (iii) of the above theorem, let \mathbf{A} denote the full subcategory of \mathbf{V} consisting of those objects A such that the functor $\mathbf{V}(A, V(-)) : \mathbf{C} \rightarrow \mathbf{Set}$ is representable, and let $G_{\mathbf{A}} : \mathbf{A} \rightarrow \mathbf{C}$ be the resulting “partial adjoint” to V . Show that $F_{\mathbf{V}}(1)$ belongs to \mathbf{A} , that \mathbf{A} is closed under small colimits, and that every object of \mathbf{V} can be obtained from the free object on one generator by iterated small colimits. Deduce that $\mathbf{A} = \mathbf{V}$.

9.5. Some corollaries and examples. Since composites of adjunctions are adjunctions (Theorem 7.3.9), the above result yields

Corollary 9.5.1. *A composite of representable functors among varieties of algebras is representable.* \square

Actually, this reasoning shows that a composite of representable functors $\mathbf{C} \rightarrow \mathbf{V} \rightarrow \mathbf{W}$, where \mathbf{V} and \mathbf{W} are varieties and \mathbf{C} any category with small colimits, is representable, but I have given the above more limited statement because of its simplicity.

What does the representing object for a composite of representable functors among varieties look like? Suppose we have

$$\begin{array}{l} \text{representing coalgebras:} \\ \text{right adjoints:} \\ \text{left adjoints:} \end{array} \quad \begin{array}{ccc} & R & S \\ & & \\ \mathbf{U} & \begin{array}{c} \xrightarrow{V} \\ \xleftarrow{D} \end{array} & \mathbf{V} & \begin{array}{c} \xleftarrow{W} \\ \xrightarrow{E} \end{array} & \mathbf{W}, \end{array}$$

so that the composite functor WV has left adjoint DE . To identify the underlying \mathbf{U} -object of the \mathbf{W} -coalgebra representing WV , we note that this object will represent the functor $U_{\mathbf{W}}WV$. The factor $U_{\mathbf{W}}W$ is represented by $|S|$, so by Theorem 7.7.1, the object representing $U_{\mathbf{W}}WV$ can be obtained by applying to $|S|$ the left adjoint of V . Thus, the underlying \mathbf{U} -object of our desired representing object is $D(|S|)$.

Let us combine this observation with the description of D in the proof of Theorem 9.4.2. D takes a \mathbf{V} -algebra A to a \mathbf{U} -algebra obtained by “pasting together” a family of copies of $|R|$ indexed by the generators in any presentation of A , using “pasting instructions” obtained from the relations in that presentation. Hence the representing object $D(|S|)$ for $U_{\mathbf{W}}WV$ can be obtained by “pasting together” a family of copies of $|R|$ in a way prescribed by any presentation of $|S|$. From this one can deduce that if $|R| = \langle X \mid Y \rangle_{\mathbf{U}}$ and $|S| = \langle X' \mid Y' \rangle_{\mathbf{V}}$, then the representing object for $U_{\mathbf{W}}WV$ can be presented in \mathbf{U} by a generating set indexed by $X \times X'$ and a set of relations indexed by $Y \times X' \sqcup X \times Y'$. (Equivalently, if we look at V as taking each \mathbf{U} -algebra A to a \mathbf{V} -algebra whose elements are X -tuples of elements of A satisfying a certain Y -tuple of equations, and similarly regard W as taking each \mathbf{V} -algebra B to a \mathbf{W} -algebra whose elements are X' -tuples of elements of B satisfying a certain Y' -tuple of equations, then their composite can be described as taking each \mathbf{U} -algebra A to a \mathbf{W} -algebra whose elements are all $X \times X'$ -tuples of elements of A that satisfy a $Y \times X' \sqcup X \times Y'$ -tuple of relations.)

Of course, we also want to know the co- \mathbf{W} structure on this object. Not unexpectedly, this arises from the co- \mathbf{W} structure on the object $|S|$. We shall see some examples of representing objects of composite functors in §9.9. I won’t work out the details of the general description of such objects, but if you are interested, you can do this, as

Exercise 9.5:1. Describe precisely how to construct a presentation of the object representing WV , and a description of its co- \mathbf{W} structure, in terms of presentations of $|R|$ and $|S|$ and their co- \mathbf{V} and co- \mathbf{W} structures.

Theorem 9.4.2 has the following consequence (noted as Exercise 8.9:8 in the last chapter); though it is unfortunate that the consequence is better known than the theorem, and is thought by many to be the “last word” on the subject!

Corollary 9.5.2. *Any functor $V: \mathbf{W} \rightarrow \mathbf{V}$ between varieties of algebras which respects underlying sets has a left adjoint.*

Proof. By Theorem 9.4.2(iii) \Rightarrow (i), to show V has a left adjoint it suffices to show that $U_{\mathbf{V}}V: \mathbf{W} \rightarrow \mathbf{Set}$ is representable. But by hypothesis, $U_{\mathbf{V}}V = U_{\mathbf{W}}$, which is clearly representable, by any of our three criteria (representing object: $F_{\mathbf{W}}(1)$; description: sends each object A to the set of 1-tuples of elements of A satisfying the empty set of relations; left adjoint:

$F_{\mathbf{W}}$). \square

This corollary applies to such constructions as (i) the underlying-set functor $U_{\mathbf{W}}: \mathbf{W} \rightarrow \mathbf{Set}$ of any variety \mathbf{W} , the left adjoint of which is, we already know, the *free algebra* construction; (ii) the inclusion of any variety \mathbf{W} in a larger variety \mathbf{V} of algebras of the same type, the left adjoint of which is the construction of “imposing the identities of \mathbf{W} ” on algebras in \mathbf{V} ; (iii) the functor $\mathbf{Set} \rightarrow G\text{-}\mathbf{Set}$ (for any group G) which takes a set A and regards it as a G -set with trivial action; this has for left adjoint the *orbit-set* functor $G\text{-}\mathbf{Set} \rightarrow \mathbf{Set}$ (cf. Exercise 7.6:1); (iv) the functor taking an associative ring A to its underlying additive group, whose left adjoint is the *tensor ring* construction, and similarly (v) the functor taking an associative ring A to its underlying multiplicative monoid, whose left adjoint is the *monoid-ring* construction (both these left adjoint constructions were discussed in terms of their universal properties in §3.12), and (vi) the “commutator brackets” functor from associative algebras over a commutative ring k to Lie algebras over k , whose left adjoint is the *universal enveloping algebra* construction (§8.7).

On the other hand, the functor $SL(n): \mathbf{CommRing}^1 \rightarrow \mathbf{Group}$ with which we began this chapter certainly does not preserve underlying sets. That was a good example for getting away from functors represented by free algebras on one generator, because the representing algebra both requires more than one generator, and requires nontrivial relations, i.e., is nonfree. There are also important examples where a representing algebra is free, but on more than one generator (equivalently, where the functor has the property that the underlying set $|V(A)|$ of the constructed algebra is a fixed power $|A|^X$ of the underlying set of the given algebra A), or can be generated by one element but subject to some relations (equivalently, where $|V(A)|$ can be described as the subset of $|A|^X$ consisting of those elements which satisfy certain equations). Among constructions of the first type are the $n \times n$ matrix ring functor $M_n: \mathbf{Ring}^1 \rightarrow \mathbf{Ring}^1$, the representing object for which is free on n^2 generators, and the formal power series functor (either $\mathbf{Ring}^1 \rightarrow \mathbf{Ring}^1$ or $\mathbf{CommRing}^1 \rightarrow \mathbf{CommRing}^1$) taking a ring A to the ring $A[[t]]$, whose representing object is free on countably many generators. The left adjoints of these have no standard names, but can be described as taking a ring B to the ring over which one has a “universal $n \times n$ matrix representation of B ”, respectively a “universal representation of B by formal power series”. A functor with representing algebra presented by one generator and a nonempty set of relations is the construction $\mathbf{CommRing}^1 \rightarrow \mathbf{Bool}^1$ taking a ring A to the set of its idempotent elements, made a Boolean ring as described in Exercise 3.14:3. The underlying ring of its representing coalgebra is presented by a generator x and the relation $x^2 = x$, and can be described as $\mathbb{Z} \times \mathbb{Z}$, with $x = (1, 0)$. Another example with one generator and a nonempty relation-set is the functor $\mathbf{Ab} \rightarrow \mathbf{Ab}$ taking any abelian group to its subgroup of elements of exponent n (for any fixed $n > 0$), represented by the cyclic group of order n . Still another is the functor $G\text{-}\mathbf{Set} \rightarrow \mathbf{Set}$ (for G any nontrivial group) represented by the one-element G -set. This takes a G -set A to the set of fixed points of the action of G ; its left adjoint is the functor $\mathbf{Set} \rightarrow G\text{-}\mathbf{Set}$ mentioned in point (iii) of the preceding paragraph, which thus has both a left and a right adjoint!

We saw in Chapter 3 that every monoid had both a universal map into a group, and a universal map of a group into it. This says that the forgetful functor

$$U: \mathbf{Group} \rightarrow \mathbf{Monoid}$$

also has both a left and a right adjoint. That it has a left adjoint is now clear from that fact that it preserves underlying sets. Our present results do not say anything about why it should have a right adjoint, but they do say that that right adjoint must be a representable functor. Let us find its

representing cogroup.

We recall that that right adjoint is the functor

$$V: \mathbf{Monoid} \rightarrow \mathbf{Group}$$

taking every monoid A to its group of invertible elements. Since the invertible elements of a monoid A form a subset of $|A|$, one would at first glance expect that $U_{\mathbf{Group}} V$, when expressed in the form described in Lemma 8.4.16(ii), should have X a singleton, i.e., should be represented by a monoid presented by one generator and some relations. But at second glance, we see that this cannot be so: the condition of invertibility on an element of a monoid is not an equation in that element alone. We can find the representing monoid for V by applying its left adjoint U to the free group on one generator. The result is this same group, regarded as a monoid, and as such, it has presentation

$$(9.5.3) \quad R = \langle x, y \mid xy = e = yx \rangle.$$

Thus for any monoid A , the description of $|V(A)|$ in the form described in Lemma 8.4.16(ii) is

$$(9.5.4) \quad \{(\xi, \eta) \in |A| \times |A| \mid \xi\eta = e = \eta\xi\}.$$

Since two-sided inverses to monoid elements are unique when they exist, every element (ξ, η) of $|V(A)|$ is determined by its first component, subject to the condition that this have an inverse. So up to functorial isomorphism, (9.5.4) is indeed the set of invertible elements of A . (We noted this example briefly in the paragraph following Lemma 8.4.16.)

Let us write down the cogroup structure on the representing monoid (9.5.3). If we write the coproduct of two copies of this monoid as

$$R \amalg R = \langle x_0, y_0, x_1, y_1 \mid x_0 y_0 = e = y_0 x_0, x_1 y_1 = e = y_1 x_1 \rangle,$$

then we find that the comultiplication is given by

$$\mathbf{m}(x) = x_0 x_1, \quad \mathbf{m}(y) = y_1 y_0.$$

(If you are uncertain how I got these formulas, stop here and think it out. If you are still not sure, ask in class! Note the reversed multiplication of the y 's, a consequence of the fact that when one multiplies two invertible monoid elements, their inverses multiply in the reverse order.) It is also easy to see that the coinverse operation $\mathbf{i}: R \rightarrow R$ is given by

$$\mathbf{i}(x) = y, \quad \mathbf{i}(y) = x,$$

and, finally, that the co-neutral-element map, from R to the initial object of \mathbf{Monoid} , namely $\{e\}$, is the unique element of $\mathbf{Monoid}(R, \{e\})$, characterized by

$$\mathbf{e}(x) = e = \mathbf{e}(y).$$

Exercise 9.5:2. Describe explicitly the co-operations of the coalgebras representing two of the other examples discussed above, as we have done for the group-of-units functor $\mathbf{Monoid} \rightarrow \mathbf{Group}$.

Exercise 9.5:3. We noted above that we might naively have expected the group-of-invertible-elements functor $\mathbf{Monoid} \rightarrow \mathbf{Group}$ to be represented by a 1-generator monoid, but that it was not. Let us look more closely at this type of situation. Suppose $W: \mathbf{V} \rightarrow \mathbf{W}$ is a representable functor among varieties of algebras, with representing \mathbf{W} -coalgebra R .

(i) Show that $U_{\mathbf{W}} W: \mathbf{V} \rightarrow \mathbf{Set}$ is isomorphic to a subfunctor of $U_{\mathbf{V}}$ if and only if there

exists a map $F_{\mathbf{V}}(1) \rightarrow |R|$ which is an *epimorphism* in \mathbf{V} (but not necessarily surjective).

- (ii) Describe the epimorphism implicit in our discussion of the group-of-invertible-elements functor.
- (iii) Generalize the result of (i) in one way or another.

We can get other examples of representable functors by composing some of those we have described. For instance, if we start with the $n \times n$ matrix ring functor $\mathbf{Ring}^1 \rightarrow \mathbf{Ring}^1$, follow it by the underlying multiplicative monoid functor $\mathbf{Ring}^1 \rightarrow \mathbf{Monoid}$, and this by the group-of-units functor $\mathbf{Monoid} \rightarrow \mathbf{Group}$, we get a functor $\mathbf{Ring}^1 \rightarrow \mathbf{Group}$ which takes every ring A to the group of all invertible $n \times n$ matrices over A , known as $GL(n, A)$.

Let us record a couple of other general results on representability of functors, equivalently, on existence of adjoints. As we noted in example (ii) following Corollary 9.5.2, that corollary implies

Corollary 9.5.5. *The inclusion of any subvariety \mathbf{U} in a variety \mathbf{V} has a left adjoint. \square*

Combining this with Corollary 9.5.1 (composites of representable functors are representable), we get

Corollary 9.5.6. *If a functor $W: \mathbf{V} \rightarrow \mathbf{W}$ between varieties of algebras is representable, then so is its restriction to any subvariety of $\mathbf{U} \subseteq \mathbf{V}$. \square*

For instance, having observed that $GL(n)$ is a representable functor on \mathbf{Ring}^1 , we know automatically that it gives a representable functor on $\mathbf{CommRing}^1$. (What is the relation between the representing objects for these two functors?)

When a functor between varieties of algebras $W: \mathbf{V} \rightarrow \mathbf{W}$ is representable, this representability is usually easy to see and to prove – the construction of the underlying set of $W(A)$ is easily expressed in the form described in Lemma 8.4.16(ii). On the other hand, when we want to prove that a functor V is *not* representable, this criterion is clearly not as helpful; the more useful criterion here is Proposition 7.10.3, which says that W is representable if and only if it respects limits and satisfies a ‘‘solution-set condition’’. As we noted in §§7.7-7.10, most cases of nonrepresentability reveal themselves through failure of the functor to respect limits of one sort or another. For example:

Exercise 9.5:4. Verify that *none* of the following covariant functors from abelian groups to abelian groups is representable:

- (i) $F(A) = A \otimes A$.
- (ii) $G(A) =$ the torsion subgroup of A (the subgroup of all elements of finite order).
- (iii) $H(A) = A/nA$ (n a fixed integer).
- (iv) $J(A) = nA$ (n a fixed integer).

In Exercises 7.10:5-7.10:6, we saw examples of the rarer situation in which some left universal construction was impossible only because the solution-set condition was not satisfied. Those examples were of nonexistence of *initial objects* and of *free objects*, so by Theorem 8.4.13, the categories in question were, necessarily, not varieties (though in one of the examples this category, that of complete lattices, failed to be a variety only in that it had a large set of operations). The following exercise shows that in the case of the criterion for *representability*, there are counterexamples where the domain *is* a variety.

Exercise 9.5:5. Let us call an object S of a variety \mathbf{V} *simple* if the only congruences on S are the trivial congruence and the total congruence (the least and the greatest equivalence relations on $|S|$).

(i) Find a variety \mathbf{V} having the properties that (a) for every cardinal α there exists a simple algebra S_α in \mathbf{V} of cardinality $\geq \alpha$, and (b) every algebra in \mathbf{V} contains a unique one-element subalgebra. (Suggestion: Show either that there are simple groups of arbitrarily large cardinalities, or that there are fields of arbitrarily large cardinalities; in the latter case you must also say how to regard fields as simple objects of a variety satisfying (b).)

Now assume we have chosen such a \mathbf{V} , and for each α some S_α , as above. For every object A of \mathbf{V} , define $V(A) = \mathbf{V}(\coprod_{\alpha \leq |A|} S_\alpha, A) \in \text{Ob}(\mathbf{Set})$; equivalently (up to natural isomorphism) $V(A) = \prod_{\alpha \leq |A|} \mathbf{V}(S_\alpha, A)$.

(ii) Show how to make V a functor, and show that this functor respects small limits, but is not representable. (You may either get these results directly, or with the help of part (iii) below.)

(iii) Recall that the variety we are writing \mathbf{V} could be more precisely written as $\mathbf{V}_{(\mathbb{U})}$, the category of \mathbb{U} -small objects of a certain type that satisfy a certain system of identities. Letting \mathbb{U}' be any universe properly larger than \mathbb{U} , show that $\mathbf{V}_{(\mathbb{U}')}$ contains an object S such that the restriction to $\mathbf{V}_{(\mathbb{U})}$ of the functor $h_S: \mathbf{V}_{(\mathbb{U}')} \rightarrow \mathbf{Set}_{(\mathbb{U}')}$ is isomorphic to the functor V of (ii) above.

Thus, intuitively, this example is based on a functor which is representable, but by an object outside our universe. What was tricky was to find such a functor which nevertheless took \mathbb{U} -small algebras to \mathbb{U} -small sets.

Curiously, in the condition from Chapter 7 for the existence of *right* adjoint functors, one *can* drop the solution-set condition when the domain category is a variety:

Exercise 9.5:6. Show that if \mathbf{V} is a variety of algebras and \mathbf{C} a category with small colimits, then every functor $F: \mathbf{V} \rightarrow \mathbf{C}$ which respects small colimits has a right adjoint; i.e., is the left adjoint to a representable functor.

Knowing that representable functors from a variety \mathbf{W} to a variety \mathbf{V} correspond to \mathbf{V} -coalgebra objects of \mathbf{W} , it is natural to try, for various choices of \mathbf{V} and \mathbf{W} , to find *all* such coalgebras, and hence all such functors. How difficult this task is depends on the varieties in question. At the easy extreme are certain large classes of cases for which we shall see in §9.10 that there can be no nontrivial representable functors. At the other end are cases such as that of representable functors from the variety of commutative rings (or commutative algebras over a fixed commutative ring k) to **Group**. Such functors are called “affine algebraic groups”, and are an important area of research in algebraic geometry.

In the next three sections, we shall tackle some cases of an intermediate level of difficulty, for which the problem is nontrivial, but where, with a reasonable amount of work, we can get a complete classification.

9.6. Representable endofunctors of Monoid. Let us consider representable functors from the variety **Monoid** into itself.

A representable functor from an arbitrary category \mathbf{C} with finite coproducts to **Monoid** is represented by a comonoid, which we shall for convenience write as a 3-tuple $(R, \mathbf{m}, \mathbf{e})$, (rather than as a pair $(R, (\mathbf{m}, \mathbf{e}))$), where R is an object of \mathbf{C} , and the other two components are a binary *comultiplication*

$$\mathbf{m}: R \rightarrow R \amalg R$$

and a zeroary *co-neutral-element*

$$e: R \rightarrow I.$$

Here I denotes the initial object of \mathbf{C} , that is, the coproduct of the empty family. These co-operations must satisfy the coassociative law, and the right and left coneutral laws. The coassociative law can be shown diagrammatically as the dual to (9.2.6); thus, it says that the diagram

$$(9.6.1) \quad \begin{array}{ccc} R & \xrightarrow{\mathbf{m}} & R \amalg R \\ \downarrow \mathbf{m} & & \downarrow \mathbf{m} \amalg \text{id}_R \\ R \amalg R & \xrightarrow{\text{id}_R \amalg \mathbf{m}} & R \amalg R \amalg R \end{array}$$

commutes. The two coneutral laws likewise say that if we write i_R for the unique map from the initial object I to R , then the composite maps

$$(9.6.2) \quad R \xrightarrow{\mathbf{m}} R \amalg R \xrightarrow{(i_R e, \text{id}_R)} R, \quad R \xrightarrow{\mathbf{m}} R \amalg R \xrightarrow{(\text{id}_R, i_R e)} R$$

are each the identity morphism of R , where in each of these latter diagrams, the parenthesized pair shown above the second arrow is an abbreviation for the morphism obtained from the two entries of that pair via the universal property of the coproduct $R \amalg R$.

Let us now specialize to the case $\mathbf{C} = \mathbf{Monoid}$. Then the initial object I is the trivial monoid $\{e\}$; hence the homomorphism e can only be the map taking every element of R to e . (Contrast this with the case of $\text{SL}(n)$ discussed in §9.1, where e had for codomain the initial object \mathbb{Z} of **CommRing**¹, and the specification of the identity matrix was nontrivial.) Nonetheless, the fact that this unique zeroary co-operation satisfies the coneutral laws (9.6.2) will be a nontrivial condition.

To study (9.6.1) and (9.6.2), we need to recall the structure of a coproduct of monoids. We noted in §3.10 that such a coproduct

$$(9.6.3) \quad \amalg_{\alpha \in I} R^\alpha$$

could be described in essentially the same way as for groups; namely, assuming for notational convenience that the sets $|R^\alpha| - \{e\}$ are disjoint, each element of (9.6.3) can be written uniquely as a product

$$(9.6.4) \quad r_0 r_1 \dots r_{h-1}, \quad \text{with } h \geq 0, \text{ each } r_i \text{ in some } |R^{\alpha_i}| - \{e\}, \\ \text{and } \alpha_i \neq \alpha_{i+1} \text{ for } 0 \leq i < h-1.$$

(Here the neutral element e of (9.6.3) is understood to be the case $h = 0$ of (9.6.4).)

However, in the case of the coproduct $R \amalg R$ we are interested in now, the two monoids being put together are *not* disjoint. Let us therefore distinguish our two canonical images of R in $R \amalg R$ as R^λ and R^ρ (the superscripts corresponding to the ‘‘left’’ and ‘‘right’’ arguments of the comultiplication we want to study). We shall thus write $R \amalg R$ as $R^\lambda \amalg R^\rho$, i.e., as the coproduct of these two copies of R , and write the images of an element $x \in |R|$ under the two coprojections $R \rightrightarrows R^\lambda \amalg R^\rho$ as x^λ and x^ρ respectively.

The coassociative law involves three variables, hence in (9.6.1), R is ultimately mapped into a three-fold coproduct of copies of itself; let us write this object $R^\lambda \amalg R^\mu \amalg R^\rho$, the μ standing for the ‘‘middle’’ variable in the associativity identity.

The first step in describing an element (9.6.4) is to specify the sequence of indices

$(\alpha_0, \dots, \alpha_{h-1})$; so let us define an *index-string* to mean a finite (possibly empty) sequence of members of $\{\lambda, \mu, \rho\}$, with no two successive terms equal. We shall call h the *length* of the index-string $(\alpha_0, \dots, \alpha_{h-1})$. For every index-string $\sigma = (\alpha_0, \dots, \alpha_{h-1})$, we shall denote by $|R|^\sigma$ the set of all products (9.6.4) with that sequence of superscripts, i.e.,

$$|R|^\sigma = (|R^{\alpha_0}| - \{e\}) \dots (|R^{\alpha_{h-1}}| - \{e\}).$$

The underlying set of each of the monoids $R^\lambda \amalg R^\rho$ and $R^\lambda \amalg R^\mu \amalg R^\rho$ is thus the disjoint union of its subsets $|R|^\sigma$. We define the *height* $\text{ht}(s)$ of $s \in |R^\lambda \amalg R^\rho|$ as the length of the unique σ such that $s \in |R|^\sigma$. Finally, to study our comultiplication \mathbf{m} , let us define the *degree* of an element of R itself by

$$\text{deg}(x) = \text{ht}(\mathbf{m}(x)).$$

We note that for each $h > 0$, there are precisely two index-strings of length h consisting only of ρ 's and λ 's: one beginning with ρ and the other beginning with λ . Thus, if $x \in |R|$ is an element of positive degree h , then $\mathbf{m}(x)$ either belongs to $|R|^{(\lambda, \rho, \lambda, \dots)}$ (h entries in the superscript) i.e., has the form $y_0^\lambda z_1^\rho y_2^\lambda \dots$, or it belongs to $|R|^{(\rho, \lambda, \rho, \dots)}$, and has the form $x_0^\rho y_1^\lambda z_2^\rho \dots$.

It is easy to see that the counital laws (9.6.2) say

$$(9.6.5) \quad \text{If } \mathbf{m}(x) = \dots y_i^\lambda z_{i+1}^\rho y_{i+2}^\lambda z_{i+3}^\rho \dots, \text{ then } x = \dots y_i y_{i+2} \dots = \dots z_{i+1} z_{i+3} \dots$$

(Note that the way we have written $\mathbf{m}(x)$ here covers both the cases $x \in |R|^{(\lambda, \rho, \lambda, \dots)}$ and $x \in |R|^{(\rho, \lambda, \rho, \dots)}$.) In particular, (9.6.5) implies

$$(9.6.6) \quad \text{If } x \neq e, \text{ then } \text{deg}(x) \geq 2.$$

On the two possible sorts of elements of degree exactly 2, we see that (9.6.5) precisely determines the action of \mathbf{m} :

$$(9.6.7) \quad \begin{cases} \text{If } \mathbf{m}(x) \in |R|^{(\lambda, \rho)}, \text{ then } \mathbf{m}(x) = x^\lambda x^\rho. \\ \text{If } \mathbf{m}(x) \in |R|^{(\rho, \lambda)}, \text{ then } \mathbf{m}(x) = x^\rho x^\lambda. \end{cases}$$

Let us also record what (9.6.5) tells us about the degree 3 case:

$$(9.6.8) \quad \begin{cases} \text{If } \mathbf{m}(x) \in |R|^{(\lambda, \rho, \lambda)}, \text{ then } \mathbf{m}(x) = y_0^\lambda x^\rho y_2^\lambda \text{ where } y_0 y_2 = x. \\ \text{If } \mathbf{m}(x) \in |R|^{(\rho, \lambda, \rho)}, \text{ then } \mathbf{m}(x) = z_0^\rho x^\lambda z_2^\rho \text{ where } z_0 z_2 = x. \end{cases}$$

We now turn to the coassociative law. This says that for any $x \in |R|$,

$$(9.6.9) \quad (\text{id}_{R^\lambda}, \mathbf{m})\mathbf{m}(x) = (\mathbf{m}, \text{id}_{R^\rho})\mathbf{m}(x) \quad \text{in } R^\lambda \amalg R^\mu \amalg R^\rho.$$

Let us note the explicit descriptions of the left-hand morphism on each side of the above equation. $(\text{id}_{R^\lambda}, \mathbf{m})$: $R^\lambda \amalg R^\rho \rightarrow R^\lambda \amalg R^\mu \amalg R^\rho$ leaves each element of the form $y^\lambda \in |R^\lambda \amalg R^\rho|$ unchanged, while it takes an element $z^\rho \in |R^\lambda \amalg R^\rho|$ to the element $\mathbf{m}(z)$, but with all the superscripts “ λ ,” changed to “ μ ,” (because of the way we label our 3-fold coproduct). Likewise, $(\mathbf{m}, \text{id}_{R^\rho})$ leaves each z^ρ unchanged, and takes each y^λ to the element $\mathbf{m}(y)$, with all superscripts “ ρ ,” changed to “ μ .”

Now let $x \in |R| - \{e\}$, suppose that $\mathbf{m}(x)$ belongs to the set $|R|^\sigma$ (σ a string of λ 's and ρ 's), and let the common value of the two sides of (9.6.9) belong to the set $|R|^\tau$ (τ a string of λ 's, μ 's and ρ 's). Note that each “ λ ” in σ yields a single λ in τ on evaluating the left-hand

side of (9.6.9), but looking at the right-hand side of (9.6.9), it gives *at least* one λ in τ , because of (9.6.6). Since the two sides of (9.6.9) are equal, all of these “at least one”s must be exactly one. For this to happen, the elements y_i in the expansion (9.6.5) must all have degree ≤ 3 . By a symmetric argument (comparing occurrences of ρ in σ and in τ) we get the same conclusion for the elements z_i . Note also that if τ begins with μ , then the right-hand side of (9.6.9) tells us σ must begin with a λ , while the left-hand side says it must begin with a ρ , a contradiction. Hence τ can only begin with a λ or a ρ . In the former case, σ must begin with a λ which expands to $\lambda\mu$ on the right-hand side of (9.6.9) (so as not to yield more than one λ); in the latter case it must begin with a ρ which expands to $\rho\mu$ on the left-hand side. In either case, we conclude that the first factor in the expansion of $\mathbf{m}(x)$ must have degree 2. The same arguments apply to the last factor. In summary:

(9.6.10) For all $x \in |R|$, all elements y_i and z_i in (9.6.5) have degree ≤ 3 ; hence by (9.6.5), every element of R is a product of elements of degree ≤ 3 . Moreover, the elements giving the *first* and *last* factors of $\mathbf{m}(x)$ have degree 2.

But the observation about first and last factors, applied to the final equation in each line of (9.6.8), gives

(9.6.11) Every element of R of degree 3 is a product of two elements of degree 2.

(9.6.10) and (9.6.11) together allow one to express every element of R as a product of elements of degree 2, showing that R is generated by these elements. We can prove still more:

Lemma 9.6.12. *Let $(R, \mathbf{m}, \mathbf{e})$ be a co-Monoid object in **Monoid**. Then every element $x \in |R|$ has an expression as a product*

$$x_0 \dots x_{h-1} \quad (h \geq 0),$$

where all x_i are of degree 2, and this expression is unique subject only to the condition that there be no two successive factors x_j, x_{j+1} such that one of $\mathbf{m}(x_j), \mathbf{m}(x_{j+1})$ belongs to $|R|^{(\lambda, \rho)}$, the other belongs to $|R|^{(\rho, \lambda)}$, and $x_j x_{j+1} = e$.

Proof. Since R is generated by elements of degree 2, and since any expression involving two successive factors whose product is e can be simplified to a shorter expression, we can clearly express every element in the indicated form subject to the conditions noted. To show that this form is unique, it suffices to prove that given an element and such an expression for it,

$$x = x_0 \dots x_{h-1} \quad (\deg(x_i) = 2, \quad i = 0, \dots, h-1),$$

we can recover the factors x_i from x . I claim in fact that if for this x we write the common value of the two sides of (9.6.9) as a reduced product of elements of R^λ, R^μ and R^ρ , i.e., as in (9.6.4), then the sequence of factors belonging to R^μ will be precisely $x_0^\mu, \dots, x_{h-1}^\mu$, recovering the x_i , as required.

Indeed, let us note that for any x such that $\mathbf{m}(x) \in |R|^{(\lambda, \rho)}$, the common value of the two sides of (9.6.9), computed using (9.6.7), is $x^\lambda x^\mu x^\rho$, while when $\mathbf{m}(x) \in |R|^{(\rho, \lambda)}$ it is $x^\rho x^\mu x^\lambda$. Hence when we evaluate the common value of the two sides of (9.6.9) for $x = x_0 \dots x_{h-1}$, the factors with superscript μ comprise, *initially*, the sequence claimed. They will continue to do so after we reduce this product to the form (9.6.4) by combining any successive factors that may belong to the same monoid $|R|^\lambda, |R|^\mu$ or $|R|^\rho$, unless, in the course of this reduction, the

factors with superscript ρ and/or λ separating some pair of successive μ -factors cancel, allowing these μ -factors to merge. Now if $\mathbf{m}(x_i)$ and $\mathbf{m}(x_{i+1})$ both belong to $|R|^{(\lambda, \rho)}$ or both belong to $|R|^{(\rho, \lambda)}$, then between x_i^μ and x_{i+1}^μ we will have exactly one λ -factor and one ρ -factor, and these cannot cancel. In the case where one belongs to $|R|^{(\lambda, \rho)}$ and the other to $|R|^{(\rho, \lambda)}$, we get adjacent factors $x_i^\rho x_{i+1}^\rho$ or $x_i^\lambda x_{i+1}^\lambda$ in the same set R^ρ or R^λ . When we merge these, they will cancel only if $x_i x_{i+1} = e$ in R ; but this is the case excluded by our hypothesis. \square

Note that in the above argument, we could have asserted that every element can be reduced to a unique product of the indicated form in which no two successive factors *whatever* have product e . However, we have proved uniqueness subject to a *weaker* condition than this, so we have a *stronger* uniqueness result. Indeed, this result implies (as the weaker uniqueness statement would not):

Corollary 9.6.13. *If (R, \mathbf{m}, e) is a co-Monoid object in **Monoid**, then the monoid R has a presentation $\langle X \mid Y \rangle$, where X is the set of elements of R having degree 2 with respect to the comultiplication \mathbf{m} , and Y is the set of all relations of the form $x_0 x_1 = e$ holding in R such that one of $\mathbf{m}(x_0)$, $\mathbf{m}(x_1)$ lies in $|R|^{(\lambda, \rho)}$, and the other in $|R|^{(\rho, \lambda)}$.*

Proof. We know that X generates R , and by definition the relations comprising Y are satisfied by these generators. It remains to verify that if two words w_0 and w_1 in the elements of X are equal in R , then this equality follows from the relations in Y .

Now if w_i ($i = 0$ or 1) contains a substring which is the left-hand side of some relation in Y , then by applying that relation, we can reduce w_i to a shorter word. Hence a finite number of applications of such relations will transform w_0 and w_1 to words w'_0 and w'_1 that contain no such substrings. The values of these words in R are still equal; hence the uniqueness statement of Lemma 9.6.12 tells us they are the same word. Thus, by applying relations in Y , we have obtained the equality of w_0 and w_1 in R , as required. \square

Clearly the next step in studying our comonoid should be to examine the properties of the set of pairs of elements of R of degree 2 satisfying $x_0 x_1 = e$. So let us make

Definition 9.6.14. *If (R, \mathbf{m}, e) is a co-Monoid object in **Monoid**, then $P(R, \mathbf{m}, e)$ will denote the 4-tuple (X^+, X^-, E, u) , where*

$$X^+ = \{x \in |R| \mid \mathbf{m}(x) = x^\lambda x^\rho\} = \{x \in |R| \mid \mathbf{m}(x) \in |R|^{(\lambda, \rho)}\} \cup \{e\},$$

$$X^- = \{x \in |R| \mid \mathbf{m}(x) = x^\rho x^\lambda\} = \{x \in |R| \mid \mathbf{m}(x) \in |R|^{(\rho, \lambda)}\} \cup \{e\},$$

$$E = \{(x_0, x_1) \in |R|^2 \mid \deg(x_0), \deg(x_1) \leq 2, x_0 x_1 = e\} \subseteq (X^+ \times X^-) \cup (X^- \times X^+),$$

and $u = e$, the neutral element of R .

Thus, X^+ and X^- are sets intersecting in the singleton $\{u\}$, and E is a binary relation on the union of these sets, which relates certain elements of X^+ to certain elements of X^- , and vice versa. We note a key property of this relation: If both (x_0, x_1) and (x_1, x_2) belong to it, then since x_1 has x_0 as a left inverse and x_2 as a right inverse in R , x_0 must equal x_2 .

Let us formalize the type of combinatorial object we have obtained.

Definition 9.6.15. An E -system will mean a 4-tuple (X^+, X^-, E, u) , where X^+ and X^- are sets and u an element such that

$$X^+ \cap X^- = \{u\},$$

and

$$E \subseteq (X^+ \times X^-) \cup (X^- \times X^+)$$

is a relation such that

$$(9.6.16) \quad u E u,$$

$$(9.6.17) \quad x_0 E x_1, x_1 E x_2 \Rightarrow x_0 = x_2.$$

A morphism of E -systems $(X^+, X^-, E, u) \rightarrow (X'^+, X'^-, E', u')$ will mean a map $X^+ \cup X^- \rightarrow X'^+ \cup X'^-$ carrying X^+ into X'^+ , X^- into X'^- , the relation E into the relation E' , and u to u' .

Thus, the objects $P(R, \mathbf{m}, \mathbf{e})$ constructed in Definition 9.6.14 are E -systems.

When an E -system (X^+, X^-, E, u) arises as in that definition from a co-**Monoid** object $(R, \mathbf{m}, \mathbf{e})$ of **Monoid**, Corollary 9.6.13 tells us how to recover the monoid R from (X^+, X^-, E, u) , and (9.6.7) tells us how to recover \mathbf{m} . (As noted earlier, there is no choice regarding \mathbf{e} .) If the concept of E -system does a good enough job of capturing the structure of the co-**Monoid** objects of **Monoid**, then every E -system should arise from such an object, and Corollary 9.6.13 and (9.6.7) should allow us to construct that object. Let us try to see whether this is so.

Given any E -system (X^+, X^-, E, u) , let us form the monoid with the presentation suggested by Corollary 9.6.13:

$$(9.6.18) \quad R = \langle X^+ \cup X^- - \{u\} \mid x_0 x_1 = e \text{ whenever } x_0 E x_1 \rangle.$$

On this monoid we have a unique zeroary co-operation \mathbf{e} , namely the trivial map $R \rightarrow \{e\}$. We would now like to define a comultiplication homomorphism from this monoid into the coproduct of two copies of itself by setting

$$(9.6.19) \quad \mathbf{m}(x) = \begin{cases} x^\lambda x^\rho & \text{if } x \in X^+ - \{u\}, \\ x^\rho x^\lambda & \text{if } x \in X^- - \{u\}. \end{cases}$$

The next two exercises will show that this construction in fact inverts that of Definition 9.6.14, a result which we will then summarize as a theorem. You should therefore read these exercises through, and think about what is involved, even if you do not work out all the details.

Exercise 9.6:1. (i) Show that for any E -system $X = (X^+, X^-, E, u)$, if we define R by (9.6.18), then (9.6.19) gives a well-defined homomorphism $\mathbf{m}: R \rightarrow R^\lambda \amalg R^\rho$.

(ii) Show that this \mathbf{m} and the trivial morphism \mathbf{e} make R a comonoid object of **Monoid**. Let us denote this object $Q(X)$.

The next observation will make some subsequent results easier to state:

(iii) Verify that the presentation (9.6.18) is equivalent to the modified presentation with u included among the generators and $u = e$ added to the relations; and that (9.6.19) then holds with the “ $-\{u\}$ ”s deleted.

(iv) Show that the construction P of Definition 9.6.14, and the above construction Q , may be made functors in obvious ways, and that Q is then left adjoint to P .

(v) Deduce from Corollary 9.6.13 that the counit of this adjunction, i.e., the canonical morphism from the functor QP to the identity functor of the category of co-**Monoid** objects of

Monoid, is an isomorphism. In particular, every comonoid object of **Monoid** arises under Q from an E -system.

We have not yet shown that every E -system arises from a comonoid. We shall prove this by showing that the *unit* of the above adjunction, i.e., the canonical morphism from the identity functor of the category of E -systems to PQ , is also an isomorphism.

(To banish any suspicion that our desired conclusion might follow automatically from (v) above, consider the analogous situation where P is the forgetful functor $\mathbf{Group} \rightarrow \mathbf{Monoid}$, and Q its left adjoint, taking every monoid to its universal enveloping group. Then the counit $QP \rightarrow \text{Id}_{\mathbf{Group}}$ is an isomorphism, but the unit $\text{Id}_{\mathbf{Monoid}} \rightarrow PQ$ is not: monoids containing noninvertible elements do not appear as values of P , and each such monoid falls together under Q with a monoid that *is* a value of P .)

We will get this result by obtaining a normal form for monoids $Q(X)$:

- Exercise 9.6.2.** (i) Show that given any E -system $X = (X^+, X^-, E, u)$, the monoid R with presentation (9.6.18) has for normal form the set of words in the indicated generators (including the empty word) that contain no subsequences x_0x_1 with x_0Ex_1 . (Suggestion: van der Waerden’s trick.)
- (ii) Deduce that the unit of the adjunction between P and Q is an isomorphism.

The above results are summarized in the first sentence of the next theorem. The second sentence translates the comonoid structure (9.6.18)-(9.6.19) into a description of the functor represented, and the final sentence follows by Corollary 9.3.6.

Theorem 9.6.20. *Every representable functor V from **Monoid** to **Monoid** is determined by an E -system. The functor corresponding to the E -system (X^+, X^-, E, u) can be described as a subfunctor (in the sense of Lemma 6.9.3 and Definition 8.4.10) of a direct product of copies of the identity functor and of the opposite-monoid functor; namely, as the construction taking each monoid A to the submonoid of $A^{(X^+ - \{u\})} \times (A^{\text{op}})^{(X^- - \{u\})}$ consisting of those elements s such that for all $(x, y) \in E - \{(u, u)\}$, the coordinate s_x is a left inverse to the coordinate s_y .*

Writing **E -System** for the category of E -systems, the above construction yields a contravariant equivalence $E\text{-System}^{\text{op}} \rightarrow \mathbf{Rep}(\mathbf{Monoid}, \mathbf{Monoid})$. \square

For the purpose of describing the morphism of representable functors induced by a given morphism of E -systems, it is actually most convenient to treat the functor $V: \mathbf{Monoid} \rightarrow \mathbf{Monoid}$ corresponding to the E -system (X^+, X^-, E, u) as taking a monoid A to a submonoid of

$$A^{(X^+ - \{u\})} \times \{e\} \times (A^{\text{op}})^{(X^- - \{u\})};$$

i.e., to introduce an extra slot, indexed by the element u of the E -system, such that the coordinate of $V(A)$ in that slot is required to be the neutral element e of A . (Cf. Exercise 9.6:1(iii).) We can then say that if $\mathbf{f}: E \rightarrow E'$ is a morphism of E -systems, and $f: V' \rightarrow V$ the corresponding morphism of representable functors, then for a monoid A and an element $\xi \in |V'(A)|$, the image $f(A)(\xi)$ has for x th coordinate the $\mathbf{f}(x)$ th coordinate of ξ , whether $\mathbf{f}(x)$ happens to be u , or to be a member of $X^+ \cup X^- - \{u\}$.

Let us look at some simple examples of E -systems and the corresponding representable functors. We shall display an E -system by showing the elements of $X^+ - \{u\}$ and $X^- - \{u\}$ respectively as points in two boxes, $\square \square$, and indicating a condition x_0Ex_1 by an arrow from the point x_0 to the point x_1 . (The element u will not be shown; it may be thought of as

embedded in the dividing line between the boxes.)

$\boxed{\cdot \mid \square}$ By (9.6.18)-(9.6.19), the comonoid R corresponding to this E -system is the free monoid on one generator x , with the comultiplication under which $\mathbf{m}(x) = x^\lambda x^\rho$. We see that the functor this represents is (up to isomorphism) the *identity* functor **Monoid** \rightarrow **Monoid**. This description of the functor represented can also be seen from the second sentence of the above theorem.

$\boxed{\square \mid \cdot}$ You should verify that this E -system similarly gives the *opposite monoid* functor.

$\boxed{\cdot \mid \cdot}$ (the relation $E - \{(u, u)\}$ still being empty). This gives the direct product of the above two functors, i.e., the functor associating to every monoid A the monoid

$$\{(\alpha, \beta) \mid \alpha, \beta \in |A|\},$$

with multiplication

$$(9.6.21) \quad (\alpha_0, \beta_0) (\alpha_1, \beta_1) = (\alpha_0 \alpha_1, \beta_1 \beta_0).$$

$\boxed{\cdot \leftarrow \cdot}$ This corresponds to the subfunctor of the preceding example determined by adding to the description of its underlying set the conditions

$$\alpha\beta = e = \beta\alpha.$$

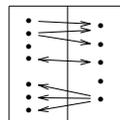
Since under these conditions α uniquely determines β , the second coordinate provides no new information, and we can describe this functor, up to isomorphism, as associating to A its *group of invertible elements* α , regarded as a monoid.

$\boxed{\cdot \rightarrow \cdot}$ As above, except that only the condition $\alpha\beta = e$, and not $\beta\alpha = e$ is imposed. Right inverses are *not* generally unique, so we must describe this functor as associating to A the monoid of elements $\alpha \in |A|$ given with a *specified* right inverse β . The multiplication is again as in (9.6.21).

$\boxed{\cdot \leftarrow \cdot}$ This associates to A the monoid of elements α given with a specified *left inverse* β , again multiplied as in (9.6.21). Set-theoretically, this construction is isomorphic to the preceding, via $(\alpha, \beta) \leftrightarrow (\beta, \alpha)$, but the monoid structures are opposite to one another. (I have indicated this in the paraphrases by naming, after the words ‘‘monoid of’’, the elements which are multiplied as in A , while those with the opposite multiplication are referred to as specified inverses of these elements.)

$\boxed{\begin{array}{c} \cdot \\ \cdot \\ \cdot \end{array} \mid \cdot}$ ‘‘The monoid of *pairs* of elements of A with a specified *common* right inverse’’.

And so on. We note that for a general diagram such as



the associated functor is the direct product of the functors associated with the graph-theoretic connected components of the diagram. Each of these components, *except* those of the form $\boxed{\cdot \leftarrow \cdot}$ must have, by (9.6.17), the property that arrows, if any, all go in the same direction, i.e., from left to right or from right to left. Subject to this restriction, the arrows are independent.

Let us pause to note the curious fact that, although for every *nonzero* cardinal r , the construction that associates to a monoid A the monoid of its right invertible elements given with a specified r -tuple of right inverses is a representable functor, this is false for $r = 0$:

Exercise 9.6:3. Let $H: \mathbf{Monoid} \rightarrow \mathbf{Monoid}$ be the functor associating to a monoid A its submonoid of right invertible elements (a subfunctor of the identity functor).

- (i) Show that H is not representable, if you did not already do so as Exercise 7.2.2(i).
- (ii) Show, however, that the composite functor HH is representable, and concisely describe this functor.
- (iii) Show that H can be written as a direct limit of representable functors. (Hint: can you write the empty set as an inverse limit of nonempty sets?)

It is natural to ask how to compose two representable functors expressed in terms of E -systems.

Exercise 9.6:4. In this exercise, “functor” will mean “representable functor $\mathbf{Monoid} \rightarrow \mathbf{Monoid}$ ”.

- (i) Define precisely what is meant by the connected components of an E -system, and prove the assertion made above that the functor associated with an E -system is the direct product of the functors associated with its connected components. Using this result, reduce the problem of describing the E -system of the composite of two functors to the case where the E -systems of the given functors are connected.
- (ii) Characterize in terms of E -systems the results of composing an arbitrary functor on the right and on the left with the functors having the diagrams $\begin{array}{|c|c|} \hline \cdot & \\ \hline \end{array}$ and $\begin{array}{|c|} \hline \cdot \\ \hline \end{array}$. (Thus, four questions are asked, though two of them are trivial to answer.)

This leaves us with the problem of describing the composite of two functors whose associated diagrams are both connected, and each have more than one element. The answer is quite simple, but the argument requires two preliminary observations:

- (iii) Show that if s, t are two left invertible elements of a monoid A , or two right invertible elements, then the condition $st = e$ implies that they are both invertible.
- (iv) Let V be a functor whose diagram is connected. Show that if some $\xi \in |V(A)|$ has an invertible element of A in at least one coordinate, then it has invertible elements in all coordinates, and these are determined by that one coordinate. Show that the set of elements ξ with these properties forms a submonoid of $V(A)$, isomorphic to the group of invertible elements of A . (In writing “at least one coordinate” above, I am understanding our description of V to be that of Theorem 9.6.20, which does not include a coordinate indexed by u .)
- (v) Deduce from (iii) and (iv) a description for the composite of any two functors whose diagrams are both connected and each have more than one element (not counting u as an element of our diagrams).

Exercise 9.6:5. Suppose $f: V \rightarrow V'$ is a morphism of representable functors $\mathbf{Monoid} \rightarrow \mathbf{Monoid}$, and W is another such functor. Assuming the results of the preceding exercise, show how to describe the map of E -systems corresponding to $f \circ W: VW \rightarrow V'W$, respectively $W \circ f: WV \rightarrow WV'$, in terms of the map of E -systems corresponding to f .

Exercise 9.6:6. We saw in the discussion following Corollary 9.5.1 that the object representing a composite of representable functors among varieties could be constructed from presentations $\langle X \mid Y \rangle_{\mathbf{U}}$ and $\langle X' \mid Y' \rangle_{\mathbf{V}}$ of representing objects for those functors, using a set of generators indexed by $X \times X'$ and a set of relations indexed by $Y \times X' \sqcup X \times Y'$. See whether you can get the results of the preceding two exercises by applying this idea to presentations of the representing objects for functors $\mathbf{Monoid} \rightarrow \mathbf{Monoid}$ induced by given E -systems. (If you did Exercise 9.5:1, you will be able to apply the results of that exercise here; if not, you can still work out the corresponding results for this particular case.)

9.7. Functors to and from some related categories. The above characterization of representable functors $\mathbf{Monoid} \rightarrow \mathbf{Monoid}$ can be used to characterize various classes of representable functors involving the category \mathbf{Group} as well.

We begin with some general observations. Let $U: \mathbf{Group} \rightarrow \mathbf{Monoid}$ denote the “forgetful” functor, let $F: \mathbf{Monoid} \rightarrow \mathbf{Group}$ denote the left adjoint of U , the “universal enveloping group” functor, and let $G: \mathbf{Monoid} \rightarrow \mathbf{Group}$ denote the right adjoint of U , the “group of invertible elements” functor. It is clear that the counit of the first adjunction and the unit of the second are isomorphisms

$$\varepsilon_{U,F}: FU \cong \text{Id}_{\mathbf{Group}} \quad \text{and} \quad \eta_{G,U}: \text{Id}_{\mathbf{Group}} \cong GU.$$

In general, when two maps compose to the identity in one order, their composite in the opposite order is a retraction of one object to a subobject isomorphic to the other. In this case, we see that the composites of our two adjoint pairs in the reverse order, UF and UG , are retractions of \mathbf{Monoid} onto $U(\mathbf{Group})$, which is clearly a full subcategory of \mathbf{Monoid} isomorphic to \mathbf{Group} . The other unit and counit of our adjunctions relate each monoid to its image in this subcategory under the corresponding retraction; let us write these

$$\eta = \eta_{U,F}: \text{Id}_{\mathbf{Monoid}} \rightarrow UF \quad \text{and} \quad \varepsilon = \varepsilon_{G,U}: UG \rightarrow \text{Id}_{\mathbf{Monoid}}$$

(breaking the convention that η and ε generally denote the unit and counit of the same adjunction). The next steps are given in the following two exercises:

Exercise 9.7:1. (i) Show that the monoids S of the form $U(A)$ (A a group) are precisely those monoids for which the universal map $\eta(S): S \rightarrow UF(S)$ is an isomorphism, and also precisely those monoids for which the universal map $\varepsilon(S): UG(S) \rightarrow S$ is an isomorphism.

(ii) Show that UF is left adjoint to UG .

(iii) Show that for any variety \mathbf{V} , the representable functors $\mathbf{Group} \rightarrow \mathbf{V}$ can be identified with the representable functors $V: \mathbf{Monoid} \rightarrow \mathbf{V}$ which are invariant under composition on the right with UG (i.e., those V such that the induced map $V\varepsilon: VUG \rightarrow V$ is an isomorphism).

(iv) Show similarly that the representable functors $\mathbf{V} \rightarrow \mathbf{Group}$ can be identified with the representable functors $\mathbf{V} \rightarrow \mathbf{Monoid}$ which are invariant under composition on the left with UG (i.e., such that the induced map $\varepsilon V: UGV \rightarrow V$ is an isomorphism).

Though we shall not need it, you may also

(v) Show that the functors $\mathbf{Group} \rightarrow \mathbf{V}$, respectively $\mathbf{V} \rightarrow \mathbf{Group}$ which have right adjoints (i.e., the left adjoints of representable functors) can be identified with the functors $\mathbf{Monoid} \rightarrow \mathbf{V}$, respectively $\mathbf{V} \rightarrow \mathbf{Monoid}$ which have right adjoints and are invariant under composition on the right, respectively on the left with UF .

Exercise 9.7:2. Using the preceding exercise,

(i) Show that every representable functor $\mathbf{Group} \rightarrow \mathbf{Monoid}$ is a power (i.e., product of copies) of the forgetful functor U . (First proved by D. Kan [82].)

(ii) Show that every representable functor $\mathbf{Monoid} \rightarrow \mathbf{Group}$ is a power of the group-of-invertible-elements functor G .

(iii) Show that every representable functor $\mathbf{Group} \rightarrow \mathbf{Group}$ is a power of the identity functor.

Thus, in each of these three cases, all representable functors arise as powers of one “basic” functor, U , G or $\text{Id}_{\mathbf{Group}}$ respectively. Calling this functor B in each case, so that the general representable functor between the categories in question has the form B^X , let us observe that for any set map $X \rightarrow Y$ we get a map $B^Y \rightarrow B^X$. Are these the only morphisms among these

functors?

Not quite. For instance, in the case of functors $\mathbf{Group} \rightarrow \mathbf{Group}$, if we take $X = Y = 1$, so that we are considering endomorphisms of the identity functor of \mathbf{Group} , there is not only the identity morphism, associating to every group its identity map, and arising from the unique set map $1 \rightarrow 1$, but also the trivial morphism, associating to every group the endomorphism under which all elements go to e . To correctly describe the morphisms among our functors, let \mathbf{Set}^{Pt} denote the category of *pointed sets*, whose objects are sets given with a single distinguished element, and whose morphisms are set maps sending distinguished element to distinguished element. (This may be identified with the variety $\Omega\text{-Alg}$ with Ω consisting of a single zeroary operation.) The next exercise shows that this is the right category for parametrizing these functors.

Exercise 9.7:3. (i) Let $L: E\text{-System} \rightarrow \mathbf{Set}^{\text{Pt}}$ denote the functor taking every E -system $X = (X^+, X^-, E, u)$ to the pointed set (X^+, u) . Show that when restricted to the full subcategory of E -systems whose “box pictures” have all connected components of the form $\boxed{\cdot \rightleftarrows \cdot}$, the functor L gives an equivalence of categories.

(ii) Deduce that in each of the cases of the preceding exercise, the indicated category of representable functors is equivalent to $(\mathbf{Set}^{\text{Pt}})^{\text{op}}$. Precisely, letting B denote the “basic” functor in each case, show that morphisms $B^X \rightarrow B^Y$ correspond to the morphisms of pointed sets $(Y \cup \{u\}, u) \rightarrow (X \cup \{u\}, u)$ where u denotes an element not in X or Y .

Let us turn back to something mentioned at the beginning of the preceding section. In the description of a comonoid object of \mathbf{Monoid} , the co-neutral-element was uniquely determined, and hence provided no information; nevertheless, the coidentities it was required to satisfy played an important role in our arguments. The next exercise shows that these coidentities were really needed for our results.

Exercise 9.7:4. Consider the following two representable functors from \mathbf{Monoid} to the variety of semigroups with a distinguished element (zeroary operation) e subject to no additional identities.

(a) The functor V taking $A \in \text{Ob}(\mathbf{Monoid})$ to the semigroup with underlying set $|A|$, multiplication given by $x*y = x$ for all x and y , and distinguished element given by the neutral element e of A .

(b) The functor W specifying the same underlying set and distinguished element, but with multiplication given by $x*y = e$.

Verify that in both cases the operation $*$ is indeed associative (so that the functors take values in the codomain variety claimed), and also that in both cases the distinguished element e is an *idempotent* with respect to $*$ (i.e., satisfies $e*e = e$). Show that in case (a), this element also satisfies the *right* neutral law, but not the left neutral law, while in case (b), neither neutral law is satisfied.

Note that in case (b) of the above exercise, the distinguished element satisfies the identities $e*x = e = x*e$. An element with this property is called a *zero* element of a semigroup, because these identities hold for 0 in the multiplicative semigroup of a ring. An element of a semigroup satisfying only the first of these identities is called a *left zero* element. We see that in case (a) every element is a left zero. The unique multiplication with the latter property on any set is called the *left zero multiplication*.

Little is known about general representable functors $\mathbf{Monoid} \rightarrow \mathbf{Semigroup}$. Dropping the zeroary co-operations \mathbf{e} , the above exercise gives examples that are noteworthy in that construction (a) used nothing about the given monoid A but its underlying set, while (b) used only its structure of set with distinguished element e . The next exercise displays some constructions

that do use the monoid operation, but in peculiar – almost random – ways.

Exercise 9.7:5. (i) Show that one can define a representable functor $\mathbf{Monoid} \rightarrow \mathbf{Semigroup}$ by associating to every monoid A the set of pairs (ξ, η) such that ξ is an invertible and η an arbitrary element of A , with the operation $(\xi, \eta)(\xi', \eta') = (e, \xi^{-1}\xi'^{-1}\xi\xi')$.

(ii) Show that if we impose on the ordered pairs in the description of the above functor the additional condition that $\xi^n = e$ for a fixed positive integer n , and/or the condition $\xi\eta = \eta$, the resulting subsets are still closed under the above operation, and hence define further representable functors.

The above observations lead to

Exercise 9.7:6. (Open question [2, Problem 21.7, p.94]) Find a description of (or other strong results about) all representable functors $\mathbf{W} \rightarrow \mathbf{Semigroup}$, where \mathbf{W} is any of the varieties \mathbf{Monoid} , \mathbf{Group} or $\mathbf{Semigroup}$.

The following questions may be easy or hard to answer; I have not thought about them:

Exercise 9.7:7. Let $V: \mathbf{Monoid} \rightarrow \mathbf{Monoid}$ be a representable functor whose E -system has a single connected component, and is not one of $\begin{matrix} \cdot & \square \\ \square & \cdot \end{matrix}$, $\begin{matrix} \square & \cdot \\ \cdot & \square \end{matrix}$, $\begin{matrix} \cdot & \square \\ \square & \cdot \\ \cdot & \square \\ \square & \cdot \end{matrix}$. What can one say about the class of monoids of the form $V(A)$ ($A \in \text{Ob}(\mathbf{Monoid})$)? How much does this class depend on the choice of V ? How does it compare with the class of monoids that are embeddable in groups? With the class of monoids $H(A)$, where H is the functor of Exercise 9.6:3?

One may likewise ask these questions for the classes of monoids arising as values of the *left adjoints* of such functors.

Some results on representable functors to and from the variety of *heaps* and related varieties are given in [2, §22].

9.8. Representable functors among categories of abelian groups and modules. Let us next consider representable functors from abelian groups to monoids. Let

$$V: \mathbf{Ab} \rightarrow \mathbf{Monoid}$$

be such a functor, with representing coalgebra $(R, \mathbf{m}, \mathbf{e})$. Since coproducts of abelian groups are direct sums, we may write the coproduct of two copies of R as $R^\lambda \oplus R^\rho$; thus, every element of this group has the form $y^\lambda + z^\rho$ for unique $y, z \in |R|$. In particular, for each $x \in |R|$ we can write

$$\mathbf{m}(x) = y^\lambda + z^\rho.$$

As in the case of functors on \mathbf{Monoid} , the co-neutral-element must be the trivial map. Applying the coneutral laws to the above equation, we immediately get $x = y = z$, that is,

$$\mathbf{m}(x) = x^\lambda + x^\rho \quad (x \in |R|).$$

Given any two elements $a, b \in |V(A)| = \mathbf{Ab}(R, A)$, this says that their ‘‘product’’ in $V(A)$ is the homomorphism taking each $x \in |R|$ to $a(x) + b(x)$. In other words, the induced ‘‘multiplication’’ of homomorphisms is just the familiar addition of homomorphisms of abelian groups.

It is clear that, conversely, for every abelian group R this operation on homomorphisms with domain R *does* make h_R a \mathbf{Monoid} -valued functor. So for each $R \in \text{Ob}(\mathbf{Ab})$, there is a unique representable functor $V: \mathbf{Ab} \rightarrow \mathbf{Monoid}$ whose representing coalgebra has underlying object R .

In view of the form V takes, it is natural to call the binary co-operation on R a ‘‘coaddition’’ rather than a ‘‘comultiplication’’. Of course, it is well known that on the sets $\mathbf{Ab}(R, A)$ addition

of maps is actually an operation of *group*, and, indeed, of *abelian group*, with the unique inverse operation described in the obvious way. Thus, our determination of all representable functors $\mathbf{Ab} \rightarrow \mathbf{Monoid}$ also determines all representable functors $\mathbf{Ab} \rightarrow \mathbf{Group}$ and $\mathbf{Ab} \rightarrow \mathbf{Ab}$. That is,

Lemma 9.8.1. *For every object R of \mathbf{Ab} , there is a unique co- \mathbf{Monoid} object, a unique co- \mathbf{Group} object, and a unique co- \mathbf{Ab} object with underlying object R . Each of these has coaddition given by the diagonal map*

$$(9.8.2) \quad \mathbf{a}(x) = x^\lambda + x^\rho \quad (x \in |R|),$$

and co-neutral-element given by $\mathbf{e}(x) = 0$ ($x \in |R|$). In the co- \mathbf{Group} and co- \mathbf{Ab} structures, the co-inverse operation is given by

$$\mathbf{i}(x) = -x. \quad \square$$

Since this result was so easy to prove, let's make some more work for ourselves, and try to generalize it!

Recall that an abelian group is equivalent to a left \mathbb{Z} -module, and that for any ring K , a left K -module M can be described as an abelian group with a family of abelian group endomorphisms, called "scalar multiplications", indexed by the elements of K , such that sums of these endomorphisms, composites of these endomorphisms, and the identity endomorphism are the endomorphisms indexed by the corresponding sums and products of elements of K and by the multiplicative neutral element $1 \in |K|$. (Unless the contrary is stated, our rings are always members of the variety \mathbf{Ring}^1 of associative rings with multiplicative neutral element 1.) We will write $K\text{-Mod}$ for the variety of left K -modules.

It is easy to see that the argument giving Lemma 9.8.1 generalizes to the case of representable functors from $K\text{-Mod}$ to the varieties \mathbf{Monoid} , \mathbf{Group} and \mathbf{Ab} .

What about functors from $K\text{-Mod}$ to $K\text{-Mod}$, or better, to $L\text{-Mod}$ for another ring L ?

To study this question, let us write out explicitly the identities for the scalar multiplication operations of $K\text{-Mod}$ which we stated above in words. The identities saying that each such multiplication is an abelian group endomorphism say that for all $c \in |K|$ and $x, x' \in |M|$,

$$(9.8.3) \quad c(x+x') = cx + cx'.$$

(We are here taking advantage of the fact that group homomorphisms can be characterized as set-maps respecting the binary group operation alone.) The identities characterizing sums and composites of scalar multiplications, and scalar multiplication by $1 \in |K|$, say that for $c, c' \in |K|$, $x \in |M|$,

$$(9.8.4) \quad (c+c')x = cx + c'x$$

$$(9.8.5) \quad (cc')x = c(c'x)$$

$$(9.8.6) \quad 1x = x.$$

Now suppose L is another ring, and $(R, \mathbf{a}, \mathbf{i}, \mathbf{e}, (\mathbf{s}_d)_{d \in |L|})$ a co- L -module object in $K\text{-Mod}$, where R is the underlying K -module, \mathbf{a} , \mathbf{i} and \mathbf{e} give the co-abelian-group structure of R , and for each $d \in |L|$, \mathbf{s}_d is the co-operation corresponding to scalar multiplication by d . The co-abelian-group structure will, as we have noted, have the form described in Lemma 9.8.1.

The s_d will be unary co-operations, i.e., K -module homomorphisms $R \rightarrow R$, which can thus be looked at as unary *operations* on the set $|R|$. We now need some basic observations:

Exercise 9.8:1. Let R be any K -module, and \mathbf{a} , \mathbf{i} , \mathbf{e} the coaddition, coinverse and cozero morphisms defining the unique co-**Ab** structure on R in $K\text{-Mod}$.

- (i) Show that every K -module endomorphism $\mathbf{s}: R \rightarrow R$ satisfies the coidentity corresponding to the identity (9.8.3); i.e., show that the unary operation induced by such an \mathbf{s} on each $h_R(A)$ is an abelian group endomorphism.
- (ii) Show that such an operation \mathbf{s} induces the identity operation on each $h_R(A)$ (cf. (9.8.6)) if and only if it is the identity endomorphism of R .
- (iii) Show that if \mathbf{s}_d , $\mathbf{s}_{d'}$ and $\mathbf{s}_{d''}$ are three endomorphisms of R , then the operations on the abelian groups $h_R(A)$ induced by \mathbf{s}_d and $\mathbf{s}_{d'}$ sum to the operation induced by $\mathbf{s}_{d''}$ if and only if $\mathbf{s}_d + \mathbf{s}_{d'} = \mathbf{s}_{d''}$.
- (iv) Show likewise that the operation induced by $\mathbf{s}_{d''}$ is the composite in a given order of the operations induced by \mathbf{s}_d and $\mathbf{s}_{d'}$ (cf. (9.8.5)) if and only if $\mathbf{s}_{d''}$ is the composite of \mathbf{s}_d and $\mathbf{s}_{d'}$ in the *opposite* order.

From the above results we deduce that

If K and L are rings, and R a left K -module, then a co-left- L -module structure on R is equivalent to a system of R -module endomorphisms $(\mathbf{s}_d)_{d \in |L|}$ which for all $d, d' \in |L|$ satisfy

$$(9.8.7) \quad (9.8.8) \quad \mathbf{s}_1 = \text{id}_R$$

$$(9.8.9) \quad \mathbf{s}_{d+d'} = \mathbf{s}_d + \mathbf{s}_{d'}$$

$$(9.8.10) \quad \mathbf{s}_{dd'} = \mathbf{s}_{d'} \mathbf{s}_d.$$

This is a nice result, but we can make it more elegant with a change of notation. The reversal of the order of composition in (9.8.10) can be cured if we write the operators \mathbf{s}_d on the *right* of their arguments, instead of on the left, and compose them accordingly. Moreover, once the operation of elements of L (by co-scalar-multiplications) is written on a different side from the operation of elements of K (by scalar multiplication), there is no real danger of confusion if we drop the symbols \mathbf{s} , i.e., replace the above notation $\mathbf{s}_d(x)$ by xd ($x \in |R|$, $d \in |L|$). We now find that the scalar multiplications by elements of K and the co-scalar-multiplications by elements of L satisfy a very symmetrical set of conditions, namely, that for all $c, c' \in |K|$, $x, x' \in |R|$, $d, d' \in |L|$,

$$(9.8.11) \quad 1x = x \qquad \qquad \qquad x1 = x$$

$$(9.8.12) \quad c(x+x') = cx+cx' \qquad \qquad (x+x')d = xd+x'd$$

$$(9.8.13) \quad (c+c')x = cx+c'x \qquad \qquad x(d+d') = xd+xd'$$

$$(9.8.14) \quad (cc')x = c(c'x) \qquad \qquad x(dd') = (xd)d'$$

$$(9.8.15) \quad c(xd) = (cx)d$$

Here (9.8.15), and the right hand equation of (9.8.12), say that the co-scalar-multiplications are endomorphisms of the K -module R . The conditions in the left-hand column, together with the identities for the abelian group structure of R , constitute the identities of a left K -module, while the remaining three conditions on the right say that the co-scalar-multiplication endomorphisms

behave as required to give a co-left- L -module structure. (Only three such conditions are needed, as against the four on the left, because of Exercise 9.8:1(i).)

We have, in fact, rediscovered a standard concept of ring theory:

Definition 9.8.16. *An abelian group on which a ring K operates by maps written on the left and a ring L operates by maps written on the right so that (9.8.11)-(9.8.15) are satisfied is called a (K, L) -bimodule.*

For given K and L , the variety of (K, L) -bimodules will be denoted $K\text{-Mod-}L$.

Note that given two arbitrary varieties of algebras \mathbf{V} and \mathbf{W} , the category of \mathbf{V} -coalgebra objects of \mathbf{W} cannot in general be regarded as a variety of algebras, because the co-operations $s: R \rightarrow \coprod_{\text{ari}(s)} R$ do not have the form of maps $|R|^\beta \rightarrow |R|$, unless $\text{ari}(s) = 1$. In the present case, it happened that the two *non-unary* co-operations of our objects, the coaddition and the cozero, were uniquely determined, so that the structure could be defined wholly by unary co-operations, and so, atypically, the category of these coalgebras could be identified with a variety of algebras, to wit, $K\text{-Mod-}L$.

Ring-theorists often write a (K, L) -bimodule R as ${}_K R_L$. Here the subscripts are not part of the “name” of the object, but reminders that K operates on the left, and L on the right. (Actually, ring-theorists more often use other letters, such as B , for “bimodule”, or M , for “module”, reserving R for rings. But in this chapter we are using R wherever possible for “representing object”.) That such a bimodule structure makes R a co- L -module in $K\text{-Mod}$ is equivalent to the result familiar to ring-theorists, that the set of left K -module homomorphisms from a (K, L) -bimodule to a left K -module,

$$(9.8.17) \quad K\text{-Mod}({}_K R_L, {}_K A)$$

has a natural structure of left L -module. Let us describe without using the language of coalgebras how this L -module structure arises. If we regard the actions of the elements of L on R as K -module endomorphisms, then the functoriality of $K\text{-Mod}(-, -)$ in its first variable turns these into endomorphisms of the abelian group $K\text{-Mod}({}_K R, {}_K A)$, and since this functoriality is contravariant, the order of composition of these endomorphisms is reversed; so from the right L -module structure on R , we get a left L -module structure on that hom-set. Explicitly, given any $f \in K\text{-Mod}(R, A)$ and $d \in |L|$, the action of d on f in this induced left L -module structure is given by

$$(9.8.18) \quad (df)(x) = f(xd).$$

This takes a more elegant form if we adopt

(*Frequent convention in ring theory.*) If possible, write homomorphisms of *left* modules on the *right* of their arguments, and homomorphisms of *right* modules on the *left* of their arguments, and use the notation for composition of such homomorphisms appropriate to the side on which they are written.

For a discussion of (9.8.19), and its advantages and weaknesses, see [47]. We have already applied this idea once, in the change of notation introduced immediately after (9.8.7). In our present situation, it suggests that we should write elements $f \in K\text{-Mod}(R, A)$ on the right of elements $x \in |A|$. When we do so, (9.8.18) takes the form

$$(9.8.20) \quad x(df) = (xd)f.$$

In summary:

Lemma 9.8.21. *If K and L are unital associative rings, then a $\text{co-}L\text{-Mod}$ object of $K\text{-Mod}$ is essentially the same as a (K, L) -bimodule ${}_K R_L$. When R is so regarded, the left L -module structure on the functor $K\text{-Mod}(R, -)$ is composed of the usual abelian group structure on hom-sets, together with the scalar multiplications (9.8.18), or in right-operator notation, (9.8.20). \square*

It is not hard to verify that the above correspondence yields an equivalence of categories $K\text{-Mod-}L \approx \mathbf{Rep}(K\text{-Mod}, L\text{-Mod})^{\text{op}}$.

9.9. More on modules: left adjoints of representable functors. Let us now find the left adjoint to the functor induced as above by a (K, L) -bimodule R . This must take a left L -module B to a left K -module A with a universal left L -module homomorphism

$$(9.9.1) \quad h: B \rightarrow K\text{-Mod}(R, A).$$

To find this object A , we will apply our standard heuristic approach: Consider an arbitrary left K -module A with an L -module homomorphism (9.9.1), and see what elements of A , and what relations among these elements, this map gives us.

For each $y \in |B|$, (9.9.1) gives a homomorphism $h(y): R \rightarrow A$; and such a homomorphism gives us, for each $x \in |R|$, an element of A . With (9.8.19) in mind, let us write this as

$$x*y = xh(y) \quad (x \in |R|, y \in |B|).$$

I claim that the conditions that these elements must satisfy are that for all $x, x' \in |R|$, $y, y' \in |B|$, $c \in |K|$, $d \in |L|$,

$$(9.9.2) \quad (x+x')*y = x*y + x'*y \quad x*(y+y') = x*y + x*y'$$

$$(9.9.3) \quad (cx)*y = c(x*y) \quad \text{---}$$

$$(9.9.4) \quad x*(dy) = (xd)*y.$$

Indeed, the two equations on the left are the conditions for the maps $h(y)$ to be left K -module homomorphisms, while the equations on the right and at the bottom are the conditions for the map (9.9.1) to be a homomorphism of left L -modules with respect to the given L -module structure on B and the operations (9.8.20) on $K\text{-Mod}(R, A)$. We note the gap on the right-hand side of (9.9.3); since nothing acts on the *right* on the L -module B , there is nothing to put there. (But do not lose heart; this gap will presently be filled.) So the universal A with a homomorphism (9.9.1) will be presented by generators $x*y$ ($x \in |R|$, $y \in |B|$) and relations (9.9.2)-(9.9.4).

Again we have discovered a standard concept. The K -module presented by this system of generators and relations is denoted

$$R \otimes_L B,$$

and called the *tensor product over L* of the (K, L) -bimodule R and the left L -module B . The generators of this module corresponding to the elements $x*y$ of the above discussion are written $x \otimes y$ ($x \in |R|$, $y \in |B|$).

We reiterate that $R \otimes_L B$ is only a left K -module. Intuitively, when we form the tensor product $({}_K R_L) \otimes_L ({}_L B)$, the operation of tensoring over L “eats up” the two L -module

structures, leaving only a K -module structure. This is dual to the situation of (9.8.17), where the construction of taking the hom-set over K “ate up” the two K -module structures, leaving only an L -module structure.

We have shown:

Lemma 9.9.5. *If we regard a (K, L) -bimodule ${}_K R_L$ as a $\text{co-}L\text{-Mod}$ object of $K\text{-Mod}$, then the left adjoint to the functor it represents,*

$$K\text{-Mod}(R, -): K\text{-Mod} \rightarrow L\text{-Mod}$$

is the functor

$$R \otimes_L -: L\text{-Mod} \rightarrow K\text{-Mod}.$$

Thus, given the bimodule R , a left K -module A , and a left L -module B , we have a functorial bijection, which is in fact an isomorphism of abelian groups

$$L\text{-Mod}(B, K\text{-Mod}(R, A)) \cong K\text{-Mod}(R \otimes_L B, A). \quad \square$$

An interesting consequence of Lemmas 9.8.21 and 9.9.5 is that every *representable* functor between module categories, and likewise the left adjoint of every such functor, respects **Ab**-structures, i.e., sends sums of morphisms to sums of morphisms. This is not true of general functors between module categories, as the reader can see from the functor $A \mapsto A \otimes A$ of Exercise 9.5:4(i).

In defining the tensor product $R \otimes_L B$, I said that one presents it as a left K -module using the relations (9.9.2)-(9.9.4). But another standard definition is to present it as an *abelian group* using only the relations corresponding to (9.9.2) and (9.9.4), and then to use (9.9.3) to define a left K -module structure on this group. Not every abelian group with elements $x*y$ ($x \in |R|, y \in |B|$) satisfying (9.9.2) and (9.9.4) has a left K -module structure satisfying (9.9.3); but the *universal* abelian group with these properties does, because the universal construction is functorial in R as a right L -module, and the left K -module structure of R constitutes a system of right- L -module endomorphisms; these induce endomorphisms of the constructed abelian group which make it a K -module.

This approach shows that the underlying abelian group structure of $R \otimes_L B$ depends only on the right L -module structure of R and the left L -module structure of B ; this is again analogous to the situation for the hom-set $K\text{-Mod}({}_K R_L, {}_K A)$, which starts out as an abelian group constructed using only the left K -module structures of R and A , and then acquires a left L -module structure from the right L -module structure of R , by functoriality.

We should now learn how to compose the representable functors we have described. Suppose we have three rings, H, K, L , and adjoint pairs determined by an (H, K) -bimodule R and a (K, L) -bimodule S :

$$(9.9.6) \quad H\text{-Mod} \begin{array}{c} \xrightarrow{H\text{-Mod}({}_H R_K, -)} \\ \xleftarrow{{}_H R_K \otimes_K -} \end{array} K\text{-Mod} \begin{array}{c} \xrightarrow{K\text{-Mod}({}_K S_L, -)} \\ \xleftarrow{{}_K S_L \otimes_L -} \end{array} L\text{-Mod}.$$

By observations we made in §9.5, the underlying left K -module of the coalgebra determining the composite adjoint pair can be gotten by applying the left adjoint functor $R \otimes_K -$ to the underlying object of the coalgebra determining the other adjoint pair; hence it is the left H -module $R \otimes_K S$. It remains to find the coalgebra structure, i.e., the right L -module structure, on this object; this

arises from the right L -module structure on S , by the same “functoriality” effect noted above for the left module structure of $R \otimes_L B$. So the composite of the adjoint pairs shown above is determined by an (H, L) -bimodule ${}_H(R \otimes_K S)_L$.

At this point, we have discussed enough kinds of structure on tensor products so that we are ready to put them all into a definition, after which we will state formally the above characterization of representing objects for composite functors.

Definition 9.9.7. *If K is a ring, R a right K -module and S a left K -module, then*

$$R \otimes_K S$$

will denote the abelian group presented by generators $x \otimes y$ ($x \in |R|$, $y \in |S|$) and the relations (for all $x, x' \in |R|$, $d \in |K|$, $y, y' \in |S|$)

$$(9.9.8) \quad (x+x') \otimes y = x \otimes y + x' \otimes y, \quad x \otimes (y+y') = x \otimes y + x \otimes y'$$

$$(9.9.9) \quad (xd) \otimes y = x \otimes (dy).$$

If R is in fact an (H, K) -bimodule, respectively if S is a (K, L) -bimodule, respectively if both are true (by which we mean, if the right K -module structure of R is given as part of an (H, K) -bimodule structure, and/or if the left K -module structure of S is given as part of a (K, L) -bimodule structure, for some rings H, L), then the abelian group $R \otimes_K S$ becomes a left H -module, respectively a right L -module, respectively an (H, L) -bimodule, with scalar multiplications characterized by (one or both of) the following identities for $c \in |H|$, $e \in |L|$:

$$(9.9.10) \quad c(x \otimes y) = (cx) \otimes y \quad (x \otimes y)e = x \otimes (ye)$$

Note that (9.9.10) has restored the symmetry that was missing in (9.9.3)! Note also that the four cases of the above definition (tensoring a right module R_K or a bimodule ${}_H R_K$ with a left module ${}_K S$ or a bimodule ${}_K S_L$) reduce to a single case if for every K we identify $\mathbf{Mod}\text{-}K$ with $\mathbb{Z}\text{-Mod}\text{-}K$ and $K\text{-Mod}$ with $K\text{-Mod}\text{-}\mathbb{Z}$, and likewise \mathbf{Ab} with $\mathbb{Z}\text{-Mod}\text{-}\mathbb{Z}$.

Let us now set down the result sketched before this definition.

Lemma 9.9.11. *In the situation shown in (9.9.6), the composite of the functors among left module categories represented by bimodules ${}_H R_K$ and ${}_K S_L$ is represented by the (H, L) -bimodule $R \otimes_K S$. \square*

Terminological note: Given bimodules ${}_H R_K$ and ${}_K S_L$, we may call a map $*$ from $|R| \times |S|$ into an (H, L) -bimodule ${}_H T_L$ satisfying the equations corresponding to (9.9.8)-(9.9.10) a “bilinear map $R \times S \rightarrow T$ ”, generalizing the term we have already used in the case of abelian groups (§3.9), so that we may describe $R \otimes_K S$ as an (H, L) -bimodule with a universal bilinear map of these bimodules into it. However, many ring theorists (perhaps the majority) feel that the term “bilinear” should logically only mean “left H -linear and right L -linear”, i.e., the conditions of (9.9.8) and (9.9.10), and they use the adjective “balanced” to express (9.9.9). So they would call $R \otimes_K S$ an (H, L) -bimodule with a universal K -balanced bilinear map of $R \times S$ into it.

The results on modules and bimodules developed above are sometimes regarded as a “model case” in terms of which to think of the general theory of representable functors among varieties of algebras, and their representing coalgebras. Thus, Freyd entitled the paper [10] in which he

introduced the theory of such functors and coalgebras “*On algebra-valued functors in general, and tensor products in particular*”, and he called the coalgebra that represents a composite of representable functors between arbitrary varieties of algebras the “tensor product” of the coalgebras representing the given functors. I recommend that paper to the interested student, though with one word of advice: Ignore the roundabout way the author treats zeroary operations, and simply consider them, as we have done, to be morphisms from the empty product (the terminal object) in a category into the object in question.

Further remarks: In the paragraph following (9.8.10), we chose a notation which “separated” the actions of elements of K and elements of L , writing them on opposite sides of elements of R . It is also worth seeing what happens if we do not separate them, but continue to write them both to the left of their arguments. The actions of elements of L will then compose in the opposite way to the multiplication of those elements in the ring L . This can be thought of as making R a left module over L^{op} , the opposite of the ring L (defined as the ring with the same underlying set and additive group structure as L , but with the order of multiplication reversed). Thus we have on R both a left K -module structure and a left L^{op} -module structure, related by the conditions that the additive group operations of the two module structures are the same, and that the scalar multiplications of the L^{op} -module structure are endomorphisms of the K -module structure. The latter condition says that the images of the elements of K and of elements of L^{op} in the endomorphism ring of the common abelian group R commute with one another. Now we saw in §3.13 that given two rings P and Q , if we form the tensor product of their underlying abelian groups, this can be given a ring structure such that the maps $p \mapsto p \otimes 1$ and $q \mapsto 1 \otimes q$ are homomorphisms of P and Q into $P \otimes Q$, whose images commute elementwise, and which is universal among rings given with such a pair of homomorphisms from P and Q . Thus, in our present situation, the mutually commuting left K -module and left L^{op} -module structures on R are equivalent to a single structure of left $K \otimes L^{\text{op}}$ -module. That is

$$(9.9.12) \quad K\text{-Mod-}L \cong (K \otimes L^{\text{op}})\text{-Mod}.$$

Hence one can study bimodules with the help of the theory of tensor products of rings, and vice versa.

This also shows us that if we want to study representable functors between categories of *bimodules*, we do not need to undertake a new investigation, but can reduce this situation to the one we have already studied by using rings $K_0 \otimes K_1^{\text{op}}$, etc., in place of K , etc..

Exercise 9.9:1. (i) If you did Exercise 3.13:4(ii), translate the results you got there to a partial or complete description of all $(\mathbb{Q}(2^{1/3}), \mathbb{Q}(2^{1/3}))$ -bimodules.

(ii) If you did Exercise 3.13:4(i), translate the results you got there to a partial or complete description of all \mathbb{R} -centralizing (\mathbb{C}, \mathbb{C}) -bimodules B , where “ \mathbb{R} -centralizing” means satisfying the identity $rx = xr$ for all $r \in \mathbb{R}$, $x \in B$.

The student familiar with the theory of modules over *commutative* rings may have been surprised at my saying earlier that when we form a hom-set $K\text{-Mod}({}_K R, {}_K A)$ or a tensor product $R_L \otimes_L L B$, the K -module structure, respectively the L -module structure, was “eaten up” in the process, since in the commutative case, these objects inherit natural K - and L -module structures. You can discover the general statement, of which these apparently contradictory observations are cases, by doing the next exercise. (The answer comes out in parts (iii) and (iv).)

Exercise 9.9:2. Let K be a ring (not assumed commutative) and M a left K -module.

- (i) Determine the *structure*, in the sense of §8.10, of the functor $K\text{-Mod}({}_K M, -): K\text{-Mod} \rightarrow \mathbf{Set}$.
- (ii) Determine similarly the structure of $K\text{-Mod}(-, {}_K M): K\text{-Mod}^{\text{op}} \rightarrow \mathbf{Set}$.
- (iii) Determine the structure of $K\text{-Mod}(-, -): K\text{-Mod}^{\text{op}} \times K\text{-Mod} \rightarrow \mathbf{Set}$.
- (iv) Answer the corresponding three questions with tensor products in place of hom-sets (with or without the help of Corollary 7.11.6).

Let us note another basic ring-theoretic tool that we can understand with the help of the results of this section. Suppose $f: L \rightarrow K$ is a ring homomorphism. Then we can make any left K -module A into a left L -module by keeping the same abelian group structure, and defining the new scalar multiplication by $d \cdot x = f(d)x$ ($d \in |L|$). This functor preserves underlying sets, hence it is representable. It is called “restriction of scalars along f ”, and its left adjoint is called “extension of scalars along f ”. (When f is the inclusion of a subring L in a ring K these are natural terms to use. The usage in the case of arbitrary homomorphisms f is a generalization from that case.) You should find the first part of the next exercise straightforward, and the second not too hard.

Exercise 9.9:3. Let $f: L \rightarrow K$ be a ring homomorphism.

- (i) Describe the bimodule representing the restriction-of-scalars functor, and get a description of the extension-of-scalars construction $K\text{-Mod} \rightarrow L\text{-Mod}$ as a tensor product operation.
- (ii) If K and L are commutative, we may also consider the “restriction of base-ring” functor from K -algebras to L -algebras, defined to preserve underlying ring-structures, and act as restriction of scalars on module structures. (You may here take “algebras” over K and L either to mean commutative algebras, or not-necessarily commutative (but as always, unless the contrary is stated, associative) algebras, depending on what you are familiar with.) We know this functor is representable. (Why?) Describe its representing coalgebra. Show that the left adjoint of this functor acts on the underlying modules of algebras by extension of scalars. How is the ring structure on the resulting modules defined?

I will close our discussion of abelian groups and modules with an observation that goes back to the beginning of the preceding section: When we determined the form of the general comonoid object of \mathbf{Ab} , our argument used only the fact that we had a binary co-operation satisfying the coneutral laws with respect to the unique zeroary co-operation – the coassociative law was never needed! Thus, if we let \mathbf{Binar}^e denote the variety of sets with a binary operation and a neutral element e for that operation, then $\text{co-}\mathbf{Binar}^e$ objects of \mathbf{Ab} are automatically $\text{co-}\mathbf{Monoid}$ objects, and even $\text{co-}\mathbf{AbMonoid}$ objects, and, as we noted, these have unique coinverse operations making them $\text{co-}\mathbf{Group}$ and $\text{co-}\mathbf{Ab}$ objects.

On the other hand, if we drop the co-neutral-element, then associativity and commutativity conditions do make a difference:

Exercise 9.9:4. Characterize all representable functors from \mathbf{Ab} to each of the following varieties:

- (i) \mathbf{Binar} , the variety of sets given with a binary operation.
- (ii) $\mathbf{Semigroup}$ (a subvariety of \mathbf{Binar}).
- (iii) $\mathbf{AbSemigroup}$ (a subvariety of $\mathbf{Semigroup}$).

In the last two cases, you should discover that every such functor decomposes as a direct sum of a small number of functors whose structures are easily described.

9.10. Some general results on representable functors, mostly negative. As we mentioned in Exercise 9.7:6, the form of the general representable functor $\mathbf{Monoid} \rightarrow \mathbf{Semigroup}$ is not known. What about representable functors going the other way, from $\mathbf{Semigroup}$ to \mathbf{Monoid} ?

It is easy to show that in this case there are no nontrivial examples. The idea is that in working in $\mathbf{Semigroup}$, one has no distinguished elements available, so there is no way to pin down a zeroary “neutral element” operation.

Before having you prove this, let me indicate the exception implied in the word “nontrivial”. If \mathbf{C} is any category with an initial object I , then the functor h_I takes every object of \mathbf{C} to a one-element set, which of course has a unique structure of \mathbf{V} -algebra for every variety \mathbf{V} ; hence for every variety \mathbf{V} , the object I admits co-operations making it a \mathbf{V} -coalgebra. Let us call the functor represented by this coalgebra, which takes every object of \mathbf{C} to the one-element algebra (the terminal object) of \mathbf{V} , the *trivial* functor $\mathbf{C} \rightarrow \mathbf{V}$.

Exercise 9.10:1. Show that if \mathbf{W} is a variety without zeroary operations, and \mathbf{V} a variety with at least one zeroary operation, then there is no nontrivial representable functor $\mathbf{W} \rightarrow \mathbf{V}$.

More generally, can you give a condition on a general category \mathbf{C} with finite coproducts that insures that there are no nontrivial representable functors from \mathbf{C} to a variety \mathbf{V} with at least one zeroary operation?

Here is another observation about specific varieties from which we can extract a general principle. We began this chapter with an example of a representable functor from rings to groups; but if one looks for a nontrivial representable functor from groups to rings, it is hard to imagine how one might be constructed, because a nontrivial ring must have distinct 0 and 1, and we have only one distinguished group element, e , to use in the coordinates of a distinguished element of a ring we are constructing. This argument can be made precise. To get the result from general considerations, we make

Definition 9.10.1. If \mathbf{C} is a category with a terminal object T , let us (as in Exercise 6.8:3) define a pointed object of \mathbf{C} to mean a pair (A, p) where A is an object of \mathbf{C} and p a morphism $T \rightarrow A$. (Thus, since T is the product of the empty family of copies of A , such a pair is an object of \mathbf{C} given with a single zeroary operation.) A morphism $(A, p) \rightarrow (A', p')$ of such objects will mean a morphism $A \rightarrow A'$ making a commuting triangle with p and p' . The category of pointed objects of \mathbf{C} , with these morphisms, will be denoted \mathbf{C}^{pt} .

Dually, if \mathbf{C} is a category with an initial object I , an augmented object of \mathbf{C} will mean a pair (A, a) where A is an object of \mathbf{C} and a is a morphism $A \rightarrow I$ (an “augmentation map”), equivalently, a zeroary co-operation on A . Again using the obvious commuting triangles as morphisms, we denote the category of augmented objects of \mathbf{C} by \mathbf{C}^{aug} .

Thus, in comma category notation, $\mathbf{C}^{\text{pt}} = (T \downarrow \mathbf{C})$, and $\mathbf{C}^{\text{aug}} = (\mathbf{C} \downarrow I)$.

A category \mathbf{C} will be called “pointed” if it has a zero object (an object that is both initial and terminal; Definition 6.8.1).

Exercise 6.8:3 shows that if \mathbf{C} is a category with a terminal object, then \mathbf{C}^{pt} is a pointed category. By duality, if \mathbf{C} is a category with an initial object, then \mathbf{C}^{aug} is likewise pointed. The next exercise begins with a few more observations of the same sort, then gets down to business.

Exercise 9.10:2. Prove the following:

- (i) Let \mathbf{C} be a category with a terminal (respectively initial) object. Then the forgetful functor $\mathbf{C}^{\text{pt}} \rightarrow \mathbf{C}$ (respectively $\mathbf{C}^{\text{aug}} \rightarrow \mathbf{C}$) is an equivalence if and only if \mathbf{C} is a pointed

category.

- (ii) If \mathbf{V} is a variety of algebras, then \mathbf{V}^{pt} is equivalent to a variety of algebras.
- (iii) A variety of algebras \mathbf{V} is a pointed category if and only if \mathbf{V} has a zeroary operation, and all derived zeroary operations of \mathbf{V} are equal.

Now suppose \mathbf{V} is a variety, and \mathbf{C} a category having small coproducts.

- (iv) Show that \mathbf{C}^{aug} also has small coproducts, and that

$$\mathbf{Rep}(\mathbf{C}, \mathbf{V}^{\text{pt}}) \approx \mathbf{Rep}(\mathbf{C}^{\text{aug}}, \mathbf{V}^{\text{pt}}) \approx \mathbf{Rep}(\mathbf{C}^{\text{aug}}, \mathbf{V}).$$

(If you don't see how to begin, you might think first about the case $\mathbf{V} = \mathbf{Set}$.)

- (v) Show that \mathbf{Group} is pointed, and that $(\mathbf{Ring}^1)^{\text{pt}}$ consists only of the trivial ring with its unique pointed structure. Deduce that there are no nontrivial functors from \mathbf{Group} or any of its subvarieties to \mathbf{Ring}^1 or any of its subvarieties (e.g., $\mathbf{CommRing}^1$).
- (vi) Deduce from (iv) the result of Exercise 9.10:1.

Incidentally, I believe the term ‘‘augmented’’ comes from ring theory, where an ‘‘augmentation’’ on a k -algebra R means a k -algebra homomorphism $\varepsilon: R \rightarrow k$. This ring-theoretic concept in turn probably originated in algebraic topology, where the cohomology of a pointed space acquires, by contravariance of the cohomology ring functor, such an augmentation.

Here is yet another sort of nonexistence result:

Exercise 9.10:3. Let R be an object of a variety \mathbf{V} , and let $\tau: R^\lambda \amalg R^\rho \rightarrow R^\lambda \amalg R^\rho$ denote the automorphism that interchanges x^λ and x^ρ for all $x \in |R|$. Denote by $\text{Sym}(R^\lambda \amalg R^\rho)$ the fixed-point algebra of τ ; i.e., the algebra of ‘‘ (λ, ρ) -symmetric’’ elements of $R^\lambda \amalg R^\rho$.

- (i) Show that a binary co-operation $\mathbf{m}: R \rightarrow R^\lambda \amalg R^\rho$ is cocommutative (i.e., satisfies the coidentity making the induced operations on all sets $\mathbf{V}(R, A)$ commutative) if and only if it carries R into $\text{Sym}(R^\lambda \amalg R^\rho)$.
- (ii) Show that in the variety \mathbf{Group} , one has $\text{Sym}(R^\lambda \amalg R^\rho) = \{e\}$ for all objects R .
- (iii) Deduce that there are no nontrivial representable functors $\mathbf{Group} \rightarrow \mathbf{Ab}$, hence also no nontrivial representable functors $\mathbf{Group} \rightarrow \mathbf{Ring}^1$; and that there are no nontrivial representable functors $\mathbf{Group} \rightarrow \mathbf{Semilattice}$, hence also no nontrivial representable functors $\mathbf{Group} \rightarrow \mathbf{Lattice}$.

Seeing that there are no nontrivial representable functors from groups to lattices, we may ask, what about functors in the reverse direction? The category $\mathbf{Lattice}$ has no zeroary operations, so there can be no functors from it or any of its subvarieties to \mathbf{Group} by Exercise 9.10:1; but suppose we get out of this hole by considering lattices with one or more distinguished elements. I do not know the answer to the first part of the next exercise, though I do know the answer to the second.

Exercise 9.10:4. (i) Is there a variety \mathbf{L} of lattices for which there exists a nontrivial representable functor $\mathbf{L}^{\text{pt}} \rightarrow \mathbf{Group}$?

- (ii) For \mathbf{C} a category with a terminal object T , let $\mathbf{C}^{2\text{-pt}}$ denote the category of 3-tuples (A, p_0, p_1) where A is an object of \mathbf{C} , and p_0, p_1 are morphisms $T \rightrightarrows A$. Is there any variety \mathbf{L} of lattices for which there exists a nontrivial representable functor $\mathbf{L}^{2\text{-pt}} \rightarrow \mathbf{Group}$?

Of course, not every plausible heuristic argument restricting the properties of representable functors is valid. For instance, every primitive operation of lattices, and hence also every derived operation of lattices, is isotone with respect to the natural ordering of the underlying set, while Boolean rings have the operation of complementation, which is not. Nevertheless, we have the construction of the following exercise.

Exercise 9.10:5. Let $\mathbf{DistLat}^{0,1}$ denote the variety of distributive lattices (Exercise 5.1:15) with least element 0 and greatest element 1. An element x of such a lattice L is called *complemented* if there exists $y \in |L|$ such that $x \wedge y = 0$ and $x \vee y = 1$.

Show that for $L \in \text{Ob}(\mathbf{DistLat}^{0,1})$, the set of complemented elements of L can be made a Boolean ring, whose natural partial ordering is the restriction of the natural partial ordering of L , and that this construction yields a representable functor $C: \mathbf{DistLat}^{0,1} \rightarrow \mathbf{Bool}$. Give a description of this functor in terms of “tuples of elements satisfying certain relations”, and describe the Boolean operations on such tuples.

Here is a triviality question of a different sort.

Exercise 9.10:6. If \mathbf{U} , \mathbf{V} , \mathbf{W} are varieties such that there exist nontrivial representable functors $\mathbf{W} \rightarrow \mathbf{V}$ and $\mathbf{V} \rightarrow \mathbf{U}$, must there exist a nontrivial representable functor $\mathbf{W} \rightarrow \mathbf{U}$?

Let us turn to positive results. We recall from Exercise 7.3:5 that every equivalence of categories is also an adjunction. Note also that a category \mathbf{C} equivalent to a variety with small colimits will have small colimits, hence in particular, will have small coproducts. Thus we can make sense of the concept of a representable functor from \mathbf{C} to varieties of algebras. We deduce

Lemma 9.10.2. Suppose $\mathbf{C} \xrightleftharpoons[i]{j} \mathbf{V}$ is an equivalence between an arbitrary category \mathbf{C} and a variety of algebras \mathbf{V} . Then $j: \mathbf{C} \rightarrow \mathbf{V}$ is representable, and has a representing coalgebra with underlying object $j(F_{\mathbf{V}}(1))$. \square

The above fact is used in [50] to study the *self-equivalences* of the variety of rings, and more generally, of the variety of algebras over a commutative ring k . (The self-equivalences of any category \mathbf{C} , modulo isomorphism of functors, form a group, called the *automorphism class group* of \mathbf{C} . When $\mathbf{C} = \mathbf{Ring}^1$, this group is shown in [50] to be isomorphic to \mathbb{Z}_2 ; the nonidentity element arises from the self-equivalence $K \mapsto K^{\text{op}}$. For k a commutative ring, the variety of k -algebras has a more complicated automorphism class group if k has nontrivial idempotent elements or nontrivial automorphisms.)

Exercise 9.10:7. We saw in Exercise 6.9:18 that for R a ring, the varieties $R\text{-Mod}$ and $M_n(R)\text{-Mod}$ were equivalent. By the above lemma, the equivalence must be representable. Determine the bimodules that yield this equivalence.

This suggests the question: Given rings K and L and an object R of $K\text{-Mod-L}$, under what conditions is the functor $K\text{-Mod} \rightarrow L\text{-Mod}$ represented by R an equivalence? In a future edition of these notes, I hope to add a section introducing *Morita theory*, which answers this question, and to give the generalization of that theory that answers the corresponding question for arbitrary varieties of algebras.

A challenging related problem is

Exercise 9.10:8. Characterize those functors between module categories, $F: K\text{-Mod} \rightarrow L\text{-Mod}$, which have both a left and a right adjoint.

Another useful result is given in

Exercise 9.10:9. Let \mathbf{C} be a category with small colimits, and \mathbf{V} a variety of algebras. Show that the category $\mathbf{Rep}(\mathbf{C}, \mathbf{V})$ is closed under taking small limits within the functor category $\mathbf{V}^{\mathbf{C}}$.

As an example, let n be a positive integer, and consider the functors $\text{GL}(n)$, $\text{GL}(1) \in \mathbf{Rep}(\mathbf{CommRing}^1, \mathbf{Group})$. (So $\text{GL}(1)$ is just the “group of units” functor.) We can

define morphisms

$$e, \det: GL(n) \rightrightarrows GL(1),$$

where the first takes every invertible matrix to 1, and the second takes every invertible matrix to its determinant. By the preceding exercise, the limit (equalizer) of this diagram of functors and morphisms is representable. This equalizer is in fact the functor $SL(n)$, with which we began this chapter.

Exercise 9.10:10. In §9.6 we described the general object of **Rep(Monoid, Monoid)**. Find a finite family S of such objects with the property that every object of this category is the limit of a system of objects in S and morphisms among these.

9.11. A few ideas and techniques. In §§9.7-9.9, we considered some cases of the problem, “Given varieties \mathbf{V} and \mathbf{W} , find all representable functors $\mathbf{W} \rightarrow \mathbf{V}$ ”. We can turn this question around and ask, “Given an object R of a variety \mathbf{W} , what kinds of algebras can we make out of the values of the functor h_R ?” This question asks for the *structure* on the set-valued functor h_R , in the sense of §8.10, i.e., for the operations admitted by that functor and the identities that they satisfy.

I gave some examples of this question in Exercise 8.10:3; we can now see what you probably discovered (without having terminology in which to state it precisely) if you did that exercise: that to find the operations on such a functor and the identities they satisfy, one needs to look for the co-operations admitted by its representing object, and the co-identities satisfied by these.

Let us work out an example here.

Suppose we are interested in the algebraic structure one can put, in a functorial way, on the set of *elements of exponent 2* in a general group G . This means we want to study the structure on the functor taking G to the set of such elements, i.e., the set-valued functor represented by the group \mathbb{Z}_2 ; so our task is equivalent to describing the clone of co-operations admitted by \mathbb{Z}_2 in **Group**.

An n -ary co-operation on \mathbb{Z}_2 means a group homomorphism $\mathbb{Z}_2 \rightarrow \mathbb{Z}_2 \amalg \dots \amalg \mathbb{Z}_2$, and hence corresponds to an element of exponent 2 in the latter group. Though in \mathbb{Z}_2 one usually uses additive notation, these coproduct groups are noncommutative, so let us write \mathbb{Z}_2 multiplicatively, calling the identity element e and the nonidentity element t . Then the coproduct of n copies of \mathbb{Z}_2 will be generated by elements t_0, \dots, t_{n-1} of exponent 2, and (by the observations in §3.6 on the structure of coproduct groups), the general element of this coproduct can be written uniquely

$$(9.11.1) \quad t_{\alpha_0} t_{\alpha_1} \dots t_{\alpha_{h-1}}, \text{ where } h \geq 0, \text{ all } \alpha_i \in n, \text{ and } \alpha_i \neq \alpha_{i+1} \text{ for } 0 \leq i < h-1.$$

Let us begin by seeing what structure on $h_{\mathbb{Z}_2}$ is apparent to the naked eye, and translating it into the above terms. Since the identity element of every group is of exponent 2, $h_{\mathbb{Z}_2}$ admits

$$(9.11.2) \quad \begin{array}{l} \text{the zeroary operation } e, \text{ determined by} \\ \text{the unique homomorphism } \mathbb{Z}_2 \rightarrow \{e\}. \end{array}$$

Also, any conjugate of an element of exponent 2 has exponent 2, hence $h_{\mathbb{Z}_2}$ admits

$$(9.11.3) \quad \begin{array}{l} \text{the binary operation } (x, y) \mapsto x^y = y^{-1}xy = yxy, \text{ determined by} \\ \text{the homomorphism } \mathbb{Z}_2 \rightarrow \mathbb{Z}_2 \amalg \mathbb{Z}_2 \text{ taking } t \text{ to } t_1 t_0 t_1. \end{array}$$

To see whether these generate all functorial operations on $h_{\mathbb{Z}_2}$, consider the general element

(9.11.1) of the n -fold coproduct of copies of \mathbb{Z}_2 . If (9.11.1) has exponent 2, all factors must cancel when we square it, which we see means that we must have $\alpha_0 = \alpha_{h-1}$, $\alpha_1 = \alpha_{h-2}$, etc.. If h is even and positive, this gives, in particular, $\alpha_{(h/2)-1} = \alpha_{h/2}$, which contradicts the final condition of (9.11.1). Hence the only element (9.11.1) of even length having exponent 2 is e . This element induces the constant n -ary operation e , and we see that for each n , this is a derived n -ary operation of the zeroary operation (9.11.2).

On the other hand, for $h = 2k+1$, we see that (9.11.1) will have exponent 2 if and only if it has the form

$$(9.11.4) \quad t_{\alpha_0} \dots t_{\alpha_{k-1}} t_{\alpha_k} t_{\alpha_{k-1}} \dots t_{\alpha_0} \quad \text{with } k \geq 0, \text{ and } \alpha_i \neq \alpha_{i+1} \text{ for } 0 \leq i < k.$$

The n -ary operation that this induces on elements of exponent 2 can clearly be expressed in terms of the operation (9.11.3); it is

$$(9.11.5) \quad (x_0, \dots, x_n) \mapsto (\dots(((x_{\alpha_k})^{x_{\alpha_{k-1}}})^{x_{\alpha_{k-2}}}) \dots)^{x_{\alpha_0}}.$$

So the operations (9.11.2) and (9.11.3) do indeed generate the clone of operations on $h_{\mathbb{Z}_2}$.

How can we find a generating set for the identities these operations satisfy? This is shown in

Exercise 9.11:1. Note that the set of all terms in the two operations (9.11.2) and (9.11.3) includes terms not of the form e or (9.11.5); e.g., e^{x_0} , $x_0^{(x_1^{x_2})}$, $(x_0^{x_1})^{x_1}$. (The last is not of the form (9.11.5) because it fails to satisfy the condition on the α 's inherited from (9.11.4)).

(i) For each of these terms, show how the resulting *derived operation* of $h_{\mathbb{Z}_2}$ can be expressed either as e or in the form (9.11.5), and extract from each such observation an identity satisfied by (9.11.2) and (9.11.3). Do the same with other such terms, until you can show that you have enough identities to reduce every term in the operations (9.11.2) and (9.11.3) either to e or to the form (9.11.5).

(ii) Deduce that all identities of the operations (9.11.2) and (9.11.3) of $h_{\mathbb{Z}_2}$ are consequences of the identities in your list.

Exercise 9.11:2. Let \mathbf{V} denote the variety defined by a zeroary operation e and a binary operation $(-)^{-}$, subject to the identities of our two operations on $h_{\mathbb{Z}_2}$, and let $V: \mathbf{Group} \rightarrow \mathbf{V}$ be the functor represented by \mathbb{Z}_2 with the co-operations defined above. Is every object of \mathbf{V} embeddable in an object of the form $V(G)$ for G a group? Translate your answer into a property of the functor V and its left adjoint (or if you don't get an answer, translate the question into a *question* about these functors).

Exercise 9.11:3. Analyze similarly the structure of $h_{\mathbb{Z}_n}$ for a general positive integer n .

Let me present, next, an interesting problem which, though not obviously related to the concepts of this chapter, turns out, like the question examined above, to be approachable by studying the structure on a functor.

If y is an element of a group G , recall that the map $x \mapsto y^{-1}xy$ is an automorphism of G , and that an automorphism that has this form for some $y \in |G|$ is called an *inner* automorphism of G . Now suppose one is handed a group G and an automorphism $\alpha \in \mathbf{Group}(G, G)$. Is it possible to say whether α is inner, using only its properties within the category \mathbf{Group} , i.e., conditions stable in terms of objects and morphisms, without reference to the ‘‘internal’’ nature of the objects?

Well, observe that if α is an inner automorphism of G , induced as above by an element $y \in |G|$, then given any homomorphism h from G to another group H , there exists an automorphism of H which yields a commuting square

$$\begin{array}{ccc} G & \xrightarrow{h} & H \\ \downarrow \alpha & & \downarrow \\ G & \xrightarrow{h} & H, \end{array}$$

namely, the inner automorphism of H induced by $h(y)$. In fact, this construction associates to every such pair (H, h) an automorphism $\alpha_{(H, h)}$ in a “coherent manner”, in the sense that given two such pairs (H_0, h_0) and (H_1, h_1) , and a morphism $f: H_0 \rightarrow H_1$ such that $h_1 = fh_0$, the automorphisms $\alpha_{(H_0, h_0)}$ of H_0 and $\alpha_{(H_1, h_1)}$ of H_1 form a commuting square with f .

I claim, conversely, that any automorphism α of a group G which can be “extended coherently”, in the above sense, to all groups H with maps of G into them, is inner. The next exercise formalizes this “coherence” property in a general category-theoretic setting, then asks you to prove this characterization of inner automorphisms of groups.

Exercise 9.11:4. Given an object C of a category \mathbf{C} , let $(C \downarrow \mathbf{C})$ denote the category whose objects are pairs (D, d) , $(D \in \text{Ob}(\mathbf{C}), d \in \mathbf{C}(C, D))$ and where morphisms $(D_0, d_0) \rightarrow (D_1, d_1)$ are morphisms $D_0 \rightarrow D_1$ making commuting triangles with d_0 and d_1 . (Cf. Definition 6.8.11.) Let $U_C: (C \downarrow \mathbf{C}) \rightarrow \mathbf{C}$ denote the forgetful functor sending (D, d) to D .

Call an endomorphism α of an object C “functorializable” if there exists an endomorphism a of this forgetful functor U_C which, when applied to the initial object of $(C \downarrow \mathbf{C})$, namely (C, id_C) , yields α .

Show that for $\mathbf{C} = \mathbf{Group}$, an automorphism of an object G is functorializable if and only if it is an inner automorphism. In fact, determine the monoid of endomorphisms of $U_G: (G \downarrow \mathbf{Group}) \rightarrow \mathbf{Group}$ and its image in the monoid of endomorphisms of G . (Suggestion: Consider a universal example of a pair (X, x) where X is an object of $(G \downarrow \mathbf{Group})$ and x an element of the underlying group of X .)

Some related questions you can also look at: How does the above monoid compare with the monoid of endomorphisms of the *identity* functor of $(G \downarrow \mathbf{Group})$? Can you characterize functorializable endomorphisms of objects of other interesting varieties?

On to another topic. The next exercise is unexpectedly hard (unless there is a trick I haven’t found), but is interesting.

Exercise 9.11:5. Let \mathbf{V} and \mathbf{W} be varieties of algebras (finitary if you wish). Show that the category $\mathbf{Rep}(\mathbf{V}, \mathbf{W})$ has an *initial object*.

(I do this in [45]. If you read that, then, of course, any solution you submit has to bring in some improvement or some alternative approach to what is in that paper.)

The next exercise develops some results and examples regarding these initial representable functors.

Exercise 9.11:6. Suppose we classify varieties into three sorts: (a) those with no zeroary operations, (b) those with a unique derived zeroary operation, and (c) those with more than one derived zeroary operation. Applying this classification to both of the varieties \mathbf{V} and \mathbf{W} in the preceding exercise, we get nine cases.

(i) Show that in *most of* these cases, the initial object of $\mathbf{Rep}(\mathbf{V}, \mathbf{W})$ must be trivial, in the weak sense that it takes every object A *either* to the one-element algebra *or* to the empty algebra.

(ii) Determine the initial object of $\mathbf{Rep}(\mathbf{Set}, \mathbf{Semigroup})$.

(iii) Determine the initial object of $\mathbf{Rep}(\mathbf{Set}, \mathbf{Binar})$, where \mathbf{Binar} is the variety of sets with a single (unrestricted) binary operation.

(iv) Interpret the result of Exercise 8.3:11 as describing the initial object of $\mathbf{Rep}(\mathbf{Binar}, \mathbf{Binar})$.

(v) The three preceding examples all belong to the same one of the nine cases referred to in (i). Give an example belonging to a different case, in which $\mathbf{Rep}(\mathbf{V}, \mathbf{W})$ also has nontrivial initial object.

When I first learned about the concept of “coidentities” in coalgebra objects of a category \mathbf{C} , I was a little disappointed that the possible coidentities merely corresponded to the identities of set-based algebras of the same type – I thought it would have been more interesting if this “exotic” version of the concept of algebra led to “exotic” sorts of identities as well. But perhaps there is still hope for something exotic, if the question is posed differently. Recall that in §8.6 we characterized varieties of algebras as those classes of algebras that were closed under three operators \mathbf{H} , \mathbf{S} and \mathbf{P} . I don’t know the answer to the question asked in

Exercise 9.11:7. Define analogs of the operators \mathbf{H} , \mathbf{S} and \mathbf{P} for classes of objects of $\mathbf{Rep}(\mathbf{C}, \Omega\text{-Alg})$. Presumably, for every variety \mathbf{V} of Ω -algebras, $\mathbf{Rep}(\mathbf{C}, \mathbf{V})$ will be closed in $\mathbf{Rep}(\mathbf{C}, \Omega\text{-Alg})$ under your operators; but will these be the only closed classes?

If not, try to characterize the classes closed under your operators (possibly assuming some restrictions on \mathbf{C} and Ω).

9.12. Contravariant representable functors. In §9.2 we defined the concept of an *algebra-object* of a category, but we immediately passed to that of a *coalgebra object* in §9.3, and showed in §9.4 that a covariant functor has a left adjoint if and only if it is represented by such an object. Let us now look at the version of this result for algebra objects, and the contravariant functors these represent. We recall from §7.12 that a contravariant adjunction involves a pair of *mutually right adjoint* or of *mutually left adjoint* functors. Putting “ \mathbf{C}^{op} ” in place of “ \mathbf{C} ” in Theorem 9.4.2, we get

Theorem 9.12.1. *Let \mathbf{C} be a category with small limits, \mathbf{V} a variety of algebras, and $V: \mathbf{C}^{\text{op}} \rightarrow \mathbf{V}$ a functor. Then the following conditions are equivalent:*

- (i) V has a right adjoint $W: \mathbf{V}^{\text{op}} \rightarrow \mathbf{C}$ (so that V and W form a pair of mutually right adjoint contravariant functors).
- (ii) $V: \mathbf{C}^{\text{op}} \rightarrow \mathbf{V}$ is representable, i.e., is isomorphic to $\mathbf{C}(-, R)$ for some \mathbf{V} -algebra object R of \mathbf{C} (Definition 9.2.9).
- (iii) The composite of V with the underlying-set functor $U_{\mathbf{V}}: \mathbf{V} \rightarrow \mathbf{Set}$ is representable, i.e., is isomorphic to $h^{|R|} = \mathbf{C}(-, |R|)$ for some object $|R|$ of \mathbf{C} . \square

Now suppose that in the above situation we take for \mathbf{C} another variety of algebras, \mathbf{W} ; what will a \mathbf{V} -object R of \mathbf{W} look like? Its \mathbf{V} -operations will be \mathbf{W} -algebra homomorphisms $t_R: |R|^{\text{ari}(t)} \rightarrow |R|$; that is, set maps $\|R\|^{\text{ari}(t)} \rightarrow \|R\|$ which respect the \mathbf{W} -operations of $|R|$. Let us write down the condition for an n -ary operation t on a set to “respect” an m -ary operation s :

$$\begin{aligned} & s(t(x_{0,0}, \dots, x_{0,n-1}), \dots, t(x_{m-1,0}, \dots, x_{m-1,n-1})) \\ & = t(s(x_{0,0}, \dots, x_{m-1,0}), \dots, s(x_{0,n-1}, \dots, x_{m-1,n-1})). \end{aligned}$$

The above equation assumes the arities m and n are natural numbers. For operations of arbitrary arities, the condition may be written

$$(9.12.2) \quad s((t(x_{ij})_{j \in \text{ari}(t)})_{i \in \text{ari}(s)}) = t((s(x_{ij})_{i \in \text{ari}(s)})_{j \in \text{ari}(t)}).$$

Note that this condition is symmetric in s and t , and that when s and t are both *unary*, it says that $s(t(x)) = t(s(x))$, i.e., that as elements of the monoid of set maps $\|R\| \rightarrow \|R\|$, s and t commute. Generalizing this term, one calls operations s and t of arbitrary arities which satisfy (9.12.2) *commuting operations*. This condition is equivalent to commutativity of the diagram

$$\begin{array}{ccc} \|R\|^{\text{ari}(s) \times \text{ari}(t)} & \xrightarrow{t^{\text{ari}(s)}} & \|R\|^{\text{ari}(s)} \\ \downarrow s^{\text{ari}(t)} & & \downarrow s \\ \|R\|^{\text{ari}(t)} & \xrightarrow{t} & \|R\| \end{array}$$

where $t^{\text{ari}(s)}$ and $s^{\text{ari}(t)}$ are understood to act in the ‘‘obvious’’ way on $\text{ari}(s) \times \text{ari}(t)$ -tuples of elements of $\|R\|$.

To get some feel for this concept, you might do

- Exercise 9.12:1.** (i) Show that two zeroary operations commute if and only if they are equal. More generally, when will an n -ary operation s commute with a zeroary operation t ?
- (ii) Verify that every zeroary or unary operation on a set commutes with itself.
- (iii) Show that not every binary operation s on a set X commutes with itself. In fact, consider the following four conditions on a binary operation s : (a) s commutes with itself, (b) s satisfies the commutative identity $s(x, y) = s(y, x)$, (c) s satisfies the associative identity $s(s(x, y), z) = s(x, s(y, z))$, and (d) there exists a neutral element $e \in X$ for s , i.e., an element satisfying the identities $s(x, e) = x = s(e, x)$. Determine which of the 16 possible combinations of truth values for these conditions can be realized.

Summarize your results as one or more implications which hold among these conditions, and such that any combination of truth-values consistent with those implications can be realized.

We see that if \mathbf{V} is a variety of Ω -algebras and \mathbf{W} a variety of Ω' -algebras, then a \mathbf{V} -algebra object of \mathbf{W} is equivalent to a set-based algebra $R = (|R|, (s_R)_{s \in |\Omega' \sqcup \Omega|})$, where the operations indexed by $|\Omega'|$, respectively, $|\Omega|$, are of the arities specified in Ω' , respectively, Ω , and satisfy the identities of \mathbf{W} , respectively \mathbf{V} , and where, moreover, for every $s \in |\Omega'|$ and $t \in |\Omega|$, the commutativity identity (9.12.2) is satisfied. Since all these conditions are identities, the category of such objects forms a variety!

Given such an object R , and an ordinary object A of \mathbf{W} , we see that the operations of the \mathbf{V} -algebra $\mathbf{W}(A, R)$ are given by ‘‘pointwise’’ application of the \mathbf{V} -operations of R to \mathbf{W} -homomorphisms $A \rightarrow R$. In general, if A and B are objects of a variety \mathbf{W} and one combines a family of algebra homomorphisms $f_\alpha : A \rightarrow B$ ($\alpha \in \beta$) by pointwise application of a β -ary operation t on the set $|B|$, the result is not a homomorphism of \mathbf{W} -algebras. What makes this true here is the fact that t is an operation on R as an object of the category \mathbf{W} , i.e., that it commutes with all the \mathbf{W} -operations.

Since the functor $\mathbf{W}(-, R) : \mathbf{W}^{\text{op}} \rightarrow \mathbf{V}$ belongs to a *mutually right adjoint* pair, its adjoint will also satisfy condition (i) of Theorem 9.12.1, and hence the other two equivalent conditions; that is, this adjoint will *also* be a representable contravariant functor, but going the other way, $\mathbf{V}^{\text{op}} \rightarrow \mathbf{W}$. As the next exercise shows, the representing object for this functor is gotten by superficially modifying the representing object R for the original functor.

Exercise 9.12:2. Let $V: \mathbf{W}^{\text{op}} \rightarrow \mathbf{V}$ be a representable contravariant functor, whose representing \mathbf{V} -algebra object R is, in the above formulation $(|R|, (s_R)_{s \in |\Omega' \sqcup \Omega|})$. Show that the right adjoint to V is the functor $\mathbf{V}(-, R')$, where R' has the same underlying set as R , and the same operations, but with the roles of the \mathbf{W} -operations and the \mathbf{V} -operations as “primary” and “secondary” interchanged, so that it becomes a \mathbf{W} -algebra object of \mathbf{V} .

A basic contrast between covariant and contravariant representable functors on a variety \mathbf{W} is that the former, as we saw in §9.3, define their operations *using* derived operations of \mathbf{W} , while the objects representing the latter have operations that must *commute* with those of \mathbf{W} . A consequence is that, generally speaking, the “richer” the structure of \mathbf{W} , the richer is the class of covariant representable functors on \mathbf{W} , and the scarcer are the contravariant representable functors. Thus, the case in which it is easiest to get contravariant representable functors is when \mathbf{V} is the variety with the least family of operations, namely **Set**.

A \mathbf{V} -algebra object of **Set** is in fact just an ordinary \mathbf{V} -algebra. Let us take the smallest nontrivial object in **Set**, and find the richest algebra structure we can put on it, and the functor this represents.

Exercise 9.12:3. (i) Show that the clone of all finitary operations on the object $2 = \{0, 1\}$ of **Set** can be described as the clone of derived operations of the *ring* \mathbb{Z}_2 , and that this is isomorphic to the clone of operations of the variety **Bool**¹.

(ii) Describe the contravariant adjunction between **Set** and **Bool**¹ determined by this **Bool**¹-structure on 2 .

As an interesting sideline,

(iii) Regarding **Bool**¹ as the variety generated by the 2-element Boolean ring, obtain a cardinality-bound for the free Boolean ring on n generators by considerations analogous to those applied to the free group on 3 generators in $\mathbf{Var}(S_3)$ in the discussion leading up to Exercise 2.3:2. If you did that exercise and Exercise 3.14:1, compare these two cases with respect to how close the resulting bounds are to the actual cardinalities of these free algebras.

More generally,

Exercise 9.12:4. For n any integer > 1 , let $\mathbf{X}^{[n]}$ denote the clone of all finitary operations on the set $n = \{0, \dots, n-1\}$.

(i) Show that for p a prime, $\mathbf{X}^{[p]}$ can be described as the clone of derived operations of the ring \mathbb{Z}_p . Show moreover that the variety $\mathbf{X}^{[p]\text{-Alg}}$, regarded as a subvariety of **CommRing**¹, is equivalent to **Bool**¹ by the “Boolean ring of idempotent elements” functor (Exercise 3.14:3). Describe the functor going the other way.

(ii) Show that if n is not a prime, then $\mathbf{X}^{[n]\text{-Alg}}$ does not coincide with the clone of derived operations of the ring \mathbb{Z}_n .

(iii) For n not a prime, is it still true that $\mathbf{X}^{[n]\text{-Alg}}$ is equivalent to **Bool**¹?

Let us look next at a contravariant representable algebra-valued functor on a category \mathbf{C} other than a variety of algebras, which nonetheless has properties similar to those of functors $\mathbf{V}^{\text{op}} \rightarrow \mathbf{W}$ as discussed above.

Exercise 9.12:5. In the category **POSet** of partially ordered sets and isotone maps, let 2 denote the object with underlying set $\{0, 1\}$, ordered so that $0 < 1$. (It is natural to speak of this as a “structure of partially ordered set” on 2 ; but beware confusion with Lawvere’s technical sense of “structure”, i.e., the operations which an object admits, which are the subject of (i) below.)

(i) Show that the finitary structure on this object of **POSet**, i.e., the clone of all operations $2^n \rightarrow 2$ that are morphisms of **POSet**, is a structure of *distributive lattice* (Exercise 5.1:15) with *least* element 0 and *greatest* element 1 , regarded as zeroary operations. Describe the

resulting functor $\mathbf{POSet}^{\text{op}} \rightarrow \mathbf{DistLat}^{0,1}$. (You will need to know the form that products take in \mathbf{POSet} ; for this see Definition 4.1.4.)

- (ii) Verify that \mathbf{POSet} has small limits, so that Theorem 9.12.1 is applicable to this functor.
- (iii) Show that the adjoint to this functor, a functor $(\mathbf{DistLat}^{0,1})^{\text{op}} \rightarrow \mathbf{POSet}$, can be characterized as taking every object of $\mathbf{DistLat}^{0,1}$ to the set of its morphisms into the object 2 , with the partial ordering on 2 being used to get a partial ordering on the set of morphisms. (Cf. Exercise 6.6:5.)
- (iv) Suppose instead that we consider $2 = \{0,1\}$ as an object of $\mathbf{POSet}^{0,1}$, the category whose objects are partially ordered sets with least and greatest elements, and whose morphisms are the isotone maps that respect those elements. Show that the structure on 2 in this category leads to a contravariant right adjunction with the variety $\mathbf{DistLat}$.

What if you start with \mathbf{POSet}^0 or \mathbf{POSet}^1 ?

It is clear from Lemma 9.10.2 that any contravariant equivalence $i: \mathbf{C}^{\text{op}} \rightarrow \mathbf{V}$, where \mathbf{C} is a category and \mathbf{V} a variety of algebras, will be representable. In such a situation, can \mathbf{C} also be a variety of algebras? This is addressed in the next exercise.

Exercise 9.12:6. Let us call a variety “nontrivial” if it does not satisfy the identity $x = y$.

- (i) Show that there can exist no contravariant equivalences between nontrivial varieties.
(Suggestion: find a condition on categories which is invariant under equivalence of categories, and is satisfied by all nontrivial varieties, but is not satisfied by the *opposite* of any nontrivial variety. Essentially, any condition on categories that does not refer to how many isomorphic copies an object has will be invariant under equivalence. What is hard is finding one that distinguishes between varieties and their opposites. I know some ways to do this, but they are not obvious. Perhaps you can find a more natural one. If you wish, take “variety” to mean “finitary variety”.)
- (ii) Does your criterion also show that $(\mathbf{POSet})^{\text{op}}$ and $(\mathbf{POSet}^{0,1})^{\text{op}}$ are not equivalent to varieties? If not, can you nonetheless prove this?

However, some of the contravariant representable functors considered above come surprisingly close to being equivalences; namely, when restricted to the *finitely generated* objects of one category they yield finitely generated objects of the other, and they give equivalences between these subcategories of finitely generated objects. In the case of duality of vector spaces, this is a category-theoretic translation of some well-known facts of linear algebra. In the cases of Boolean rings (Exercise 9.12:3) and of distributive lattices (Exercise 9.12:5), the results in question are likewise translations of classical fundamental results about these two kinds of object ([3, §III.3]; cf. also Exercise 6.9:17 above). For the variety \mathbf{Ab} the functor $\mathbf{Ab}(-, \mathbb{Q}/\mathbb{Z})$ is a self-duality on the category of finite (though not on the category of finitely generated) abelian groups (see [24, §4.6], noting the comment after [*ibid.* Theorem 6.2]).

It turns out, moreover, that the above dualities on finite objects can be extended to equivalences between *all* objects of one category and certain *topologized* objects of the other. The reader interested in learning about a large class of such results might look at [36], and at [83], which generalizes the results of the former paper and puts them in category-theoretic language. The result on $\mathbf{Ab}(-, \mathbb{Q}/\mathbb{Z})$ does not fall within the scope of those papers, but it, too, has a generalization to topological abelian groups, the theory of *Pontryagin duality* of locally compact abelian groups, via morphisms into the topological group \mathbb{R}/\mathbb{Z} [111]. The topological approach to duality of not necessarily finite-dimensional vector spaces is implicit in Exercises 5.5:5 and 7.5:18. A book on dualities, which I have not yet looked through, is [58].

- Exercise 9.12:7.** (i) Show from Exercise 3.14:5 that our functors connecting \mathbf{Bool}^1 and \mathbf{Set} do indeed induce a contravariant equivalence between the subcategories of finite objects.
 (ii) Deduce that if \mathbf{V} is any variety of finitary algebras, and A a finite object of \mathbf{V} , then there exists a \mathbf{V} -coalgebra object R of \mathbf{Bool}^1 such that $\mathbf{Bool}^1(R, 2) \cong A$.

If you or the class succeeded in characterizing derived operations of the “majority vote function” M_3 on $\{0,1\}$ in Exercise 1.7:1, you can now try:

- Exercise 9.12:8.** (i) Can you find some structure (in the nontechnical sense, i.e., not necessarily given by operations!) on $\{0,1\}$, such that the clone of operations generated by the majority vote function M_3 is precisely the clone of finitary operations respecting that structure?
 (ii) Does one in fact have a duality result, to the effect that the set $\{0,1\}$, with this structure on the one hand, and with the operation M_3 on the other, induces an adjunction, which, when restricted to finite objects, gives a contravariant equivalence between finite algebras in the variety generated by $(\{0,1\}, M_3)$, and finite objects of an appropriate category?

I have not thought hard about the following question:

- Exercise 9.12:9.** Suppose \mathbf{V} and \mathbf{W} are varieties, and we have a contravariant equivalence between their subcategories of finite (finitely generated? finitely presented?) objects. Will this necessarily be the restriction of a pair of mutually right adjoint representable functors between all of \mathbf{V} and all of \mathbf{W} ?

What can we say about *composites* involving contravariant representable functors? We know that for adjoint pairs of *covariant* functors

$$\mathbf{C} \begin{array}{c} \xrightarrow{U} \\ \xleftarrow{F} \end{array} \mathbf{D} \begin{array}{c} \xrightarrow{V} \\ \xleftarrow{G} \end{array} \mathbf{E},$$

the composites $\mathbf{C} \begin{array}{c} \xrightarrow{VU} \\ \xleftarrow{FG} \end{array} \mathbf{E}$ are also adjoint; so let us look at the results we get on replacing some subset of the three categories \mathbf{C} , \mathbf{D} , \mathbf{E} in this result by their opposites. This will give 8 statements, saying that composites of certain combinations of covariant adjoint pairs, contravariant right adjoint pairs, and contravariant left adjoint pairs are again adjoint pairs of one sort or another.

These statements will break into pairs of statements which have the same translations after some relabeling, because Theorem 7.3.9 itself is invariant under replacing all three categories by their opposites and interchanging the roles of \mathbf{C} and \mathbf{E} . Of the resulting four statements, one is, of course, the original Theorem 7.3.9. Two of the others involve contravariant *left* adjunctions, of which, as I have mentioned, there are no interesting cases among varieties of algebras [42]. I state the one remaining case as the next corollary. In that corollary, for a functor between arbitrary categories, $A: \mathbf{C} \rightarrow \mathbf{D}$, the “same” functor regarded as going from \mathbf{C}^{op} to \mathbf{D}^{op} is written A^{op} (though for most purposes, it is safe to write this A).

Corollary 9.12.3 (to Theorem 7.3.9). *Suppose*

$$\begin{array}{ccc} \mathbf{C}^{\text{op}} & \xrightarrow{V} & \mathbf{D} \\ \mathbf{C} & \xleftarrow{V'} & \mathbf{D}^{\text{op}} \end{array}$$

is a pair of mutually right adjoint contravariant functors, and

$$\mathbf{D} \begin{array}{c} \xrightarrow{U} \\ \xleftarrow{F} \end{array} \mathbf{E}$$

a pair of covariant adjoint functors (U the right adjoint and F the left adjoint). Then the

composite functors UV and $V'F^{\text{op}}$ (in less discriminating notation, $V'F$):

$$\begin{array}{ccccc} \mathbf{C}^{\text{op}} & \xrightarrow{V} & \mathbf{D} & \xrightarrow{U} & \mathbf{E} \\ \mathbf{C} & \xleftarrow{V'} & \mathbf{D}^{\text{op}} & \xleftarrow{F^{\text{op}}} & \mathbf{E}^{\text{op}} \end{array}$$

are also mutually right adjoint contravariant functors.

In particular, the class of contravariant functors admitting right adjoints is closed under postcomposition with right adjoint covariant functors, and under precomposition with left adjoint covariant functors. \square

Exercise 9.12:10. (i) Derive the above result from Theorem 7.3.9, and also derive the two other statements referred to in the paragraph before the corollary which involve contravariant left adjunctions.

(ii) Give a (nontrivial) example of Corollary 9.12.3, verifying directly the adjointness.

Exercise 9.12:11. Suppose in the context of the above corollary that \mathbf{C} and \mathbf{E} are both varieties of algebras. Thus the pair of mutually right adjoint functors UV and $V'F^{\text{op}}$ are induced by some object with commuting \mathbf{C} - and \mathbf{E} -algebra structures. Describe this object and its \mathbf{C} - and \mathbf{E} -algebra structures in terms of the representing objects R and S for the given functors $V: \mathbf{C}^{\text{op}} \rightarrow \mathbf{D}$ and $U: \mathbf{D} \rightarrow \mathbf{E}$.

Corollary 9.12.3 does *not* say anything about a composite of two contravariant representable functors. This will be a covariant functor, but as the first part of the next exercise shows, it need not have an adjoint on either side.

Exercise 9.12:12. (i) Let K be a field, and $V: (K\text{-Mod})^{\text{op}} \rightarrow K\text{-Mod}$ the contravariant representable functor taking each K -vector space to its dual. Show that the composite of V with itself, VV , or more accurately, VV^{op} , a covariant functor $K\text{-Mod} \rightarrow K\text{-Mod}$, has no left or right adjoint.

(ii) Show by examples that the class of representable contravariant functors between varieties is closed neither under precomposition with right adjoint covariant functors nor under postcomposition with left adjoint covariant functors.

The “double dual” functor of part (i) above *does* belong to a class of functors which have interesting properties, namely, composites of functors (covariant or contravariant) with their own adjoints. I hope to develop some of these properties in the next chapter, when I have time to write it.

I have mentioned the principle that the richer the structure of a variety of algebras, the more covariant representable functors it admits, and the fewer contravariant representable functors, and we then looked at contravariant representable functors on the variety with the least algebraic structure. In the opposite direction, rings have a particularly rich structure; thus, as the next exercise shows, they are quite poor when it comes to contravariant representable functors.

Exercise 9.12:13. Let R be a nonzero ring (commutative if you wish).

(i) Show that if R has no zero divisors, then any finitary operation $R^n \rightarrow R$ can be expressed as a composite $ap_{i,n}$ where a is an endomorphism of R , and $p_{i,n}$ is the i th projection map on R^n . Deduce that any clone of finitary operations on R as an object of \mathbf{Ring}^1 or of $\mathbf{CommRing}^1$ is generated by unary operations.

(ii) Can you generalize these observations to a wider class of rings than those without zero divisors?

(iii) Choose a simple example of a ring R with zero divisors for which the conclusion of (i)

fails, and see whether you can describe the clone of operations on that ring.

9.13. More on commuting operations. We have seen that for varieties \mathbf{V} and \mathbf{W} , the \mathbf{V} -algebra objects of \mathbf{W} correspond to sets given with two families of operations which commute with one another in the sense of (9.12.2). Let us look further at this concept of commuting operations.

Lemma 9.13.1. *If s is an operation on a set A , then the set of operations on A which commute with s forms a clone.*

Idea of Proof. If the map $s: A^{\text{ari}(s)} \rightarrow A$ is a homomorphism for all members of some set T of operations on A , it will clearly be a homomorphism for all derived operations of that family. \square

Exercise 9.13.1. Give a detailed proof of the above lemma. (Remember that proving a set of operations to be a clone includes proving that it contains the projection maps.)

Definition 9.13.2. *If s is an operation on a set A , then the clone of operations on A which commute with s will be called the centralizer of s . If S is a set of operations on A , the intersection of the centralizers of these operations will be called the centralizer of S .*

If C is a clone of operations on A and S a set of operations on A (which may or may not be contained in C), then the intersection of C with the centralizer of S will be called the centralizer of S in C . The centralizer of C in C will be called the center of C . A clone which is its own center will be called commutative.

Let us fix a notation for a construction we defined in the preceding section.

Definition 9.13.3. *If Ω and Ω' are types, then $\Omega \sqcup \Omega'$ will denote the type whose set of operation-symbols is $|\Omega| \sqcup |\Omega'|$, and where the arity function on this set is induced in the obvious way by the arity functions of Ω and Ω' .*

If \mathbf{V} and \mathbf{W} are varieties of algebras, of types Ω and Ω' respectively, then the variety of algebras of type $\Omega' \sqcup \Omega$ such that the operations from Ω satisfy the identities of \mathbf{V} , the operations from Ω' satisfy identities of \mathbf{W} , and all \mathbf{V} -operations commute with all \mathbf{W} -operations, will be denoted $\mathbf{V} \circ \mathbf{W}$.

Note that in the above definition, \mathbf{V} and \mathbf{W} are specified as *varieties*, i.e., in terms of given primitive operations. However, even if we are not interested in distinguishing “primitive” from “derived” operations, e.g., if we are interested in varieties as categories of representations of given clonal categories, the above construction “ \circ ” also induces a construction on these, since by Lemma 9.13.1, the property that two sets of primitive operations commute is equivalent to the property that their sets of derived operations commute. Likewise, if we consider varieties merely to be a certain class of *concrete categories*, “ \circ ” yields a construction on these, since the “Structure” functor of §8.10 allows us to recover their clones of operations from these concrete categories, and so apply the preceding observation. Finally, if we are interested in varieties only up to equivalence as categories, without reference to concretization (e.g., if we are not interested in distinguishing the varieties $K\text{-Mod}$ and $M_n(K)\text{-Mod}$), then $\mathbf{V} \circ \mathbf{W}$ is also determined up to equivalence on these, namely, as the category of contravariant right adjunctions between \mathbf{V} and \mathbf{W} .

(Freyd introduces essentially the concept we have called $\mathbf{V} \circ \mathbf{W}$ in [10, pp.93-95], but rather

than naming the resulting variety, he names its clonal theory $T_1 \otimes T_2$, where T_1 and T_2 are the clonal theories of the given varieties. We have made the opposite choice so as to minimize the dependence of this chapter on the view of a variety as the category of representations of a clonal theory.)

In the case of *covariant* representable functors, we saw in §9.10 that certain *differences* between two varieties \mathbf{V} and \mathbf{W} regarding the number of derived zeroary operations led to restrictions on representable functors between these varieties. For *contravariant* functors, on the other hand, it is when both varieties *have* such operations that one gets a restriction:

Lemma 9.13.4 ([10, p.94]). *Suppose \mathbf{V} and \mathbf{W} are varieties of algebras, each having at least one zeroary operation. Then $\mathbf{V} \circ \mathbf{W}$ satisfies identities saying that all derived zeroary operations of \mathbf{V} and all derived zeroary operations of \mathbf{W} fall together. The resulting derived zeroary operation of $\mathbf{V} \circ \mathbf{W}$ defines a one-element subalgebra of every $\mathbf{V} \circ \mathbf{W}$ -object.*

Proof. The fact that each derived zeroary operation coming from \mathbf{V} commutes with each derived zeroary operation coming from \mathbf{W} means that each of the former is equal to each of the latter (Exercise 9.12:1(i)). Hence, as both families are nonempty, all of these derived zeroary operations are equal. Since zeroary operations from \mathbf{V} commute with arbitrary operations from \mathbf{W} and vice versa, the resulting zeroary operation of $\mathbf{V} \circ \mathbf{W}$ is central. It is easy to verify that this means that it defines a one-element subalgebra of every algebra, equivalently, is the unique *derived* zeroary operation of $\mathbf{V} \circ \mathbf{W}$. \square

Exercise 9.13:2. Deduce from the above lemma that if \mathbf{V} is a variety having at least one zeroary operation, then the variety $\mathbf{V} \circ \mathbf{Ring}^1$ is trivial; equivalently, that there is no nontrivial contravariant representable functor $\mathbf{V}^{op} \rightarrow \mathbf{Ring}^1$ or $(\mathbf{Ring}^1)^{op} \rightarrow \mathbf{V}$. (So, for instance, there is no nontrivial contravariant representable functor $(\mathbf{Ring}^1)^{op} \rightarrow \mathbf{Ring}^1$.)

The next result shows a similar phenomenon for binary operations with neutral element.

Lemma 9.13.5 ([10, p.94]). *Suppose \mathbf{V} and \mathbf{W} are varieties of algebras, each having at least one derived binary operation with a neutral zeroary operation. Then in $\mathbf{V} \circ \mathbf{W}$, the operations induced by all such binary operations of \mathbf{V} and all such binary operations of \mathbf{W} fall together, and give the unique binary operation with neutral element in this clone. The resulting binary operation and neutral element constitute a structure of abelian monoid, which is central in the clone of operations of $\mathbf{V} \circ \mathbf{W}$.*

Proof. We shall show that if in any variety a binary operation $*$ with a neutral element and a binary operation \circ with a neutral element commute, and their neutral elements likewise commute, then $*$ = \circ , and their common value satisfies the commutative and associative identities. The remaining assertions follow as in the proof of the preceding lemma.

The neutral elements of $*$ and \circ , being commuting zeroary operations, are equal; let us write e for their common value. We now write down several cases of the commutativity of $*$ with \circ . The equation $(x*e)^\circ(e*y) = (x^\circ e)*(e^\circ y)$ reduces to $x^\circ y = x*y$, proving equality of the two operations. On the other hand, $(e*x)^\circ(y*e) = (e^\circ y)*(x^\circ e)$ reduces to $x^\circ y = y*x$, so the common value of $*$ and \circ is abelian. Finally, $(x*y)^\circ(e*z) = (x^\circ e)*(y^\circ z)$ yields associativity. \square

The above result fails without the assumption that *both* binary operations have a neutral

element. E.g., the variety **Set** has the binary “derived operation” $p_{2,0}$ (projection of an ordered pair on its first component); but it is easy to see that for every variety \mathbf{V} , one has $\mathbf{V} \circ \mathbf{Set} \cong \mathbf{V}$; so a binary operation of \mathbf{V} with neutral element is not forced in $\mathbf{V} \circ \mathbf{Set}$ to become associative, or become commutative, or to fall together with $p_{2,0}$.

Recall that we denote the variety of algebras with a single binary operation with neutral element by \mathbf{Binar}^e .

Corollary 9.13.6. *If each of \mathbf{V} and \mathbf{W} is one of \mathbf{Binar}^e , \mathbf{Monoid} , $\mathbf{AbBinar}^e$ or $\mathbf{AbMonoid}$, then $\mathbf{V} \circ \mathbf{W} \cong \mathbf{AbMonoid}$.*

Proof. Applying the preceding lemma, we see that the given zeroary and binary operations of \mathbf{V} and \mathbf{W} fall together in $\mathbf{V} \circ \mathbf{W}$ to give a single zeroary and a single binary operation that generate the clone of operations of $\mathbf{V} \circ \mathbf{W}$ and satisfy the identities of $\mathbf{AbMonoid}$. To show that $\mathbf{V} \circ \mathbf{W}$ satisfies no other identities, it suffices to note that the multiplication and neutral element of $\mathbf{AbMonoid}$ satisfy all the identities of \mathbf{V} and of \mathbf{W} (clear in each case), and commute with themselves and one another (a quick calculation). \square

The above corollary shows that the representing object for any *contravariant representable functor* between any two of the varieties listed is essentially an abelian monoid.

The next result concerns the case where our abelian monoid structures turn out to give abelian group structures. If a binary derived operation $*$ of a variety has a neutral element e , then a *left inverse operation*, respectively *right inverse operation* for $*$ will mean a unary operation ι satisfying the identity $\iota(x)*x = e$, respectively $x*\iota(x) = e$.

Theorem 9.13.7 (cf. [10, p.95]). *Suppose \mathbf{V} and \mathbf{W} are varieties of algebras, each having at least one binary operation with a neutral element, and such that at least one such operation of \mathbf{V} or of \mathbf{W} has a right or left inverse operation ι . Then in $\mathbf{V} \circ \mathbf{W}$, ι becomes a 2-sided inverse to the unique $\mathbf{AbMonoid}$ operation of this variety, making this an \mathbf{Ab} structure, again central in the clone of operations.*

Moreover, any clone of finitary operations admitting a homomorphism of the clone of operations of \mathbf{Ab} into its center is, up to isomorphism, the clone of operations of a variety $K\text{-Mod}$, where K is the set of unary operations of the clone, made a ring in a natural way. \square

Exercise 9.13.3. Prove the above theorem, with the help of previous results.

Where the above result characterizes clones with a central image of \mathbf{Ab} , Freyd [10, p.95] gives the analogous characterization of clones with a central image of $\mathbf{AbMonoid}$, with “half-ring” in place of ring. (The term “half-ring” is not standard. He presumably means an abelian monoid given with a bilinear multiplication having a neutral element 1; the more common term would be “semiring with 0 and 1”. A module over such a semiring K means an abelian monoid R with a 0- and 1-respecting homomorphism of K into its semiring of endomorphisms.)

Exercise 9.13.4. (i) Deduce from Theorem 9.13.7 that $\mathbf{Group} \circ \mathbf{Group} \cong \mathbf{Ab}$. Translate this result into a description of all representable functors $\mathbf{Group}^{\text{op}} \rightarrow \mathbf{Group}$.

(ii) Your proof of (i) should also show that $\mathbf{Ab} \circ \mathbf{Ab} \cong \mathbf{Ab}$. Thus, every abelian group yields a contravariant right adjunction between \mathbf{Ab} and \mathbf{Ab} . Describe the functors involved, and express the universal property of the adjunction as a certain bijection of hom-sets.

- Exercise 9.13:5.** (i) If K, L are rings, describe $(K\text{-Mod}) \circ (L\text{-Mod})$, and determine the general form of a representable functor $(K\text{-Mod})^{\text{op}} \rightarrow L\text{-Mod}$.
- (ii) Bring the above result into conformity with (9.8.19) by turning it into a characterization of representable functors $(K\text{-Mod})^{\text{op}} \rightarrow \mathbf{Mod}\text{-}L$. Write the associated contravariant right adjunctions as functorial isomorphisms of hom-sets.
- (iii) If K is any ring, the natural (K, K) -bimodule structure of $|K|$ induces, via the result of (ii), a functor $(K\text{-Mod})^{\text{op}} \rightarrow \mathbf{Mod}\text{-}K$. Describe this functor, and show that in the case where K is a field, it is ordinary “duality of vector spaces”.
- (iv) Given any pair of contravariant mutually right adjoint functors among categories, $U: \mathbf{C}^{\text{op}} \rightarrow \mathbf{D}$, $V: \mathbf{D}^{\text{op}} \rightarrow \mathbf{C}$, one has universal maps $\text{Id}_{\mathbf{C}} \rightarrow VU$, $\text{Id}_{\mathbf{D}} \rightarrow UV$. Determine these in case (iii) above.

Here is an important way of getting sets with two mutually commuting algebra structures.

Lemma 9.13.8. *Let \mathbf{V} and \mathbf{W} be varieties of algebras in which all operations have arities less than some regular cardinal γ , let \mathbf{C} be any category having $< \gamma$ -fold products and $< \gamma$ -fold coproducts, and let R and S be a \mathbf{V} -coalgebra object and a \mathbf{W} -algebra object of \mathbf{C} respectively. Then $\mathbf{C}(|R|, |S|)$ has a natural structure of $\mathbf{V} \circ \mathbf{W}$ -algebra (which we may denote $\mathbf{C}(R, S)$). \square*

- Exercise 9.13:6.** (i) Prove the above lemma.
- (ii) If you are familiar with basic algebraic topology, deduce from that lemma and Theorem 9.13.7 and Exercise 9.3:1(ii) that the fundamental group of any topological group is abelian.
- (You will need to verify that a topological group induces a group object of $\mathbf{HtpTop}^{(\text{pt})}$. The key fact to use is that the forgetful functor $\mathbf{Top}^{\text{pt}} \rightarrow \mathbf{HtpTop}^{(\text{pt})}$ respects products.)

In fact, the method of part (ii) above shows that all \mathbf{Binar}^e -objects of $\mathbf{HtpTop}^{(\text{pt})}$ (called “H-spaces” by topologists) have abelian fundamental group. For a brute force proof see [77, Proposition II.11.4, p.81].

Exercise 9.13:7. Describe $\mathbf{Heap} \circ \mathbf{Heap}$. (Hint: If A is a nonempty object of $\mathbf{Heap} \circ \mathbf{Heap}$, show that any choice of a zeroary operation allows one to regard A as an object of $\mathbf{Group} \circ \mathbf{Group}$.)

If possible, generalize your result; i.e., show that conditions weaker than the heap identities are enough to force two commuting ternary operations on a set to coincide, and to satisfy the identities you established for $\mathbf{Heap} \circ \mathbf{Heap}$.

Exercise 9.13:8. Recall that $\mathbf{Semilattice}$ denotes the variety of sets with a single idempotent commutative associative binary operation.

- (i) Show that in $\mathbf{Semilattice} \circ \mathbf{Semilattice}$, the two binary operations fall together.
- (ii) Deduce that $\mathbf{Semilattice} \circ \mathbf{Lattice}$ and $\mathbf{Lattice} \circ \mathbf{Lattice}$ are trivial.
- (iii) Show that $\mathbf{Semilattice} \circ \mathbf{AbMonoid} \cong \mathbf{Semilattice}^0$, the variety of semilattices with neutral element (which we are writing as a least element, arbitrarily interpreting the semilattice operation as “join”).
- (iv) Again, can you get similar results using a smaller set of identities than the full identities of $\mathbf{Semilattice}$ and/or $\mathbf{AbMonoid}$?

In this section we have seen several parallel results; let us put in abstract form what they involve.

Exercise 9.13:9. Let **CommClone** denote the full subcategory of **Clone** consisting of all *commutative* clonal categories (Definition 9.13.2). Show that for any variety \mathbf{V} , the following conditions are equivalent:

- (i) The two underlying-set-preserving functors $\mathbf{V} \circ \mathbf{V} \rightarrow \mathbf{V}$, which act by forgetting the one or the other of this family of \mathbf{V} -operations, are equivalences.
- (ii) The clone of operations of \mathbf{V} is commutative, and is an epimorph of the initial object in **CommClone** (i.e., the morphism from the initial object to that object is an epimorphism).

In connection with the last point, we note

Exercise 9.13:10. Show that the initial objects of **CommClone** and **Clone** are the same, but that this initial object has no proper epimorphs in **Clone** other than the theory of the trivial variety, but does have nontrivial proper epimorphs in **CommClone**.

(This contrasts with the result proved for rings in Exercise 6.7:9(ii).)

Let us call a variety “ \circ -idempotent” if it satisfies the equivalent conditions of Exercise 9.13:9. It would be interesting to see whether one can determine all such varieties. The epimorphs of the clone of operations of **Ab** in **CommClone** are the clones of operations of the varieties **K-Mod** for all epimorphs K of \mathbb{Z} in **CommRing**¹ (cf. Exercise 6.7:8(i)). For a nice classification of these rings K , of which there are uncountably many, see [57]. More generally, if K is a semiring with 0 and 1 (cf. paragraph following Exercise 9.13:3) which is an epimorph of the semiring \mathbb{N} of natural numbers in the category of such semirings, then the clonal theory of the variety of K -modules is an epimorph of the clonal theory of **AbMonoid**. This class of clonal theories includes those arising from epimorphs of \mathbb{Z} , since \mathbb{Z} is an epimorph of \mathbb{N} in the semiring category.

In most of the results in this section that yielded \circ -idempotent varieties \mathbf{V} , we also found larger classes of varieties, necessarily noncommutative, whose \circ -products with themselves and each other gave \mathbf{V} . I don’t know what is going on here; the phenomenon is described in

Exercise 9.13:11. If \mathbf{V} is a variety of algebras, let \mathbf{V}^{ab} denote the subvariety obtained by imposing on \mathbf{V} the identities making all operations of \mathbf{V} commute, and $A(\mathbf{V}): \mathbf{V}^{\text{ab}} \rightarrow \mathbf{V}$ the inclusion functor. For each positive integer n , let $\mathbf{V}^{\circ n}$ denote the variety $\mathbf{V} \circ \dots \circ \mathbf{V}$ with n “ \mathbf{V} ”s, and $Q(\mathbf{V}, n, i)$ ($i = 0, \dots, n-1$) be the natural n -tuple of forgetful functors $\mathbf{V}^{\circ n} \rightarrow \mathbf{V}$.

Show that for any variety \mathbf{V} and integer $n > 1$ the following conditions are equivalent:

- (a) $Q(\mathbf{V}, n, 0) = Q(\mathbf{V}, n, 1)$.
- (b) \mathbf{V}^{ab} and $\mathbf{V}^{\circ n}$ are isomorphic, by a functor making a commuting triangle with the functors $A(\mathbf{V})$ and $Q(\mathbf{V}, n, 0)$.

Show that when these conditions hold, \mathbf{V}^{ab} is \circ -idempotent.

A question I also don’t know the answer to is

Exercise 9.13:12. (i) If \mathbf{V} is a variety such that $\mathbf{V}^{\circ 3}$ has commutative clone of operations, must the clone of operations of $\mathbf{V}^{\circ 2}$ also be commutative?

On an easier note, recall from Exercise 6.9:6 that the *monoid of endomorphisms* of the identity functor of any category was commutative. This generalizes to

Lemma 9.13.9. *If \mathbf{C} is a category with finite products, then the clone of operations of the identity functor of \mathbf{C} is commutative. \square*

Exercise 9.13:13. Prove the above lemma, and characterize the clone in question in the case where \mathbf{C} is a variety.

9.14. Some further reading on representable functors, and on General Algebra. Covariant representable functors among particular varieties of algebras are studied extensively in [2]. Indeed, §§9.1-9.5 above were adapted from the introductory sections of [2], and §§9.6-9.7 from a couple of later sections. Most of [2] deals with representable functors on varieties of associative and commutative rings; for the former case, the representable functors to many other varieties are precisely determined. Thus, [2] may be considered a natural sequel to this chapter. Many open questions are also noted there. (The notation, language, and viewpoint of [2] are close to those of these notes. One difference is that where I here use the word “monoid”, in that work my coauthor and I wrote “semigroup with neutral element”, and called the variety of those objects **Semigp**^e. Also, the references in that work to these notes refer to an earlier version; to find the corrections, you can go to my web page <http://math.berkeley.edu/~gbergman>, click on “publications”, find the title of [2] near the top, and click on “errata & updates”.)

I sketched some of the material I hope eventually to include in Chapter 10 of these notes many years ago in [1, §§4-7]. I can give you a reprint of that if you are interested. Some further ideas that I might include in that chapter are found in [2, §§63-64].

Several other texts in General Algebra, most of which include some major topics not covered in these notes, were listed in the first paragraph of §0.6. A classical topic I have not touched on which is particularly surprising and useful is that of ultraproducts and ultrapowers, and can be found in most General Algebra texts.

An area of rapid research in recent decades has been the study of lattices of congruences of algebras, and their relationship to the general properties of varieties of algebras [8], [13].