

**Classical and Quantum Logic Gates:**  
**An Introduction to Quantum Computing**

Quantum Information Seminar

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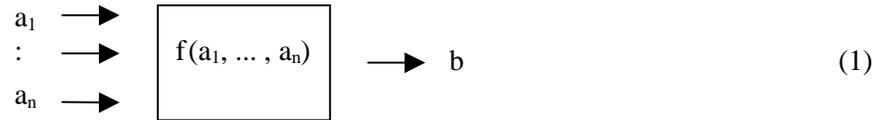
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## I.A. Irreversible Classical Logic

Classical computation theory began for the most part when Church and Turing independently published their inquiries into the nature of computability in 1936 [1]. For our purposes, it will suffice to take as our model for classical discrete computation, a block diagram of the form,



where  $b = f(a_1, a_2, \dots, a_n)$  describes a single-valued function on  $n$  discrete inputs. We will assume that such a function can be simulated or computed physically. As is usually done in classical computation, we can use base-2 arithmetic to describe the inputs and outputs, in which case  $a_1, \dots, a_n$ , and  $b$  become binary variables, or bits, taking on one of two values, 0 or 1. In this case, the function  $f(a_1, a_2, \dots, a_n)$  is known as an  $n$ -bit Boolean function.

The central problem that we will concern ourselves with repeatedly in these notes is the problem of universality. That is, given an arbitrarily large function  $f$ , is it possible to identify a universal set of simple functions – called gates – that can be used repeatedly in sequence to simulate  $f$  on its inputs. The gate functions would be restricted to operating on a small number of inputs, say two or three at a time, taken from  $a_1, a_2, \dots, a_n$ . These gates would be done in sequence, creating a composite function that represents  $f$  on all of its  $n$  inputs.

It has been known for a long time that the two-bit gates, AND and OR, and the one-bit gate NOT, are universal for classical computation, in the sense that they are sufficient to simulate any function of the form illustrated in diagram (1). Gate functions in classical logic are often represented using truth tables. The AND and OR gates have two inputs and one output, while the NOT gate has one input and one output. A truth table lists all possible combinations of the input bits and the corresponding output value for each gate.

$a_1$	$a_2$	$a_1$ AND $a_2$	$a_1$	$a_2$	$a_1$ OR $a_2$	$a_1$	NOT $a_1$	(2)
0	0	0	0	0	0	0	1	
0	1	0	0	1	1	1	0	
1	0	0	1	0	1	1	0	
1	1	1	1	1	1	1	0	

The AND gate outputs 1 if and only if both inputs are 1, the OR gate outputs 1 if either or both inputs are 1, and the NOT gate inverts the input. To illustrate that these three gates suffice to

build arbitrary Boolean functions, consider the three-bit Boolean function  $f(a_1, a_2, a_3)$  given by a truth table, whose first three lines look as follows,

$a_1$	$a_2$	$a_3$	$f(a_1, a_2, a_3)$
0	0	0	1
0	0	1	0
0	1	0	1
⋮	⋮	⋮	⋮

(3)

$f$  has the output value 1 in the first line of its truth table. The function,  $f_1 = (\text{NOT } a_1) \text{ AND } (\text{NOT } a_2) \text{ AND } (\text{NOT } a_3)$ , also has the value 1 when  $a_1 = 0$ ,  $a_2 = 0$ , and  $a_3 = 0$ , but has the value 0 for all other combinations of  $a_1$ ,  $a_2$ , and  $a_3$ . So  $f_1$  mimics the output of  $f$ , namely 1, for inputs taken from the first line of the truth table for  $f$ . Similarly, we can construct functions that mimic  $f$  for each line in the truth table for which  $f = 1$ . After the first line,  $f = 1$  again in the third line of the truth table. So for this line, we can construct the function  $f_3 = (\text{NOT } a_1) \text{ AND } (a_2) \text{ AND } (\text{NOT } a_3)$ , which has the value 1 when  $a_1 = 0$ ,  $a_2 = 0$ , and  $a_3 = 1$ . Since the functions  $f_1, f_3, \dots$  are disjoint in the sense that their 1-outputs correspond to distinct lines in the truth table, we can reconstruct  $f$  for all the lines in the truth table by disjunctively combining all of the functions  $f_1, f_3, \dots$ ,

$$f = f_1 \text{ OR } f_3 \text{ OR } \dots \tag{4}$$

which are no more than eight in number in this case since there are eight lines in the truth table for the three-bit function  $f$ . The expression on the right side of Eq. (4) is referred to in the texts as the “disjunctive normal form” of the function  $f$ . Generalizing this procedure, we see that we can simulate arbitrary Boolean functions using only the gates AND, OR, and NOT. So these three gates are universal for classical Boolean logic. As a further simplification, these three gates can be reduced to a single gate, the NAND gate,

$a_1$	$a_2$	$a_1 \text{ NAND } a_2$
0	0	1
0	1	1
1	0	1
1	1	0

(5)

Representing the NAND operation by the symbol  $\uparrow$ , it can be verified that

$$\begin{aligned} a \text{ OR } b &= (a \uparrow a) \uparrow (b \uparrow b) \\ a \text{ AND } b &= (a \uparrow b) \uparrow (a \uparrow b) \\ \text{NOT } a &= a \uparrow a \end{aligned} \tag{6}$$

Hence, we have the remarkable result that the two-bit NAND gate alone is sufficient for classical Boolean logic. However, the procedure illustrated in Eq. (4) for decomposing a Boolean function

into two-bit gates, is inefficient. That is, the number of NAND gates needed to simulate a function with  $n$  inputs this way would scale exponentially in  $n$ . This is essentially because an  $n$ -bit Boolean function has  $2^n$  lines in its truth table. Various minimization techniques have been discovered over the years – the Veitch-Karnaugh map and the Quine-McCluskey procedure are two of these – that allow more sophisticated decompositions of a function than we have illustrated, and which have proved invaluable to the design and implementation of circuits for classical computation (see ref. [2] for a simple treatment of these methods).

## **I.B. Reversible Classical Logic**

The first concerns about the reversibility of computation were raised in the 1970s. There were two related issues, logical reversibility and physical reversibility, which were intimately connected. Logical reversibility refers to the ability to reconstruct the input from the output of a computation, or gate function. For instance, the NAND gate is explicitly irreversible, taking two inputs to one output, while the NOT gate is reversible (it is its' own inverse). The connection to physical reversibility is usually made as follows. Since the NAND gate has only one output, one of its' inputs has effectively been erased in the process, whose information has been irretrievably lost. The change in entropy that we would associate with the lost of one bit of information is  $\ln 2$ , which, thermodynamically, corresponds to an energy increase of  $kT \ln 2$ , where  $k$  is Boltzman's constant and  $T$  is the temperature. The heat dissipated during a process is usually taken to be a sign of physical irreversibility, that the microscopic physical state of the system cannot be restored exactly as it was before the process took place.

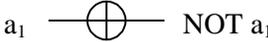
In the 70s, people were thus asking themselves two questions, which were related. One was whether a computation can be done in a logically reversible fashion (unlike one that uses NAND gates, for example), and the other was whether any heat needs to be dissipated during a computation. Both of these issues were quite academic however, since as Feynman pointed out [6], an actual transistor dissipates close to  $10^{10}$   $kT$  of heat, and even the DNA copying mechanism in a human cell dissipates about 100  $kT$  of heat per bit copied [which is understandable from a consideration of the chemical bonds that need to be broken in the process], both far from the ideal limit of  $kT \ln 2$  for irreversible computing.

That classical computation can be done reversibly with no energy dissipated per computational step was discovered by Bennett in 1973 [3]. He showed this by constructing a reversible model of the Turing machine – a symbolic model for computation introduced by Turing in 1936 [1] –

and showing that any problem that can be simulated on the original irreversible machine can also be simulated with the same efficiency on the reversible model. The logical reversibility inherent in the reversible model implied that an implementation of such a machine would also be physically reversible. This started the search for physical models for reversible classical computation, a review of which is given in [5].

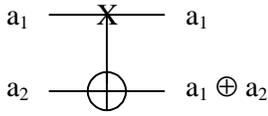
Our model for reversible computation will be similar to that of diagram (1), except that the number of inputs and outputs of the function  $f$  will be the same, and  $f$  will be required to be a one-to-one Boolean function. Likewise, we can pose the problem of universality as before, and ask for a set of universal reversible logic gates that can simulate arbitrary reversible Boolean functions. Since reversible logic gates are symmetric with respect to the number of inputs and outputs, we can represent them in ways other than the truth table, that emphasizes this symmetry. We have already encountered the reversible NOT gate, whose truth table was given in tables (2). We could also write this in the form of a matrix, or as a graphic,

NOT	$\langle 0 \rangle$	$\langle 1 \rangle$
$\langle 0 \rangle$	0	1
$\langle 1 \rangle$	1	0


(7)

The matrix form lists the lines in the truth table in the form  $\langle 0 \rangle, \langle 1 \rangle$ . The input lines are listed horizontally on the top and the output lines are listed vertically along the side, in the same order. We fill in the matrix with 1's and 0's such that each horizontal or vertical line has exactly one 1, which is to be interpreted as a one-to-one mapping of the input to the output. For example, a 1 in column one, row two in the NOT means that a  $\langle 0 \rangle$  input gets taken to a  $\langle 1 \rangle$  output. The graphical representation to the right of the table in (7) is a condensed representation of the NOT gate. The horizontal line represents a bit, whose initial variable value,  $a_1$ , is listed on the left and whose final value, NOT  $a_1$ , is listed on the right. The operation of the NOT gate in the middle is symbolised by the  $\oplus$  sign. A two-bit gate closely related to the NOT gate is the two-bit Controlled-NOT (or C-NOT) gate,

XOR = C-NOT	$\langle 00 \rangle$	$\langle 01 \rangle$	$\langle 10 \rangle$	$\langle 11 \rangle$
$\langle 00 \rangle$	1	0	0	0
$\langle 01 \rangle$	0	1	0	0
$\langle 10 \rangle$	0	0	0	1
$\langle 11 \rangle$	0	0	1	0

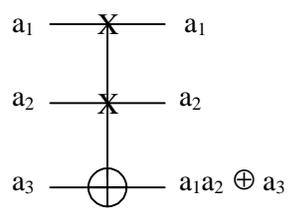

(8)

which performs a NOT on the second bit if the first bit is  $\langle 1 \rangle$ , but otherwise has no effect. The C-NOT is sometimes also called XOR, since it performs an exclusive OR operation on the two input

bits and writes the output to the second bit. The graphical representation of this gate has two horizontal lines representing the two variable bits, and the conditionality of the operation is represented by the addition of a vertical line originating from the first bit and terminating with a NOT symbol on the second bit. The X symbol on the first bit indicates that this bit is being checked to see whether it is in  $\langle 1 \rangle$ , before performing NOT on the second bit.

It turns out that classical reversible logic differs in one significant respect from classical irreversible logic in that two-bit gates are not sufficient for universal reversible computation. However, a three-bit gate is sufficient. A universal three-bit gate was identified by Toffoli [4] in 1981, called the Controlled-Controlled-NOT (or CC-NOT),

$T_3 = \text{CC-NOT}$	$\langle 000 \rangle$ ...	$\langle 110 \rangle$	$\langle 111 \rangle$	
$\langle 000 \rangle$	1 0 0 0 0 0 0 0			
$\langle 001 \rangle$	0 1 0 0 0 0 0 0			
$\langle 010 \rangle$	0 0 1 0 0 0 0 0			
$\langle 011 \rangle$	0 0 0 1 0 0 0 0			
$\langle 100 \rangle$	0 0 0 0 1 0 0 0			
$\langle 101 \rangle$	0 0 0 0 0 1 0 0			
$\langle 110 \rangle$	0 0 0 0 0 0 0 1			
$\langle 111 \rangle$	0 0 0 0 0 0 1 0			


(9)

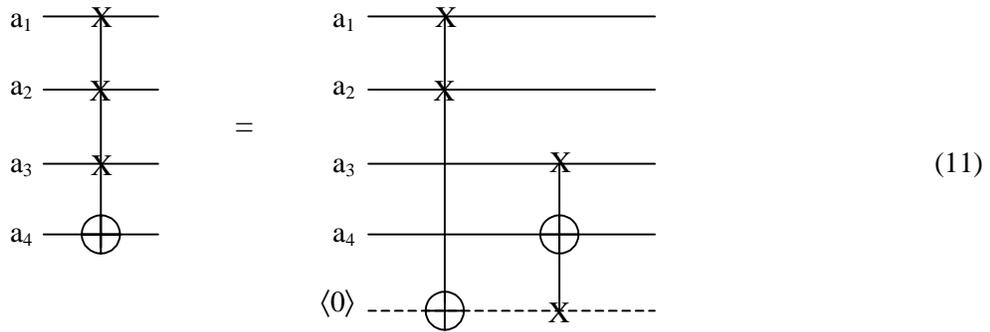
or simply the three-bit Toffoli gate,  $T_3$ . It applies a NOT to the third bit if the first two bits are in  $\langle 11 \rangle$ , but otherwise having no effect. The graphical representation of this conditional three-bit gate is given to the right of the table in (9), where both  $a_1$  and  $a_2$  are checked to see if they are in  $\langle 1 \rangle$  - denoted by the two solid circles on these bits - before performing NOT on  $a_3$ . The Toffoli gate is known to be universal for reversible Boolean logic, the argument for which is based on the fact that the Toffoli gate contains the NAND gate within it. When the third bit is fixed to be 1, the Toffoli gate writes the NAND of the first two bits on the third,  $T_3: a_1, a_2, 1 \rightarrow a_1, a_2, a_1 \uparrow a_2$ .

Lastly, we would like to consider why two-bit gates are not sufficient for reversible Boolean logic. First note that the two-bit NAND gate cannot be made reversible without introducing a third bit. If we were to retain one of the input bits in the output, and produce the NAND of the two inputs in the second output bit, as this truth table illustrates,

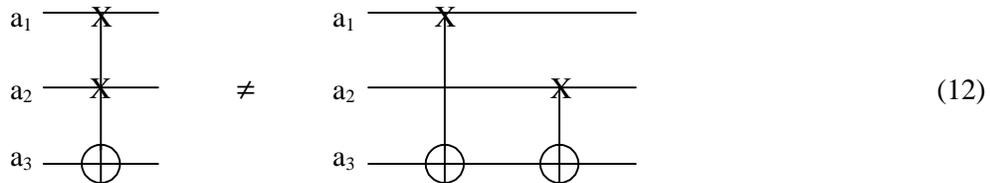
$a_1$	$a_2$	$a_1$	$a_1 \uparrow a_2$	
0	0	0	1	
0	1	0	1	
1	0	1	1	
1	1	1	0	

(10)

the gate remains irreversible still, since the first two inputs are taken to the same output. To make the NAND gate reversible, we really need to copy both the input bits into the output bit, and use a third bit to store the result of the NAND operation, as is done in the Toffoli gate. To illustrate why three-bit gates are needed for decomposing reversible Boolean functions, consider writing the four-bit CCC-NOT =  $T_4$  gate in terms of the three-bit Toffoli gates,



where we have made use of an auxiliary bit initialized to  $\langle 0 \rangle$ , and the two gates on the right side of the equation are performed from left to right. The gate on the left side is a four-bit gate that applies NOT to the fourth bit when the first three bits are in  $\langle 111 \rangle$ . The first Toffoli gate on the right side checks the bits  $a_1$  and  $a_2$  to see if they are in  $\langle 11 \rangle$ , and writes this information to the single auxiliary bit. The auxiliary bit becomes  $\langle 1 \rangle$  only when  $a_1$  and  $a_2$  are in  $\langle 11 \rangle$ . The second Toffoli gate collects this information from the auxiliary bit, and conditional on this bit and the bit  $a_3$ , performs NOT on the bit  $a_4$ , thus simulating the four-bit gate on the left side. The key feature of the Toffoli gate that allowed this decomposition is its ability to condense the information from two bits into one, something that cannot be accomplished by anything less than a three-bit gate. One might think that the Toffoli gate  $T_3$  itself might be simulated by two two-bit C-NOTs (or XORs), but as this example illustrates,



such a simulation is not possible. The conditionality offered by the two XOR gates on the right side is in an OR sense, where a NOT is performed on  $a_3$  when *either*  $a_1$  or  $a_2$  is in  $\langle 1 \rangle$ , which is distinct from the conditionality inherent in the Toffoli gate on the left side, where a NOT is performed on  $a_3$  only when *both*  $a_1$  and  $a_2$  are in  $\langle 1 \rangle$ . We will see later that while such a classical decomposition of the Toffoli gate in Eq. (12) is impossible, a quantum decomposition is possible.

## II.A. Quantum vs. Classical Logic

The first papers on quantum models for computation were published in the 1980s. Similar to the research into reversible models, the motivation was mostly academic at the time, an exploration of the ultimate limits of computation. The real payoff for quantum computing did not come until 1994, when Shor announced his quantum algorithm for factoring large numbers with an efficiency unparalleled by any classical algorithm preceding it [9]. The factoring problem is used widely to encrypt messages in public key cryptography, which made the feasibility of quantum computing an urgent issue in the years to follow.

The issues we will address in this section are the same as those that dogged early researchers in quantum computing in the 80s, that of what constitutes a quantum model for computation, if such a thing were possible, and whether there are qualitative differences between such a model and the preceding classical models for computation. We will summarize below some of the key conceptual discoveries that resulted from this research. A landmark paper on this subject, which considered some of these issues as well as the existence of a quantum analog to the Turing machine, was published in 1985 by Deutsch [7]. The two important issues to address when it comes to building a computer model is that of storage and processing. That is, what will be the memory units of our quantum computer, and how the information contained in them will be processed to do the computation. This will give us an idea of what constraints are placed on the kind of computations that can take place in this system.

The memory units of a classical discrete computer are usually taken as bits, two-valued classical variables, as we saw earlier. By analogy, we consider a two-valued quantum variable, called a qubit, which may be simulated by a two-level quantum system. Suppose the levels, or eigenstates of the quantum variable, are labeled  $|0\rangle$  and  $|1\rangle$ . This has a direct correspondence with the discrete states of a classical bit, 0 and 1. However, a qubit is a quantum state, and as such can be in a superposition state also. That is, in addition to  $|0\rangle$  and  $|1\rangle$ , a qubit can exist more generally in the state,  $c_0|0\rangle + c_1|1\rangle$ , where  $c_0$  and  $c_1$  are complex coefficients normalized to 1. This is the main distinction between classical and quantum memory, in that even though a qubit has discrete eigenstates, there is something analog about it also in the continuous range of superposition states that it can take on. In fact, there are really an infinite number of states possible to a qubit, not just two, although only two of these are orthogonal or linearly independent.

There is another aspect of quantum mechanics, namely measurement collapse, that affects how the information is read out from a qubit. Even though a qubit can exist in an arbitrary superposition state, a measurement on it will always find it in one of the two eigenstates,  $|0\rangle$  or  $|1\rangle$ , according to the measurement postulate of quantum mechanics. This means that upon measurement, a qubit loses its quantum character and reduces effectively to a bit. The irreversibility of measurement has important consequences for quantum computation, as we will see shortly.

If we have more than one qubit in our quantum system, we can express its state in terms of product eigenstates. For example, a two-qubit system has the basis states,  $|00\rangle$ ,  $|01\rangle$ ,  $|10\rangle$ ,  $|11\rangle$ , which are the quantum analogs of the input lines in the truth table for a classical logic gate. Unlike classical bits however, two or more qubits can interfere with one another, creating a macroscopically coherent superposition, of the form  $c_{00}|00\rangle + c_{01}|01\rangle + c_{10}|10\rangle + c_{11}|11\rangle$ . More generally,  $n$  qubits will have  $2^n$  product eigenstates, or dimensions, in their Hilbert space. These will form the computational basis of an  $n$ -qubit quantum computer.

It has been suggested that there are superficial similarities between a qubit and the classical nat, the analog unit of memory used in classical continuous computation, since a qubit can also exist in a continuous range of superposition states as we saw. There are a couple of important differences between a qubit and a nat however. The first is that upon measurement, a qubit always reduces to a bit, while a nat may retain its analog character after measurement. The more substantive difference between the two has to do with entanglement. Entanglement is another name for the quantum correlation between two observables. While a nat may be able to represent products of a superposition state of two qubits, of the form  $[c_0|0\rangle_1 + c_1|1\rangle_1][c_0'|0\rangle_2 + c_1'|1\rangle_2]$ , it cannot fully represent entangled states like  $(1/\sqrt{2})[|01\rangle + |10\rangle]$ , where the state of one qubit is determined by a measurement on the other. When the observables describing the two qubits are classically distinguishable, or macroscopically separated, measurement of entangled states enables quantum nonlocality, an issue that has significance in fundamental quantum theory as well as in some recent applications of quantum communication (like quantum teleportation).

Once we have established the memory units of a quantum computer, the next question to address is what constitutes the computation itself. That is, what kind of functions can be computed, or problems solved, using quantum mechanics. We saw that in classical reversible computing, the types of functions that can be computed are reversible Boolean functions, with equal numbers of inputs and outputs. The identifying feature of a quantum transformation is that it has to be

unitary. This was early recognized by researchers in quantum computing. The prototypical example of a unitary operator is quantum time evolution, where the Hamiltonian describing the quantum system is exponentiated to produce a unitary operator,

$$U(t_f, t_i) = \exp[i \int_{t_i}^{t_f} H(t) dt] \quad (13)$$

So our model for how a quantum computer processes information is that some unitary operation carries the input qubits to the output qubits. This can be represented in the form of a block diagram, similar to diagram (1),

$$\begin{array}{ccc}
 q_1 & \longrightarrow & \boxed{U(q_1, \dots, q_n)} & \longrightarrow & q'_1 \\
 \vdots & \longrightarrow & & \longrightarrow & \vdots \\
 q_n & \longrightarrow & & \longrightarrow & q'_n
 \end{array} \quad (14)$$

The integral in Eq. (13) illustrates that in the case of time-evolution, it is possible to achieve the same unitary transformation with different Hamiltonians, since the former is only concerned with the input and output of the transformation and not the intermediate state of the system. This is one reason for why there are so many different proposals for performing quantum computation in different physical settings, using trapped-ions (which we will describe later), cavity-QED, solid-state and superconducting devices, to name a few.

Since  $U^{-1} = U^\dagger$ , quantum computation is a reversible process, logically and physically. Indeed, all reversible Boolean functions are special cases of unitary transformations. For example, if a three-bit function mapped  $|001\rangle$  to  $|010\rangle$ , the inverse would map  $|010\rangle$  back to  $|001\rangle$ , which is what we would get if we took the transpose (or hermitian adjoint) of the corresponding unitary matrix. Hence, loosely speaking, any problem that can be simulated classically can also be simulated quantum mechanically. However, the new features of the quantum computer – superposition, interference between qubits, entanglement, and measurement – allow quantum computers to solve certain problems, like the factoring and database search problems, exponentially faster than can be done on any classical computer.

While unitary matrices are reversible operations on a system of qubits, the read-out process is a quantum-mechanical measurement that collapses the system to an eigenstate, an inherently irreversible process. This places severe constraints on algorithms that solve problems on a quantum computer. A quantum algorithm with classical inputs has to find a way to evolve them

into near-classical outputs again for efficient read-out, even though the intermediate state of the system will be decidedly unclassical. This is achieved dramatically in Shor's quantum algorithm for factoring numbers.

The problem of universality can be posed for quantum computation as well, in asking whether arbitrary unitary operations can be broken down into simpler ones. Similar to classical logic, quantum logic gates exist that operate on a handful of qubits at a time, and that are able to simulate arbitrary unitary operations. Note that it is a property of unitary matrices that a product of two of them remains unitary, hence a product of unitary logic gates will also be unitary. Before we describe the universal quantum logic gates, we introduce the terminology with three well known quantum gates, the NOT, the XOR, and the Walsh-Hadamaard gates. We have already described the NOT and XOR gates in classical reversible logic [see tables (7, 8)]. A straight-forward quantum generalization of these gates are the unitary matrices,

$$U_{\text{NOT}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{matrix} |0\rangle \\ |1\rangle \end{matrix} \quad U_{\text{XOR}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{matrix} |00\rangle \\ |01\rangle \\ |10\rangle \\ |11\rangle \end{matrix} \quad (15)$$

which describe the evolution of the two product eigenstates of the one-qubit NOT gate and the four products eigenstates of the two-qubit XOR gate. Note that unlike their classical counterparts, these quantum gates can transform superposition states as well. For example, operations of the form,  $U_{\text{NOT}}: c_0|0\rangle + c_1|1\rangle \rightarrow c_0|1\rangle + c_1|0\rangle$ , are also possible with the quantum NOT gate. Indeed, quantum superpositions can be created from eigenstates and vice versa, by the much discussed Walsh-Hadamaard transform, which in its one-qubit guise, takes the form

$$U_{\text{W-H}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{matrix} |0\rangle \\ |1\rangle \end{matrix} \quad (16)$$

For example,  $U_{\text{W-H}}$  maps the eigenstate  $|1\rangle$  to  $2^{-1/2} [|0\rangle - |1\rangle]$ , and vice versa. The Walsh-Hadamaard transform is a quantum gate that has no classical analog. It takes each input eigenstate to more than one eigenstate, with different alternating phases (+1 or -1).

## II.B. Universal Quantum Logic Gates

### Three-qubit gates

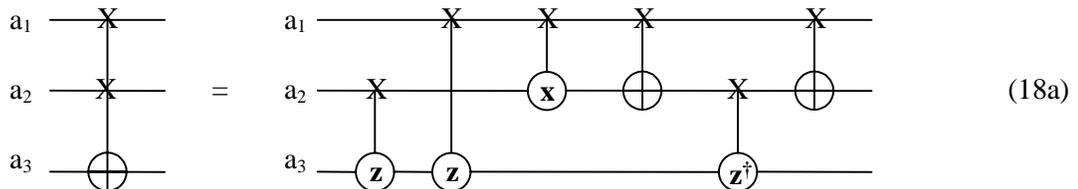
The quantum gates in Eqs. (15) and Eq. (16) are not sufficient for generating all unitary matrices. The first progress toward identifying universal quantum gates was made in 1989 by Deutsch [8]. Generalizing the three-bit Toffoli gate, which is known to be universal for reversible Boolean logic, he identified a three-qubit gate that is universal for quantum logic. The Deutsch gate operates on three qubits at a time,

$$D(\alpha) = \begin{bmatrix} 1 & & & & & & & & \\ & 1 & & & & & & & \\ & & 1 & & & & & & \\ & & & 1 & & & & & \\ & & & & 1 & & & & \\ & & & & & i \cos \alpha & \sin \alpha & & \\ & & & & & \sin \alpha & i \cos \alpha & & \\ & & & & & & & & 1 \\ & & & & & & & & & 1 \end{bmatrix} \begin{matrix} |000\rangle \\ |001\rangle \\ |010\rangle \\ |011\rangle \\ |100\rangle \\ |101\rangle \\ |110\rangle \\ |111\rangle \end{matrix} \quad (17)$$

As a function of the parameter  $\alpha$ , Eq. (17) really describes a family of gates. However, due to the property that  $D(\alpha) D(\alpha') = i D(\alpha + \alpha')$ , the parameter  $\alpha$  can be fixed as an irrational multiple of  $\pi$ , isolating one gate that can generate the whole family of gates,  $D(\alpha)$ , asymptotically. Note that  $D(\pi/2) = T_3$ , the three-bit Toffoli gate shown in table (9). The Deutsch gate is universal for quantum logic in the sense that any unitary transform on an arbitrary number of qubits can be simulated by repeated applications of  $D(\alpha)$  on three qubits at a time.

### Two-qubit gates

In 1995, several researchers working independently announced that the three-qubit Deutsch gate is not elementary, and that it can further be decomposed into two-qubit gates [10-12]. This came as a big surprise, since it was known that classical reversible logic required three-bit gates for universality. In Eq. (12), we illustrated why the three-bit Toffoli gate cannot be constructed from two-bit XOR gates. However, it turns out that the three-bit Toffoli gate  $T_3$ , a special case of the Deutsch gate, can indeed be simulated quantum mechanically using only two-qubit gates,



where the gates on the right side are done left to right, and  $x$  and  $z$  are the single-qubit gates,

$$x = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \begin{array}{l} |0\rangle \\ |1\rangle \end{array} \quad z = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix} \begin{array}{l} |0\rangle \\ |1\rangle \end{array} \quad (18b)$$

and the graphical symbol consisting of a vertical line starting from a solid circle and ending in an encircled  $x$  or  $z$  represents a conditional two-qubit gate which applies  $x$  or  $z$  to the second qubit conditional on the first qubit being in  $|1\rangle$ .  $x$  and  $z$  are both non-classical gates, since  $x$  describes a phase rotation and  $z$  transforms the inputs to superpositions. Comparing Eq. (18a) with Eq. (12), we see that the quantum features of complex coefficients and superposition states enabled the quantum decomposition of the three-bit Toffoli gate in Eq. (18a) while a purely classical decomposition was not possible in Eq. (12).

There are a few different ways of specifying the two-qubit gates universal for quantum computation (of which  $x$  and  $z$  are special cases). A family of such gates exist that can be described using three parameters [12], similar to the Deutsch gate being described by one parameter. To see this, note that a general way of writing a 2-dimensional unitary matrix, except for an overall phase factor, is [13]

$$y(\lambda, \nu, \phi) = \begin{bmatrix} \cos \lambda & -e^{i\nu} \sin \lambda \\ e^{i(\phi-\nu)} \sin \lambda & e^{i\phi} \cos \lambda \end{bmatrix} \quad (19)$$

The gates  $x$  and  $z$ , and indeed all single-qubit transforms, are special cases of Eq. (19). To see this, first note that  $y$  has the property of being able to map an arbitrary qubit state to the eigenstate  $|1\rangle$ . For a state,  $c_0|0\rangle + c_1|1\rangle$ , choosing  $\cos \lambda = |c_0|$ ,  $\nu = \arg\{c_0/c_1\}$ , and  $\phi = \arg\{c_1\}$  yields the transformation,  $y(\lambda, \nu, \phi): c_0|0\rangle + c_1|1\rangle \rightarrow |1\rangle$ . Since  $y$  is unitary and reversible, it follows from this any state of the qubit can be taken to any other using  $y$ , establishing the sufficiency of  $y$  for all single-qubit unitary operations. A family of universal two-qubit gates can be built using  $y$ ,

$$\Gamma_2[y] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos \lambda & -e^{i\nu} \sin \lambda \\ 0 & 0 & e^{i(\phi-\nu)} \sin \lambda & e^{i\phi} \cos \lambda \end{bmatrix} \quad \begin{array}{c} \text{---} \times \text{---} \\ | \\ \text{---} \textcircled{y} \text{---} \end{array} \quad (20)$$

The notation  $\Gamma_2[y]$  means that this is a 2-qubit gate which applies  $y$  to the second qubit conditional on the first qubit being in  $|1\rangle$ . Note that  $\Gamma_2[y]$  contains  $y$  since an operation on a

single qubit can be performed conditional on another qubit being in  $|0\rangle$  first, and then in  $|1\rangle$ . The graphical illustration of  $\Gamma_2[y]$  is also shown in Eq. (20). The set of gates,  $\Gamma_2[y(\lambda, \nu, \phi)]$ , is universal for quantum logic in the sense that any unitary transform, on any number of qubits, can be simulated by these gates acting on only two qubits at a time. The three parameters in  $\Gamma_2[y]$  can be irrational multiples of  $\pi$  and of each other for the purpose of isolating a single universal gate.

Before proving the universality of  $\Gamma_2[y]$ , we should clarify one point. Even though  $\Gamma_2[y]$  is a two-qubit gate, we see that the unitary matrix in Eq. (20) is effectively two-dimensional, not four-dimensional, because of the identity matrix acting on the first two states,  $|00\rangle$  and  $|01\rangle$ . Since any unitary transform can be built from this gate, this implies that 2-dimensional unitary matrices are universal. However, it does not follow from this that one-qubit gates are universal, as indeed they are not. The distinction between the number of qubits and the number of dimensions affected by a transformation can be illustrated by an example. Consider a NOT operation on the second qubit of a two-qubit system. This is represented by the unitary matrix,

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{array}{l} |00\rangle \\ |01\rangle \\ |10\rangle \\ |11\rangle \end{array} \quad (21)$$

which is four-dimensional, in the sense that all four states are affected by the transformation, even though the operation represented is a single-qubit NOT. In general, a one-qubit operation need not be two-dimensional, and similarly, a two-qubit operation need not be four-dimensional, etc.

*Proof that two-qubit gates are universal for quantum computation*

We will sketch one proof of why the two-qubit gates,  $\Gamma_2[y]$ , are universal for quantum logic. We need to show that an arbitrary unitary matrix can be simulated in terms of these gates. First, we extend the notation to define an  $m$ -qubit gate,  $\Gamma_m[y]$ , which applies  $y$  – any one of the family of gates in Eq. (19) – to the  $m$ th qubit, conditional on the first  $m-1$  qubits all being in  $|1\rangle$ . When  $m = 1$ , we have just a single-qubit  $y$  gate,  $\Gamma_1[y] = y$ . The matrix representation of  $\Gamma_m[y]$  will have 1's along the main diagonal except for the last two product states,  $|1\dots 10\rangle$  and  $|1\dots 11\rangle$ , which will be transformed by  $y$  [see Eq. (20) for an illustration of  $m = 2$ ]. Recall that  $y$  is universal for single-qubit transforms, and this allows  $\Gamma_m[y]$  to mimic its operation on a two-dimensional subspace of the  $n$ -qubit system. In particular, we showed [see after Eq. (19)] that  $y$  had the property of being

able to transform any single-qubit superposition to the eigenstate  $|1\rangle$ . In particular,  $y$  contains the phase gate  $x$  defined in Eqs. (18b) for arbitrary phases,  $y(0, 0, \phi): |0\rangle \rightarrow |0\rangle, |1\rangle \rightarrow e^{i\phi}|1\rangle$ .

Our state space will consist of  $n$  qubits, or  $2^n$  dimensions, and we let  $U$  be an arbitrary unitary operator on these  $n$  qubits. Let  $|\Psi_m\rangle$  and  $e^{i\psi_m}$  be the eigenstates and eigenvalues of  $U$ , for  $1 \leq m \leq 2^n$ . We can express one such eigenstate,  $|\Psi_m\rangle$ , in the basis of product states,

$$|\Psi_m\rangle = c_{0\dots 00} |0\dots 00\rangle + c_{0\dots 01} |0\dots 01\rangle + \dots + c_{1\dots 11} |1\dots 11\rangle \quad (22)$$

We begin the simulation by reducing  $|\Psi_m\rangle$  to the last product state,  $|1\dots 11\rangle$ . This is done by permuting each product state in the  $|\Psi_m\rangle$  expansion with the last-but-one product state,  $|1\dots 10\rangle$ , and applying  $\Gamma_n[y]$  to reduce the last qubit, spanned by  $|1\dots 10\rangle$  and  $|1\dots 11\rangle$ , to the eigenstate  $|1\dots 11\rangle$ . This is admittedly not an efficient process, since we have to do this for each of the  $2^n$  product states, but our aim is not to find a feasible means of simulation but rather any means, in analogy to the classical case where we used  $O[n2^n]$  NAND gates to simulate an  $n$ -bit Boolean function [see after Eqs. (6)].

To reduce  $|\Psi_m\rangle$  to  $|1\dots 11\rangle$ , we see that we need two kinds of operations, the  $n$ -qubit  $\Gamma_n[y]$  gate, and a 2-dimensional permutation operator that permutes any two of the  $2^n$  product states while leaving the others untouched. As we illustrated in expression (21), such a permutation operator need not be confined to act within a single qubit, but can in general involve all  $n$  qubits [eg. permuting  $|0\dots 00\rangle$  and  $|1\dots 11\rangle$ ]. We now show that such a permutation can itself be simulated using  $\Gamma_n[y]$  and single-qubit  $y$  gates. Suppose we want to permute  $|a_1, a_2, \dots, a_n\rangle$  with  $|b_1, b_2, \dots, b_n\rangle$ , where  $a_1, \dots, a_n$  and  $b_1, \dots, b_n$  are each 0 or 1. We do this one qubit at a time, starting with the first. We apply  $y$  gates to all but the first qubit to send them to  $|1\rangle$ , then apply  $\Gamma_n[y]$  to convert the first qubit from  $|a_1\rangle$  to  $|b_1\rangle$  conditional on all the other qubits being in  $|1\rangle$ , and finish by applying the inverse of the  $n-1$   $y$  gates to restore the rest of the qubits to their original values,

$$|a_1, a_2, \dots, a_n\rangle \xrightarrow{\{y\}} |a_1, 1, \dots, 1\rangle \xrightarrow{\Gamma_n[y]} |b_1, 1, \dots, 1\rangle \xrightarrow{\{y\}} |b_1, a_2, \dots, a_n\rangle \quad (23)$$

Repeating this procedure for each qubit, one at a time, simulates the whole permutation. Thus, we have shown that an eigenstate  $|\Psi_m\rangle$ , an eigenstate of  $U$ , can be reduced to  $|1\dots 11\rangle$  using only the  $n$ -qubit gate  $\Gamma_n[y]$ , supplemented by one-qubit gates  $y$ . Let us label this reduction by the operator  $W_m$ ,

$$W_m: |\Psi_m\rangle = c_{0\dots 00} |0\dots 00\rangle + \dots + c_{1\dots 11} |1\dots 11\rangle \rightarrow |1\dots 11\rangle \quad (24)$$

$W_m$  is contained in  $\Gamma_n[y]$ . Similarly, the n-qubit phase gate,

$$X_m = \Gamma_n[y(0, 0, \Psi_m)]: |1\dots 11\rangle \rightarrow e^{i\Psi_m} |1\dots 11\rangle \quad (25)$$

where is the eigenphase of  $U$  corresponding to  $|\Psi_m\rangle$ , is also contained in  $\Gamma_n[y]$ . Now following an argument by Deutsch [8], we consider the combination of transforms,  $W_m^\dagger X_m W_m$ , and show that  $W_m^\dagger X_m W_m$  maps  $|\Psi_m\rangle$  to  $e^{i\Psi_m} |\Psi_m\rangle$ , and  $|\Psi_{m'}\rangle$  to  $|\Psi_{m'}\rangle$  for  $m' \neq m$ . First, when  $m' = m$ , we know that  $X_m W_m$  maps  $|\Psi_m\rangle$  to  $e^{i\Psi_m} |1\dots 11\rangle$ . Since  $W_m$  is unitary, this means that  $W_m^\dagger = W_m^{-1}$  maps  $|1\dots 11\rangle$  to  $|\Psi_m\rangle$ . When  $m' \neq m$ , note that

$$\langle 1\dots 11 | W_m |\Psi_{m'}\rangle = \langle \Psi_m | W_m^\dagger W_m |\Psi_{m'}\rangle = \langle \Psi_m | \Psi_{m'}\rangle = 0 \quad (26)$$

where we have used the unitarity of  $W_m$  to recognize  $W_m^\dagger W_m$  as the identity, and the orthogonality of  $\{|\Psi_m\rangle\}$ . Eq. (26) shows that  $W_m |\Psi_{m'}\rangle$  has no component along  $|1\dots 11\rangle$ , which means that  $X_m$  will not affect this state. Hence  $W_m^\dagger X_m W_m |\Psi_{m'}\rangle = W_m^\dagger W_m |\Psi_{m'}\rangle = |\Psi_{m'}\rangle$  when  $m' = m$ . Since  $W_m^\dagger X_m W_m$  acts as an identity to all states  $|\Psi_{m'}\rangle$  except for  $m' = m$ , we have that

$$\prod_{m=1}^{2^n} (W_m^\dagger X_m W_m) = \sum_{m=1}^{2^n} e^{i\Psi_m} |\Psi_m\rangle \langle \Psi_m| = U \quad (27)$$

expressing that the product operator on the left shares the same eigenvectors and eigenvalues, and hence must be the same as,  $U$ . Since  $W_m$  and  $X_m$  are contained in  $\Gamma_n[y]$ , we have that  $\Gamma_n[y]$  simulates  $U$ .

It remains to show that the n-qubit  $\Gamma_n[y]$  gates can be simulated with the two-qubit gates,  $\Gamma_2[y]$ . We do this by showing that  $\Gamma_n[y(\lambda, v, \phi)]$ , can be written in terms of  $\Gamma_2[y(\lambda, v, \phi)]$  and the n-bit version of the Toffoli gate,  $T_n = \Gamma_n[y(\pi/2, \pi, \pi)]$ , which applies a NOT to the nth qubit when the first n-1 qubits are in  $|1\rangle$ . Since  $T_n$  is a Boolean gate, we know that it can be reduced to the three-bit Toffoli gate  $T_3$ , which is univereal for classical reversible logic [see section I.B], and we have already shown [see Eq. (18a)] that  $T_3$  is contained in  $\Gamma_2[y]$ . We now show how  $\Gamma_n[y]$  can be built from  $T_n$  and  $\Gamma_2[y]$  graphically. The  $\Gamma_n[y]$  gate is represented by a vertical line containing n-1 control  $X$ 's and one encircled  $y$  symbol [see left side of Eq. (28)], signifying that the first n-1 qubits are checked to see if they are in  $|1\rangle$  before performing  $y$  on the nth qubit. Such a gate can be simulated using two  $T_n$  gates and one  $\Gamma_2[y]$  gate,

(28)

where the dashed horizontal line represents an auxiliary qubit which is initialized to  $|0\rangle$  and is returned to  $|0\rangle$  at the end. The first Toffoli gate  $T_n$  on the right side checks the first  $n-1$  qubits to see if they are in  $|1\rangle$  and stores this result in the auxiliary qubit by converting it to  $|1\rangle$  conditionally. The two-bit  $\Gamma_2[y]$  gate that follows collects this information from the auxiliary qubit and performs  $y$  on the  $n$ th qubit conditionally. The last  $T_n$  is to restore the auxiliary qubit to  $|0\rangle$  for possible re-use.

### *Properties of the XOR gate*

Before considering how two-qubit gates may be implemented physically in the laboratory, we emphasize the importance of one of these gates, the XOR gate, shown in table (8) and Eqs. (15). The XOR gate is useful not only for its quantum logic properties but also for its use in quantum error correction and teleportation. Note that the XOR gate is a special case of the universal two-qubit family of gates,  $U_{\text{XOR}} = \Gamma_2[y(\pi/2, \pi, \pi)]$ . It performs a C-NOT operation, and is the two-bit version of the three-bit Toffoli gate,  $U_{\text{XOR}} = T_2$ . The properties of the XOR gate that we describe below are summarized in ref. [15]

We showed above that the two-qubit family of gates,  $\Gamma_2[y(\lambda, v, \phi)]$ , is universal for quantum logic. Soon after this was known, it was determined that the XOR gate alone, together with single-qubit unitary operations  $y$ , is universal for quantum logic [13]. That is,  $\Gamma_2[y(\lambda, v, \phi)]$  can be written in terms of the XOR and one-qubit gates,

(29)



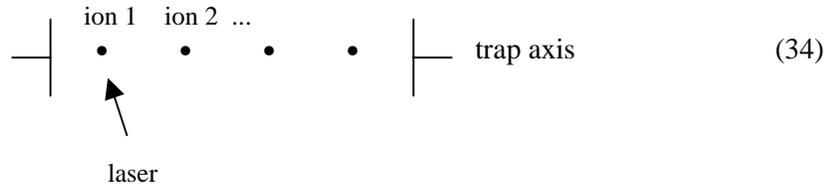
and subsequently measuring this auxillary qubit, we can force the state of  $a_1$  and  $a_2$  to having components only along even (or odd) product states. For example, the above two XORs transform

$$[c_{00}|00\rangle + c_{01}|01\rangle + c_{10}|10\rangle + c_{11}|11\rangle] |0\rangle_{\text{aux}} \rightarrow [c_{00}|00\rangle + c_{11}|11\rangle] |0\rangle_{\text{aux}} + [c_{01}|01\rangle + c_{10}|10\rangle] |1\rangle_{\text{aux}} \quad (33)$$

where the value of the auxillary qubit at the end labels the parity of the computational states. Such a parity-measurement is “non-demolitional” only in the sense that the coherence within a basis of a given parity is unaffected by the measurement of the auxillary qubit. This gives a sense of why the XOR is useful in quantum error correction routines, where the coherence of the computational qubits that we are correcting should not be affected by direct measurement.

### *Implementaton of two-qubit gates using the linear ion-trap*

The first proposal for implementing two-qubit quantum logic gates was made by Cirac and Zoller [14] in 1995. They considered two-level ions laser-cooled and trapped in a harmonic linear trap, where the ions are confined to moving along one dimension and the trap potential is quadratic in distance.



In the so-called Lamb-Dicke Limit (LDL), where each ion vibrates less than an optical wavelength, the external motion of the ions can be described by considering only the first normal mode of oscillation, called the center-of-mass mode, in which oscillate in unison at the trap frequency. The ions are assumed cooled to the ground state of this trap mode, and during the course of the computation, reach at most the first excited state of the trap mode.

Each ion is considered as a two-level system, effectively a qubit, and the trap ground and excited states are used as a “bus” to carry information from one ion to another for the purpose of a two-qubit gate. The unitary operations in this scheme are carried out by lasers applied to one ion at a time, as shown in diagram (34). Lasers applied at resonance to each ion produces Rabi oscillations that allow any single-qubit unitary operation to take place in that ion. Lasers that are detuned from ionic resonance by the trap frequency couple the internal states of the ion to the

external trap states of all the ions. This coupling takes place if the laser field has spatial dependence, as in the case of standing-wave field, and the ion's dipole moment couples to the field amplitude, which in turn depends on the position of the ion in the trap. Two-qubit operations are achieved in this scheme in two stages, first entangling the internal states of one ion with the trap states using a detuned standing-wave laser, and then transferring this information from the trap to the second ion by a similar field applied to it.

### **II.C. Multi-valued Quantum Logic** [For the continuous case, see [18, 19]]

Very recently, some progress has been made [16, 17] in generalizing quantum logic operations to the multi-valued domain, where the fundamental memory units are no longer two-state qubits, but rather  $d$ -valued *qudits*. Although a qudit can be said to have the same information as  $\log_2 d$  qubits, since they span the same Hilbert space, the measurement of a qudit is assumed to yield only one value, not  $\log_2 d$  values, corresponding to the eigenstate that the  $d$ -level quantum system collapsed to.

The main motivation for making the transition from binary to multi-valued quantum logic is to avail of the greater information capacity of multi-level atomic systems. Using more than two levels in each ion in the linear ion trap scheme for example, we could reduce the number of ions needed to be stored in the trap. This is an advantage because the bottleneck for implementing this scheme, and in many others, is the maintenance of a macroscopically coherent state of all the ions for a sufficiently long time before, subject to environmental noise, the coherence vanishes and the computation is corrupted. Another difference between binary and multi-valued logic that applies equally well to the classical and quantum domains is the trade-off in processing time – executing a large number of small (2 or 4-dim) binary gates versus a small number of large ( $d$  or  $d^2$ -dim) multi-valued gates. If large single-qubit operations are more viable than doing many small ones in sequence, then the multi-valued case will have an advantage.

The problem of universality has been addressed in the multi-valued domain [16] and it has been found that similar to the binary case, two-qudit operations suffice for performing arbitrary unitary operations on any number of qudits. Fault-tolerant error correction schemes have also been advanced for multi-valued quantum logic [17].

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