

A 3-Stranded Quantum Algorithm for the Jones Polynomial

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ABSTRACT

Let K be a 3-stranded knot (or link), and let L denote the number of crossings in K . Let ϵ_1 and ϵ_2 be two positive real numbers such that $\epsilon_2 \leq 1$.

In this paper, we create two algorithms for computing the value of the Jones polynomial $V_K(t)$ at all points $t = \exp(i\varphi)$ of the unit circle in the complex plane such that $|\varphi| \leq 2\pi/3$.

The first algorithm, called the **classical 3-stranded braid (3-SB) algorithm**, is a classical deterministic algorithm that has time complexity $O(L)$. The second, called the **quantum 3-SB algorithm**, is a quantum algorithm that computes an estimate of $V_K(\exp(i\varphi))$ within a precision of ϵ_1 with a probability of success bounded below by $1 - \epsilon_2$. The **execution time complexity** of this algorithm is $O(nL)$, where n is the ceiling function of $(\ln(4/\epsilon_2))/2\epsilon_1^2$. The **compilation time complexity**, i.e., an asymptotic measure of the amount of time to assemble the hardware that executes the algorithm, is $O(L)$.

1. INTRODUCTION

Let K be a 3-stranded knot (or link), i.e., a knot formed by the closure \bar{b} of a 3-stranded braid b , i.e., a braid $b \in B_3$. Let L be the length of the braid word b , i.e., the number of crossings in the knot (or link) K . Let ϵ_1 and ϵ_2 be two positive real numbers such that $\epsilon_2 \leq 1$.

In this paper, we create two algorithms for computing the value of the Jones polynomial $V_K(t)$ at all points $t = e^{i\varphi}$ of the unit circle in the complex plane such that $|\varphi| \leq \frac{2\pi}{3}$.

The first algorithm, called the **classical 3-stranded braid (3-SB) algorithm**, is a classical deterministic algorithm that has time complexity $O(L)$. The second, called the **quantum 3-SB algorithm**, is a quantum algorithm that computes an estimate of $V_K(e^{i\varphi})$ within a precision of ϵ_1 with a probability of success bounded below by $1 - \epsilon_2$. The **execution time complexity** of this algorithm is $O(nL)$, where n is the ceiling function of $\frac{\ln(4/\epsilon_2)}{2\epsilon_1^2}$. The **compilation time complexity**, i.e., an asymptotic measure of the amount of time to assemble the hardware that executes the algorithm, is $O(L)$.

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2. THE BRAID GROUP

The the n -stranded braid group B_n is the group generated by the symbols

$$b_1, b_2, \dots, b_{n-1}$$

subject to the following complete set of defining relations

$$\begin{cases} b_i b_j = b_j b_i & \text{for } |i - j| > 1 \\ b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1} & \text{for } 1 \leq i < n \end{cases}$$

This group can be described more informally in terms of diagrammatics as follows: We think of each braid as a hatbox with n black dots on top and another n on the bottom, and with each top black dot connected by a red string (i.e., a strand) to a bottom black dot. The strands are neither permitted to intersect nor to touch. Two such hatboxes (i.e., braids) are said to be equal if it is possible to continuously transform the strands of one braid into those of the other, without leaving the hatbox, without cutting and reconnecting the strands, and without permitting one strand to pass through or touch another. The product of two braids b and b' is defined by simply stacking the hatbox b on top of the hatbox b' , thereby producing a new braid $b \cdot b'$. Please refer to Figure 1. The generators b_i are illustrated in Figure 2. Moreover, the defining relations for the braid group B_n are shown in Figures 3. The reader should take care to note that the hatbox is frequently not drawn, but is nonetheless understood to be there.

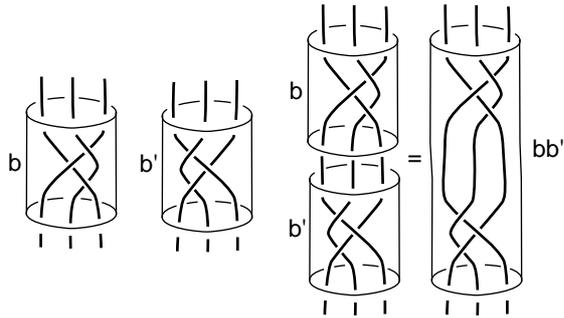


Figure 1. The product of two braids

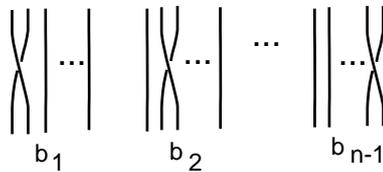


Figure 2. The generators of the n -stranded braid group B_n .

$$b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1} \quad b_i b_j = b_j b_i \quad \text{if } |i - j| > 1$$

Figure 3. A complete set of defining relations for the braid group B_n .

Every braid b in the braid group B_n can be written as a product of braid generators b_1, b_2, \dots, b_{n-1} and their inverses $b_1^{-1}, b_2^{-1}, \dots, b_{n-1}^{-1}$, i.e., every braid b can be written in the form

$$b = \prod_{i=1}^L b_{j(i)}^{\epsilon(i)} = b_{j(1)}^{\epsilon(1)} b_{j(2)}^{\epsilon(2)} \dots b_{j(L)}^{\epsilon(L)},$$

where $\epsilon(i) = \pm 1$. We call such a product a **braid word**.

Remark. We will later see that each such braid word can be thought of as a computer program which is to be compiled into an executable program. This resulting compiled program will in turn be executed to produce an approximation of the value of the Jones polynomial $J_K(t)$ at a chosen point $e^{i\varphi}$ on the unit circle.

We define

DEFINITION 2.1. The **writhe** of a braid b , written $Writhe(b)$, is defined as the sum of the exponents of a braid word representing the braid. In other words,

$$Writhe \left(\prod_{i=1}^L b_{j(i)}^{\epsilon(i)} \right) = \sum_{i=1}^L \epsilon(i)$$

For readers interested in learning more about the braid group, we refer the reader to Emil Artin's original defining papers,^{2 3 4} as well as to the many books on braids and knot theory, such as for example.^{5 6 15 29}

3. HOW KNOTS AND BRAIDS ARE RELATED

As one might suspect, knots and braids are very closely related to one another.

Every braid b can be transformed into a knot K by forming the **closed braid** \bar{b} as shown in Figure 4.

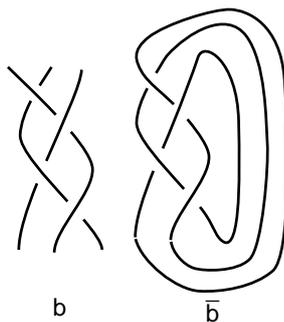


Figure 4. A braid b together with its closure \bar{b} .

This process can also be reversed. For Alexander developed a polytime algorithm for transforming an arbitrary knot K into a braid b having K as its closure.

THEOREM 3.1 (ALEXANDER). *Every knot (or link) is the closure of a braid. Such a braid can be found by a polynomial time algorithm*

Remark. *Every gardener who neatly puts away his garden hose should no doubt be familiar with this algorithm.*

We should mention that it is possible that the closures of two different braids will produce the same knot. But this non-uniqueness is well understood.

THEOREM 3.2 (MARKOV). *Two braids under braid closure produce the same knot (or link) if and only if one can be transformed into the other by applying a finite sequence of Markov moves*

We will not describe the Markov moves in this paper. For the reader interested in learning more about these moves, we suggest any one of the many books on knot theory.^{15 29}

4. THE TEMPERLEY-LIEB ALGEBRA

Let d and A be indeterminate complex numbers such that $d = -A^2 - A^{-2}$, and let

$$\mathbb{Z}[A, A^{-1}]$$

be the ring of Laurent polynomials with integer coefficients in the indeterminate A . Then the **Temperley-Lieb algebra** $TL_n(d)$ is the algebra with identity 1 over the Laurent ring $\mathbb{Z}[A, A^{-1}]$ generated by

$$1, U_1, U_2, \dots, U_{n-1}$$

subject to the following complete set of defining relations

$$\left\{ \begin{array}{l} U_i U_j = U_j U_i \quad \text{for } |i - j| > 1 \\ U_i U_{i \pm 1} U_i = U_i \\ U_i^2 = d U_i \end{array} \right.$$

This algebra can be described more informally in much the same fashion as we did for the braid group: We think of the generators $1, U_1, U_2, \dots, U_{n-1}$ as rectangles with n top and n bottom black dots, and with n disjoint

red strings (i.e., strands) connecting distinct pairs of black points. The red strings are neither permitted to intersect nor to touch one another. However, they are now allowed to connect two top black dots or two bottom black dots, as well as connect a top black dot with a bottom black dot. The generators $1, U_1, U_2, \dots, U_{n-1}$ of the Temperley-Lieb algebra $TL_n(d)$ are shown in Figure 5. The reader should take care to note that the rectangle is frequently not drawn, but is nonetheless understood to be there.

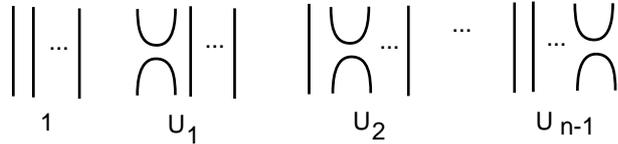


Figure 5. The generators of the Temperley-Lieb algebra $TL_n(d)$.

As we did with braids, the product ‘ \cdot ’ of two such red stringed rectangles is defined simply by stacking one rectangle on top of another. However, unlike the braid group, there is one additional ingredient in the definition of the product. Each disjoint circle resulting from this process is removed from the rectangle, and replaced by multiplying the rectangle by the indeterminate d . In this way, we can construct all the red stringed boxes corresponding to all possible finite products of the generators $1, U_1, U_2, \dots, U_{n-1}$. As before, two such red stringed rectangles are said to be equal if it is possible to continuously transform the strands of one rectangle into those of the other, without leaving the rectangle, without cutting and reconnecting the strands, and without letting one strand pass through another. Please refer to Figure 6.

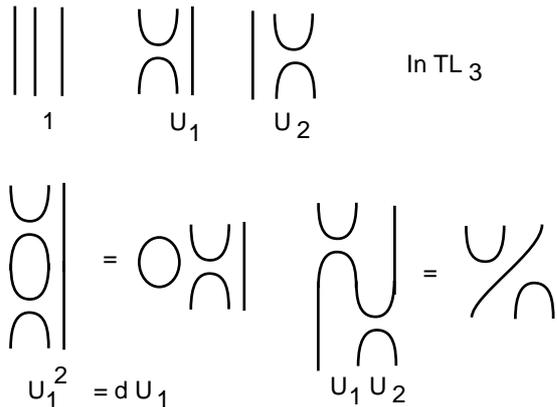


Figure 6. Two examples of the product of Temperley-Lieb generators.

Since $TL_n(d)$ is an algebra, we also need to define what is meant by the sum ‘ $+$ ’ (linear combination) of two or more rectangles. This is done simply by formally writing down linear combinations of rectangles over the Laurent ring $\mathbb{Z}[A, A^{-1}]$, and then assuming that addition ‘ $+$ ’ distributes with respect to the product ‘ \cdot ’, and that the scalar elements, i.e., the elements of the Laurent ring $\mathbb{Z}[A, A^{-1}]$, commute with all the rectangles and all the formal linear combinations of these rectangles. An example of one such linear combination is,

$$(2A^2 - 3A^{-4}) 1 + (-5 + 7A^2) U_1 + (1 + A^{-6} - A^{-10}) U_1 U_2 ,$$

We should also mention that there exists a trace

$$Tr_M : TL_n(d) \longrightarrow \mathbb{Z}[A, A^{-1}] ,$$

called the **Markov trace**, from the Temperley-Lieb algebra $TL_n(d)$ into the Laurent ring $\mathbb{Z}[A, A^{-1}]$. This trace is defined by sending each rectangle to d^{k-1} , where k denotes the number of disjoint circles that occur when the closure of the rectangle is taken as indicated in Fig. 7.

For readers interested in learning more about the Temperley-Lieb algebra $TL_n(d)$, we refer them to the many books on knot theory, such as for example.^{15 16}

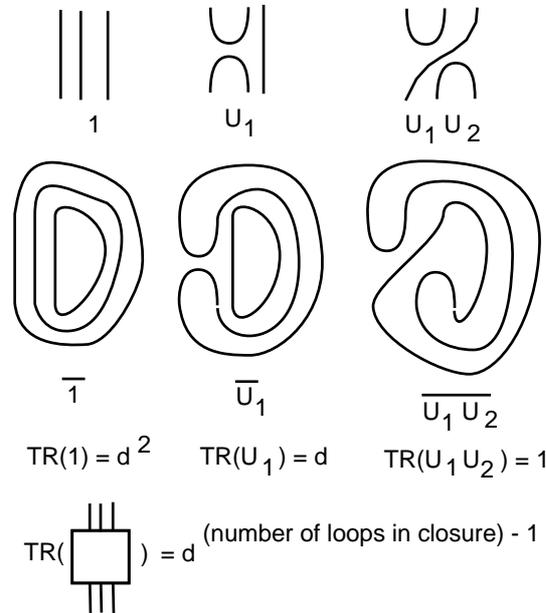


Figure 7. The Markov trace $Tr_M : TL_n(d) \rightarrow \mathbb{Z}[A, A^{-1}]$.

5. THE JONES REPRESENTATION

Vaughn Jones, using purely algebraic methods, constructed his **Jones representation**

$$J : B_n \rightarrow TL_n(d)$$

of the braid group B_n into the Temperley-Lieb algebra $TL_n(n)$ by mapping each braid generator b_i and its inverse b_i^{-1} into $TL_n(d)$ as follows*

$$\begin{cases} b_i & \mapsto A1 + A^{-1}U_i \\ b_i^{-1} & \mapsto A^{-1}1 + AU_i \end{cases}$$

He then used his representation J and the Markov trace Tr_M to construct the **Jones polynomial** $V(t)$ of a knot K (given by the closure \bar{b} of a braid b) as

$$V(t) = (-A^3)^{W_{rith}(b)} Tr_M(J(b))$$

where $t = A^{-4}$.

*Actually to be perfectly correct, Jones wrote his original representation in a variable t which is related to our variable A by the equation $t = A^{-4}$.

Later, Kauffman created the now well known diagrammatic approach to the Temperley-Lieb algebra $TL_n(d)$ and showed that his **bracket polynomial** $\langle \bar{b} \rangle$ was intimately connected to the Jones polynomial via the formula

$$\langle \bar{b} \rangle = Tr_M (J(\bar{b}))$$

For readers interested in learning more about these topic, we refer them to the many books on knot theory, such as for example.^{11 12 15 16 29}

6. THE TEMPERLEY-LIEB ALGEBRA $TL_3(D)$

We now describe a method for creating degree two representations of the Temperley-Lieb algebra $TL_3(d)$. These representation will in turn be used to create a unitary representation of the braid group B_3 , and ultimately be used to construct a quantum algorithm for computing approximations of the values of the Jones polynomial on a large portion of the unit circle in the complex plane.

From a previous section of this paper, we know that the 3 stranded Temperley-Lieb algebra $TL_3(d)$ is generated by

$$1, U_1, U_2$$

with the complete set of defining relations given by

$$\begin{cases} U_1^2 = dU_1 & \text{and} & U_2^2 = dU_2 \\ U_1U_2U_1 = U_1 & \text{and} & U_2U_1U_2 = U_2 \end{cases}$$

Moreover, the reader can verify the following proposition.

PROPOSITION 6.1. *The elements*

$$1, U_1, U_2, U_1U_2, U_2U_1$$

form a basis of $TL_3(d)$ as a module over the ring $\mathbb{Z}[A, A^{-1}]$. In other words, every element ω of $TL_3(d)$ can be written as a linear combination of the form

$$\omega = \omega_0 1 + \omega_1 U_1 + \omega_2 U_2 + \omega_{12} U_1 U_2 + \omega_{21} U_2 U_1 = \omega_0 1 + \omega_+ ,$$

where

$$\omega_0, \omega_1, \omega_2, \omega_{12}, \omega_{21}$$

are uniquely determined elements of the ring $\mathbb{Z}[A, A^{-1}]$.

7. A DEGREE 2 REPRESENTATION OF THE TEMPERLEY-LIEB ALGEBRA $TL_3(D)$

We construct a degree 2 representation of the Temperley-Lieb algebra $TL_3(d)$ as follows:

Let $|e_1\rangle$ and $|e_2\rangle$ be non-orthogonal unit length vectors from a two dimensional Hilbert space \mathcal{H} . From Schwartz's inequality, we immediately know that

$$0 < |\langle e_1 | e_2 \rangle| \leq 1$$

Let $\delta = \pm |\langle e_1 | e_2 \rangle|^{-1}$. It immediately follows that

$$1 \leq |\delta| < \infty$$

Moreover, let α denote a complex number such that $\delta = -\alpha^2 - \alpha^{-2}$.

We temporarily digress to state a technical lemma that will be needed later in this paper. We leave the proof as an exercise for the reader.

LEMMA 7.1. *Let δ be a real number of magnitude $|\delta| \geq 1$, and let α be a complex number such that $\delta = -\alpha^2 - \alpha^{-2}$. Then each of the following is a necessary and sufficient condition for α to lie on the unit circle:*

- δ is a real number such that $1 \leq |\delta| \leq 2$.
- There exist a $\theta \in [0, 2\pi]$ such that $\delta = -2 \cos(2\theta)$.

Thus,

$$\{\alpha \in \mathbb{C} : \exists \delta \text{ such that } 1 \leq |\delta| \leq 2 \text{ and } \delta = -\alpha^2 - \alpha^{-2}\}$$

is equal to the following set of points on the unit circle

$$\left\{ e^{i\theta} : \theta \in \left[0, \frac{\pi}{6}\right] \sqcup \left[\frac{\pi}{3}, \frac{2\pi}{3}\right] \sqcup \left[\frac{5\pi}{6}, \frac{7\pi}{6}\right] \sqcup \left[\frac{4\pi}{3}, \frac{5\pi}{3}\right] \sqcup \left[\frac{11\pi}{6}, 2\pi\right] \right\}$$

Also, as δ ranges over all values such that $1 \leq |\delta| \leq 2$, α^{-4} ranges over two thirds of the unit circle, i.e.,

$$\left\{ \alpha^{-4} : \exists \delta \text{ such that } 1 \leq |\delta| \leq 2 \text{ and } \delta = -\alpha^2 - \alpha^{-2} \right\} = \left\{ e^{i\varphi} : |\varphi| \leq \frac{2\pi}{2} \right\}$$

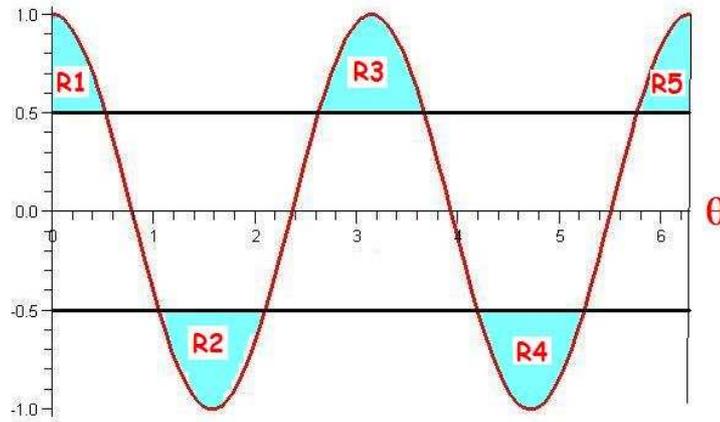


Figure 8. A plot of $\cos(2\theta)$ for $0 \leq \theta \leq 2\pi$.

We continue with the construction of our representation by using the unit length vectors $|e_1\rangle$ and $|e_2\rangle$ to create projection operators

$$E_1 = |e_1\rangle \langle e_1| \quad \text{and} \quad E_2 = |e_2\rangle \langle e_2|$$

These linear operators E_1 and E_2 are elements of the endomorphism ring $\text{End}(\mathcal{H}) \cong \text{Mat}(2, 2; \mathbb{C})$ of the Hilbert space \mathcal{H} . Since they are projection operators, they are Hermitian. By construction, they are of unit trace, i.e.,

$$\text{tr}(E_1) = 1 = \text{tr}(E_2)$$

where tr denotes the standard trace on $End(\mathcal{H}) \cong Mat(2, 2; \mathbb{C})$. The reader can also readily verify that

$$tr(E_1 E_2) = \delta^{-2} = tr(E_2 E_1)$$

and that E_1 and E_2 satisfy the relations

$$\begin{cases} E_1^2 = E_1 & \text{and} & E_2^2 = E_2 \\ E_1 E_2 E_1 = \delta^{-2} E_1 & \text{and} & E_2 E_1 E_2 = \delta^{-2} E_2 \end{cases}$$

It now follows that

THEOREM 7.2. *Let $\delta = \pm |\langle e_1 | e_2 \rangle|^{-1}$ (hence, $|\delta| \geq 1$), and let α be a complex number such that $\delta = -\alpha^2 - \alpha^{-2}$. Then the map*

$$\begin{array}{ccc} \Phi_\alpha : TL_3(d) & \longrightarrow & End(\mathcal{H}) \cong Mat(2, 2; \mathbb{C}) \\ U_j & \longmapsto & \delta E_j \\ d & \longmapsto & \delta \\ A & \longmapsto & \alpha \end{array}$$

is a well defined degree 2 representation of the Temperley-Lieb algebra $TL_3(d)$. Moreover, we have

$$\begin{aligned} tr(\Phi_\alpha(U_1)) &= \delta = tr(\Phi_\alpha(U_2)) \\ &\text{and} \\ tr(\Phi_\alpha(U_1 U_2)) &= 1 = tr(\Phi_\alpha(U_2 U_1)) \end{aligned}$$

PROPOSITION 7.3. *Let $\delta = \pm |\langle e_1 | e_2 \rangle|^{-1}$ (hence, $|\delta| \geq 1$), and let α be a complex number such that $\delta = -\alpha^2 - \alpha^{-2}$. Moreover, let $eval_\alpha : \mathbb{Z}[A, A^{-1}] \longrightarrow \mathbb{C}$ be the map defined by $A \longmapsto \alpha$. Then the diagram*

$$\begin{array}{ccc} TL_3(d) & \xrightarrow{\Phi_\alpha} & Mat(2, 2; \mathbb{C}) \\ Tr_M \downarrow & & \downarrow tr \\ \mathbb{Z}[A, A^{-1}] & \xrightarrow{eval_\alpha} & \mathbb{C} \end{array}$$

is **almost commutative** in the sense that, for each element $\omega \in TL_3(d)$,

$$eval_\alpha \circ Tr_M(\omega) = tr \circ \Phi_\alpha(\omega) + (\delta - 2)\omega_0,$$

where ω_0 denotes the coefficient of the generator 1 in ω .

8. A DEGREE 2 UNITARY REPRESENTATION OF THE THREE STRANDED BRAID GROUP B_3

In this section, we compose the above constructed representation Φ_α with the Jones representation J to create a representation of the three stranded braid group B_3 . We then determine when this representation Φ_α is unitary.

We begin by quickly recalling that the 3-stranded braid group B_3 is generated by the standard braid generators

$$b_1, b_2$$

with, in this case, the single defining relation

$$b_1 b_2 b_1 = b_2 b_1 b_2$$

We also recall that the Jones representation

$$B_3 \xrightarrow{J} TL_3(d)$$

is defined by

$$\begin{cases} b_j & \mapsto A1 + A^{-1}U_j \\ b_j^{-1} & \mapsto A^{-1}1 + AU_j \end{cases}$$

where A is a indeterminate satisfying $d = -A^2 - A^{-2}$. Thus, if we let δ and α be as defined in the previous section, we have that

$$\begin{aligned} B_3 & \xrightarrow{\Phi_\alpha \circ J} \text{End}(\mathcal{H}) \cong \text{Mat}(2, 2; \mathbb{C}) \\ b_j & \mapsto \alpha I + \alpha^{-1}dE_j \\ b_j & \mapsto \alpha^{-1}I + \alpha dE_j \end{aligned}$$

is a degree 2 representation of the braid group B_3 . Moreover, we have

PROPOSITION 8.1. *Let $\delta = \pm |\langle e_1 | e_2 \rangle|^{-1}$ (hence, $|\delta| \geq 1$), and let α be a complex number such that $\delta = -\alpha^2 - \alpha^{-2}$. Then the degree 2 representation*

$$B_3 \xrightarrow{\Phi_\alpha \circ J} \text{End}(\mathcal{H}) \cong \text{Mat}(2, 2; \mathbb{C})$$

is a unitary representation of the braid group B_3 if and only if α lies on the unit circle in the complex plane.

Proof. Since d is real and E_j is Hermitian, we have

$$(\alpha I + \alpha^{-1}\delta E_j)^\dagger = \bar{\alpha}I + \bar{\alpha}^{-1}\delta E_j$$

So for unitarity, we must have

$$\bar{\alpha}I + \bar{\alpha}^{-1}\delta E_j = \alpha^{-1}I + \alpha\delta E_j$$

It now follows from the linear independence of I, E_1, E_2 that $\Phi \circ J$ is unitary if and only if

$$\bar{\alpha} = \alpha^{-1}$$

□

From lemma 7.1, we have the following

COROLLARY 8.2. *Let $\delta = \pm |\langle e_1 | e_2 \rangle|^{-1}$ (hence, $|\delta| \geq 1$), and let α be a complex number such that $\delta = -\alpha^2 - \alpha^{-2}$. Then the representation $\Phi_\alpha \circ J$ is unitary if and only if $\alpha = e^{i\theta}$, where θ lies in the set*

$$\left\{ \theta \in [0, 2\pi] : |\cos(2\theta)| \geq \frac{1}{2} \right\} = \left[0, \frac{\pi}{6}\right] \cup \left[\frac{\pi}{3}, \frac{2\pi}{3}\right] \cup \left[\frac{5\pi}{6}, \frac{7\pi}{6}\right] \cup \left[\frac{4\pi}{3}, \frac{5\pi}{3}\right] \cup \left[\frac{11\pi}{6}, 2\pi\right]$$

9. COMPUTING THE JONES POLYNOMIAL

Recall that the Jones polynomial $V(t)$ of a knot (or link) K given by the closure \bar{b} of a braid word b is defined as

$$V(t) = (-A^3)^{\text{Writh}(b)} \text{Tr}_M(J(b)) ,$$

where $t = A^{-4}$.

Thus, from Proposition 7.3, we know that the value of the Jones polynomial at a point $t = e^{i\varphi}$ on the unit circle is given by

$$V(e^{i\varphi}) = (-e^{3i\theta})^{\text{Writh}(b)} \text{eval}_{e^{i\theta}} \circ \text{Tr}_M \circ J(b) = (-e^{3i\theta})^{\text{Writh}(b)} (\text{tr} \circ \Phi_{e^{i\theta}} \circ J)(b) + (\delta - 2) (-e^{4i\theta})^{\text{Writh}(b)} ,$$

where $e^{i\theta}$ is a point on the unit circle such that $e^{i\varphi} = (e^{i\theta})^{-4} = e^{-4i\theta}$. From lemma 1, we know that $\Phi_{e^{i\theta}}$ is only defined when $|\cos(2\theta)| \geq \frac{1}{2}$. Moreover, since $\varphi = -4\theta \pmod{2\pi}$, it also follows from lemma 1 that $\Phi_{e^{i\theta}}$ is only defined when $|\varphi| \leq \frac{2\pi}{3}$.

THEOREM 9.1. *Let φ be a real number such that $|\varphi| \leq \frac{2\pi}{3}$, and let θ be a real number such that $\varphi = -4\theta \pmod{2\pi}$. Let K be a knot (or link) given by the closure \bar{b} of a 3-stranded braid $b \in B_3$. Then the value of the Jones polynomial $V(t)$ for the knot (or link) K at $t = e^{i\varphi}$ is given by*

$$V(e^{i\varphi}) = \text{tr}(U(b)) + (\delta - 2) e^{i\theta \text{Writh}(b)} ,$$

where U is the unitary transformation

$$U = U(b) = (\Phi_{e^{i\theta}} \circ J)(b) .$$

Let us now assume that $|\varphi| \leq \frac{2\pi}{3}$ and that $\varphi = -4\theta \pmod{2\pi}$. Hence, $U = U(b)$ is unitary. Thus, if the knot (or link) K is given by the closure \bar{b} of a braid b defined by a word

$$b = \prod_{k=1}^L b_{j(k)}^{\epsilon(k)} = b_{j(1)}^{\epsilon(1)} b_{j(2)}^{\epsilon(2)} \cdots b_{j(L)}^{\epsilon(L)} ,$$

where b_1, b_2 are the generators of the braid group B_3 , and where $\epsilon(k) = \pm 1$ for $k = 1, 2, \dots, L$, then the unitary transformation $U = U(b)$ can be rewritten as

$$U = \prod_{k=1}^L \left(U^{(j(k))} \right)^{\epsilon(k)} ,$$

where $U^{(j)}$ denotes the unitary transformation (called an **elementary gate**) given by

$$\begin{aligned} U^{(j)} &= (\Phi_{e^{i\theta}} \circ J)(b_j) = \Phi_{e^{i\theta}}(A1 + A - 1U_j) = e^{i\theta} I - 2e^{-i\theta} \cos(2\theta) |e_j\rangle \langle e_j| \\ &= e^{i\theta} I - 2e^{-i\theta} \cos(2\theta) E_j \end{aligned}$$

In summary, we have:

COROLLARY 9.2. Let $t = e^{i\varphi}$ be an arbitrary point on the unit circle in the complex plane. Let b be a 3-stranded braid (i.e., a braid b in B_3) given by a braid word

$$b = \prod_{k=1}^L b_{j(k)}^{\epsilon(k)} = b_{j(1)}^{\epsilon(1)} b_{j(2)}^{\epsilon(2)} \cdots b_{j(L)}^{\epsilon(L)},$$

and let K be the knot (or link) given by the closure \bar{b} of the braid b . Then the value of the Jones polynomial $V(t)$ of K at $t = e^{i\varphi}$ is given by

$$V(e^{i\varphi}) = \left((-e^{3i\theta})^{\sum_{k=1}^L \epsilon(k)} \right) \text{tr} \left(\prod_{k=1}^L \left(U^{(j(k))} \right)^{\epsilon(k)} \right) + (\delta - 2) \left((-e^{4i\theta})^{\sum_{k=1}^L \epsilon(k)} \right),$$

where $U^{(j)}$ ($j = 1, 2$) is the linear transformation

$$U^{(j)} = e^{i\theta} I - 2e^{-i\theta} \cos(2\theta) E_j,$$

where I denotes the 2×2 identity matrix, and where E_j is the 2×2 Hermitian matrix $|e_j\rangle\langle e_j|$. We have also shown that the linear transformations $U^{(1)}$, $U^{(2)}$, and $U = \prod_{k=1}^L \left(U^{(j(k))} \right)^{\epsilon(k)}$ are unitary if and only if $|\varphi| \leq \frac{2\pi}{3}$. When $|\varphi| \leq \frac{2\pi}{3}$, we will call $U^{(1)}$ and $U^{(2)}$ **elementary gates**.

Remark. Thus, the task of determining the value of the Jones polynomial at any point $t = e^{i\varphi}$ such that $|\varphi| \leq \frac{2\pi}{3}$ reduces to the task of devising a quantum algorithm that computes the trace of the unitary transformation

$$U = U(b) = \prod_{k=1}^L \left(U^{(j(k))} \right)^{\epsilon(k)}.$$

COROLLARY 9.3. Let K be a **3-stranded knot** (or link), i.e., a knot (or link) given by the closure \bar{b} of a 3-stranded braid b , i.e., a braid $b \in B_3$. Then the formula found in the previous corollary gives a **deterministic classical algorithm** for computing the value of the Jones polynomial of K **at all points of the unit circle** in the complex plane of the form $e^{i\varphi}$, where $|\varphi| \leq \frac{2\pi}{3}$. Moreover, the time complexity of this algorithm is $O(L)$, where L is the length of the word b , i.e., where L is the number of crossings in the knot (or link) K . We will call this algorithm the **classical 3-stranded braid (3-SB) algorithm**.

10. TRACE ESTIMATION VIA THE HADAMARD TEST.

In the past section, we have shown how to create the classical 3-SB algorithm that computes the values of the Jones polynomial of a 3-stranded knot K on two thirds of the unit circle. In this section, we will show how to transform this classical algorithm into a corresponding quantum algorithm.

We will now assume that $|\varphi| \leq \frac{2\pi}{3}$ so that the elementary gates $U^{(1)}$ and $U^{(2)}$, and also the gate $U = U(b) = \prod_{k=1}^L \left(U^{(j(k))} \right)^{\epsilon(k)}$ are unitary. We know from the previous section that all we need to do to create a quantum 3-SB algorithm is to devise a quantum procedure for estimating the trace $\text{trace}(U)$ of the unitary transformation U .

To accomplish this, we will use a trace estimation procedure called the Hadamard test.

Let \mathcal{H} be the two dimensional Hilbert space associated with the the unitary transformations U , and let $\{|k\rangle : k = 0, 1\}$ be a corresponding chosen orthonormal basis. Moreover, let \mathcal{K} denote the two dimensional Hilbert space associated with an ancillary qubit with chosen orthonormal basis $\{|0\rangle, |1\rangle\}$. Then the trace estimation procedure, called the **Hadamard test**, is essentially defined by the two wiring diagrams found in Figures 9 and 10. The wiring diagrams found in Figures 9 and 10 are two basic quantum algorithmic primitives for determining respectively the real part $\text{Re}(\text{trace}(U))$ and the imaginary part $\text{Im}(\text{trace}(U))$ of the trace $\text{trace}(U)$ of U . The top qubit in each of these wiring diagrams denotes the ancillary qubit, and the bottom qubit $|k\rangle$ denotes a basis element of the Hilbert space \mathcal{H} associated with U . (The top wire in each wiring diagram denotes the ancilla qubit.) Each box labeled by an ‘ H ’ denotes the Hadamard gate

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} .$$

The box labeled by an ‘ S ’ denotes the phase gate

$$S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} .$$

And finally, the controlled- U gate is given by the standard notation.

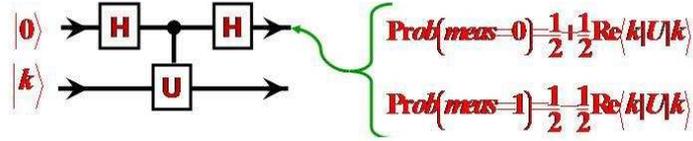


Figure 9. A quantum system for computing the real part of the diagonal element U_{kk} .
 $\text{Re}(U_{kk}) = \text{Prob}(\text{meas} = 0) - \text{Prob}(\text{meas} = 1)$.

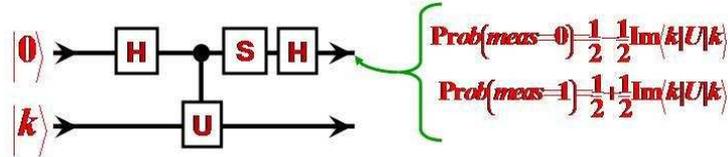


Figure 10. A quantum system for computing the imaginary part of the diagonal element U_{kk} .
 $\text{Re}(U_{kk}) = \text{Prob}(\text{meas} = 1) - \text{Prob}(\text{meas} = 0)$.

The wiring diagram found in Figure 9 has been so designed as to compute the real part $\text{Re}(U_{kk})$ of the k -th diagonal entry U_{kk} of U . For this wiring diagram has been so engineered that, when the output ancilla qubit is measured, then the resulting measured 0 or 1 occurs with probability given by

$$\begin{cases} \text{Prob}(\text{meas} = 0) &= \frac{1}{2} + \frac{1}{2} \text{Re} \langle k|U|k \rangle \\ \text{Prob}(\text{meas} = 1) &= \frac{1}{2} - \frac{1}{2} \text{Re} \langle k|U|k \rangle \end{cases}$$

Thus, the difference of these two probabilities is the real part of the k -th diagonal entry

$$\text{Prob}(\text{meas} = 0) - \text{Prob}(\text{meas} = 1) = \text{Re} \langle k|U|k \rangle = \text{Re}(U_{kk}) .$$

If this procedure (i.e., preparation of the state $|0\rangle|k\rangle$, application of the unitary transformation $(H \otimes 1) \cdot \text{Contr} - U \cdot (H \otimes 1)$, and measurement of the output ancilla qubit) is repeated n times, then the normalized number of 0's minus the number of 1's, i.e.,

$$\frac{\#0\text{'s} - \#1\text{'s}}{n},$$

becomes an ever better estimate of the real part $\text{Re}(U_{kk})$ of the k -th diagonal entry U_{kk} as the number of trials n becomes larger and larger. We will make this statement even more precise later.

In like manner, the wiring diagram found in Figure 10 has been so designed to compute the imaginary part $\text{Im}(U_{kk})$ of the k -th diagonal entry U_{kk} of U . This wiring diagram has been engineered so that, if the output ancilla qubit is measured, then the resulting measured 0 and 1 occur with probabilities given by

$$\begin{cases} \text{Prob}(meas = 0) &= \frac{1}{2} - \frac{1}{2} \text{Re} \langle k|U|k \rangle \\ \text{Prob}(meas = 1) &= \frac{1}{2} + \frac{1}{2} \text{Re} \langle k|U|k \rangle \end{cases}$$

Thus, the difference of these two probabilities is the real part of the k -th diagonal entry

$$\text{Prob}(meas = 1) - \text{Prob}(meas = 0) = \text{Im} \langle k|U|k \rangle = \text{Im}(U_{kk})$$

Much like as before, if this procedure (i.e., preparation of the state $|0\rangle|k\rangle$, application of the unitary transformation $(H \otimes 1) \cdot \text{Contr} - U \cdot S \cdot (H \otimes 1)$, and measurement of the output ancilla qubit) is repeated n times, then the normalized number of 1's minus the number of 0's, i.e.,

$$\frac{\#1\text{'s} - \#0\text{'s}}{n}$$

becomes an ever better estimate of the imaginary part $\text{Im}(U_{kk})$ of the k -th diagonal entry U_{kk} as the number of trials n increases.

We now focus entirely on the first wiring diagram, i.e., Figure 9. But all that we will say can easily be rephrased for the second wiring diagram found in Figure 10.

We continue by more formally reexpressing the wiring diagram of Figure 9 as the quantum subroutine $\text{QRE}_U(k)$ given below:

QUANTUM SUBROUTINE $\text{QRE}_U(k)$

Step 0 Initialization

$$|\psi_0\rangle = |0\rangle|k\rangle$$

Step 1 Application of $H \otimes I$

$$|\psi_1\rangle = (H \otimes I) |\psi_0\rangle = \frac{1}{\sqrt{2}} (|0\rangle|k\rangle + |1\rangle|k\rangle)$$

Step 2 Application of $\text{CONTR-}U$

$$|\psi_2\rangle = (\text{CONTR-}U) |\psi_1\rangle = \frac{1}{\sqrt{2}} (|0\rangle|k\rangle + |1\rangle U|k\rangle)$$

Step 3 Application of $H \otimes I$

$$|\psi_3\rangle = (H \otimes I) |\psi_1\rangle = |0\rangle \left(\frac{1+U}{2} \right) |k\rangle + |1\rangle \left(\frac{1-U}{2} \right) |k\rangle$$

Step 4 Measure the ancilla qubit

Resulting Measured Bit \mathbf{b}	Probability	Resulting State $ \psi_4\rangle$
0	$\frac{1}{2} + \frac{1}{2} \operatorname{Re}(\mathbf{U}_{kk})$	$\frac{ 0\rangle \left(\frac{1+U}{2} \right) k\rangle}{\sqrt{\frac{1}{2} + \frac{1}{2} \operatorname{Re}(\mathbf{U}_{kk})}}$
1	$\frac{1}{2} - \frac{1}{2} \operatorname{Re}(\mathbf{U}_{kk})$	$\frac{ 0\rangle \left(\frac{1-U}{2} \right) k\rangle}{\sqrt{\frac{1}{2} - \frac{1}{2} \operatorname{Re}(\mathbf{U}_{kk})}}$

Step 5 Output the classical bit \mathbf{b} and STOP

Next we formalize the iteration procedure by defining the following quantum subroutine:

APPROX-RE-TRACE $_U$ (n)

 LOOP $k = 1..2$

 APPROX-DIAG-ENTRY(k) = 0

 LOOP $j = 1..n$

$b = \text{QRE}_U(k)$

 APPROX-DIAG-ENTRY(k) = APPROX-DIAG-ENTRY(k) + $(-1)^b$

 END LOOP j

 END LOOP k

 OUTPUT (APPROX-DIAG-ENTRY(1) + APPROX-DIAG-ENTRY(2))

END

As mentioned earlier, quantum subroutines $\text{QIM}_U(k)$ and $\text{APPROX-IM-TRACE}_U(n)$ can be defined in a similar manor.

We continue by recognizing that there is a certain amount of computational effort involved in creating the subroutine $\text{QRE}_U(k)$. For this, we need the following formal definition:

DEFINITION 10.1. *The **compilation time** of a quantum algorithm is defined as the amount of time (computational effort) required to assemble algorithm into hardware.*

Since the compilation time to assemble the gate U is asymptotically the number of elementary gates $U^{(j)}$ in the product $\prod_{k=1}^L (U^{(j(k))})^{\epsilon(k)}$, we have

THEOREM 10.2. *Let b be a 3-stranded braid, i.e., $b \in B_3$, and let K be the knot (or link) formed from the closure \bar{b} of the braid b . Then the time complexity of compiling the braid word b into the quantum subroutine $\text{QRE}_U(\)$ is $O(L)$, where L is the length of the braid word b , i.e., where L is the number of crossings in the knot (or link)*

K. Moreover, the running time complexity of $\text{QRE}_U(\)$ is also $O(L)$. The same is true for the quantum subroutine $\text{QIM}_U(\)$.

COROLLARY 10.3. *The quantum subroutine $\text{APPROX-RE-TRACE}_U(n)$ and $\text{APPROX-IM-TRACE}_U(n)$ are each of compile time complexity $O(nL)$ and of run time complexity $O(nL)$.*

THEOREM 10.4. *Let b be a 3-stranded braid, i.e., $b \in B_3$, and let K be the knot (or link) formed from the closure \bar{b} of the braid b . Let ϵ_1 and ϵ_2 be to arbitrary chosen positive real numbers such that $\epsilon_2 \leq 1$. Let n be an integer such that*

$$n \geq \frac{\ln(2/\epsilon_2)}{\epsilon_1^2} .$$

Then with time complexity $O(nL)$, the quantum algorithm $\text{APPROX-RE-TRACE}_U(n)$ will produce a random real number $S_n^{(\text{Re})}$ such that

$$\text{Prob} \left(\left| S_n^{(\text{Re})} - \text{Re}(\text{trace}(U)) \right| \geq \epsilon_1 \right) \leq \epsilon_2$$

In other words, the probability that $\text{APPROX-RE-TRACE}_U(n)$ will output a random real number S_n within ϵ_1 of the real part $\text{Re}(\text{trace}(U))$ of the trace $\text{trace}(U)$ is greater than $1 - \epsilon_2$. The same is true for the quantum subroutine $\text{APPROX-IM-TRACE}_U(n)$.

Proof. Let X_1, X_2, \dots, X_n be the n random variables corresponding to the n random output bits resulting from n independent executions of $\text{QRE}_U(1)$, and in like manner, let $X_{n+1}, X_{n+2}, \dots, X_{2n}$ be the n random variables corresponding to the n random output bits resulting from n independent executions of $\text{QRE}_U(2)$.

Thus, each of the first n random variables have the same probability $p_0^{(1)}$ of being zero and the same probability $p_1^{(1)}$ of being 1. In like manner, the last n of these random variables have the same probabilities $p_0^{(2)}$ and $p_1^{(2)}$ of being 0 or 1, respectively. Moreover, it is important to emphasize that the $2n$ random variables $X_1, X_2, \dots, X_n, X_{n+1}, X_{n+2}, \dots, X_{2n}$ are stochastically independent.

The random variable associated with the random number

$$\frac{\#0's - \#1's}{n}$$

is

$$S_n^{(\text{Re})} = \sum_{j=1}^{2n} (-1)^{X_j} .$$

The reader can easily verify the mean μ of S_n is given by

$$\mu = p_0^{(1)} - p_1^{(1)} + p_0^{(2)} - p_1^{(2)} = \text{Re}(U_{11}) + \text{Re}(U_{22}) = \text{Re}(\text{trace}(U))$$

From Hoeffding's inequality,¹⁰ it follows that

$$\text{Prob} \left(\left| S_n^{(\text{Re})} - \text{Re}(\text{trace}(U)) \right| \geq \epsilon_1 \right) \leq 2e^{-2(2n)^2\epsilon_1^2/(\sum_{j=1}^{2n} 4)} = 2e^{-n\epsilon_1^2} .$$

Thus, when

$$n \geq \frac{\ln(2/\epsilon_2)}{\epsilon_1^2} ,$$

we have that

$$\text{Prob} \left(\left| S_n^{(\text{Re})} - \text{Re}(\text{trace}(U)) \right| \geq \epsilon_1 \right) \leq \epsilon_2 .$$

In like manner, a similar result can be proved for QIM_U . \square

As a corollary, we have

COROLLARY 10.5. *Let b be a 3-stranded braid, i.e., $b \in B_3$, and let K be the knot (or link) formed from the closure \bar{b} of the braid b . Let ϵ_1 and ϵ_2 be to arbitrary chosen positive real numbers such that $\epsilon_2 \leq 1$. Let n be an integer such that*

$$n \geq \frac{\ln(4/\epsilon_2)}{2\epsilon_1^2}.$$

Then with time complexity $O(nL)$, the quantum algorithms APPROX-RE-TRACE $_U$ (n) and APPROX-IM-TRACE $_U$ (n) will jointly produce random real numbers $S_n^{(\text{Re})}$ and $S_n^{(\text{Im})}$ such that

$$\text{Prob} \left(\left| S_n^{(\text{Re})} - \text{Re}(\text{trace}(U)) \right| \geq \epsilon_1 \text{ and } \left| S_n^{(\text{Im})} - \text{Im}(\text{trace}(U)) \right| \geq \epsilon_1 \right) \leq \epsilon_2$$

In other words, the probability that both APPROX-RE-TRACE $_U$ (n) and APPROX-IM-TRACE $_U$ (n) will output respectively random real number $S_n^{(\text{Re})}$ and $S_n^{(\text{Im})}$ within ϵ_1 of the real and imaginary parts of the trace $\text{trace}(U)$ is greater than $1 - \epsilon_2$.

11. SUMMARY AND CONCLUSION

Let K be a 3-stranded knot (or link), i.e., a knot formed by the closure \bar{b} of a 3-stranded braid b , i.e., a braid $b \in B_3$. Let L be the length of the braid word b , i.e., the number of crossings in the knot (or link) K . Let ϵ_1 and ϵ_2 be two positive real numbers such that $\epsilon_2 \leq 1$.

Then in summary, we have created two algorithms for computing the value of the Jones polynomial $V_K(t)$ at all points $t = e^{i\varphi}$ of the unit circle in the complex plane such that $|\varphi| \leq \frac{2\pi}{3}$.

The first algorithm, called the **classical 3-SB algorithm**, is a classical deterministic algorithm that has time complexity $O(L)$. The second, called the **quantum 3-SB algorithm**, is a quantum algorithm that computes an estimate of $V_K(e^{i\varphi})$ within a precision of ϵ_1 with a probability of success bounded below by $1 - \epsilon_2$. The **execution time complexity** of this algorithm is $O(nL)$, where n is the ceiling function of $\frac{\ln(4/\epsilon_2)}{2\epsilon_1^2}$. The **compilation time complexity**, i.e., an asymptotic measure of the amount of time to assemble the hardware that executes the algorithm, is $O(L)$. A pseudo code description of the quantum 3-stranded braid algorithm is given below.

QUANTUM-3-SB-ALGORITHM($b, \varphi, \epsilon_1, \epsilon_2$)

COMMENT: b = braid word representing a 3-stranded braid s.t. $K = \bar{b}$

COMMENT: φ real number s.t. $|\varphi| \leq \frac{2\pi}{3}$

COMMENT: ϵ_1 lower bound on the precision of the output

COMMENT: ϵ_2 upper bound on the probability that the

COMMENT: output is not within precision ϵ_1

COMMENT: The output of this algorithm is with probability $\geq 1 - \epsilon_2$

COMMENT: a complex number within ϵ_1 of $V_K(e^{i\varphi})$

$$n = \left\lceil \frac{\ln(4/\epsilon_2)}{2\epsilon_1^2} \right\rceil$$

$U = \text{GATE-COMPILE}(b)$

APPROX-RE-TRACE $_U$ = REAL-PART-TRACE-COMPILE(U)

```

APPROX-IM-TRACEU = IMAGINARY-PART-TRACE-COMPILE(U)
APPROXReTr = APPROX-RE-TRACEU (n)
APPROXImTr = APPROX-IM-TRACEU (n)
W = Writhe (b)
θ = -φ/4
δ = -2 * cos (2 * θ)
REEXP3 = cos (3 * θ * W)
IMEXP3 = sin (3 * θ * W)
REJONES = REEXP3*APPROXRETR-IMEXP3 * APPROXIMTR
REJONES = (-1)W * ( REJONES + (δ - 2) * cos (φ * W) )
IMJONES = IMEXP3*APPROXRETR+REEXP3 * APPROXIMTR
IMJONES = (-1)W * ( IMJONES - (δ - 2) * sin (φ * W) )
OUTPUT( REJONES, IMJONES )

```

END

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