

Classical and quantum representation theory

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Abstract

These notes present an introduction to an analytic version of deformation quantization. The central point is to study algebras of physical observables and their irreducible representations. In classical mechanics one deals with real Poisson algebras, whereas in quantum mechanics the observables have the structure of a real non-associative Jordan-Lie algebra. The non-associativity is proportional to \hbar^2 , hence for $\hbar \rightarrow 0$ one recovers a real Poisson algebra. This observation lies at the basis of ‘strict’ deformation quantization, where one deforms a given Poisson algebra into a C^* -algebra, in such a way that the basic algebraic structures are preserved.

Our main interest lies in degenerate Poisson algebras and their quantization by non-simple Jordan-Lie algebras. The traditional symplectic manifolds of classical mechanics, and their quantum counterparts (Hilbert spaces and operator algebras which act irreducibly) emerge from a generalized representation theory. This two-step procedure sheds considerable light on the subject.

We discuss a large class of examples, in which the Poisson algebra canonically associated to an (integrable) Lie algebroid is deformed into the Jordan-Lie algebra of the corresponding Lie groupoid. A special case of this construction, which involves the gauge groupoid of a principal fibre bundle, describes the classical and quantum mechanics of a particle moving in an external gravitational and Yang-Mills field.

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1 Introduction

In quantization theory one tries to establish a correspondence between a classical mechanical system, and a quantum one. The traditional method, already contained in the work of Heisenberg and Dirac, is canonical quantization. Attempts to generalize this procedure, and put it on a solid mathematical footing have led to geometric quantization [49, 24, 20]. This is a certain algorithm which still contains many gaps, and for various reasons cannot be considered satisfactory [44]. The same comment applies to path integral quantization, but we hasten to remark that both techniques have led to many examples, constructions, and insights, in physics as well as mathematics, that would have been hard to reach otherwise, and still provide the main testing ground for alternative methods.

One such alternative method is deformation quantization. The version that we use (and partly propose) employs techniques from algebra, differential geometry, and functional analysis, and appears to be very interesting from a mathematical point of view. One attempts to relate Poisson algebras to C^* -algebras in a way specified below, and as such it is possible to relate to, and exploit the phenomenal progress made in both subjects over the last decade. This progress has consisted of discovering and understanding general structures through specific examples, and in a certain sense a unification of the three mathematical disciplines mentioned above has been achieved, under the name of non-commutative geometry. On the operator-algebraic side this includes cyclic cohomology of operator algebras [14] and operator K-theory (non-commutative topology) [42], which have found interesting applications (highly relevant to quantization theory!) in foliation theory and generalized index theorems [32]. As to Poisson algebras, we mention Poisson cohomology [23] and the theory of symplectic groupoids [48].

From the point of view of physics we wish to stress that the quantization procedure discussed here is very satisfactory in that it places physical notions like observables and states at the forefront (inspired by algebraic quantum field theory [21]), plays down the (quite unnecessary) use of complex numbers in quantum mechanics,

and accurately describes a large class of examples relevant to Nature. Moreover, it brings classical and quantum mechanics very closely together and highlights their common structures.

We will introduce the relevant mathematical structures step by step, on the basis of the familiar Weyl quantization of a particle moving on \mathbb{R}^n . This will lead us to Poisson algebras, Jordan-Lie algebras, and C^* -algebras. We then introduce the appropriate notion of a representation of each of these objects, and motivate an irreducibility condition. Lie groups form a rich class of examples on which to illustrate the general theory, but since these only describe particles with nothing but an internal degree of freedom, we must look elsewhere for structures describing genuine physics. A rich structure that is tractable by our methods, and at the same time describes real physical systems, is that of a Lie groupoid [31, 16]. It has an associated ‘infinitesimal’ object (a Lie algebroid), and, as we will explain, the passage from an algebroid to a groupoid essentially amounts to quantization.

2 Classical mechanics and Poisson algebras

2.1 Introductory example: particle on flat space

Consider a particle moving on the configuration space $Q = \mathbb{R}^n$. We use canonical coordinates (x^μ, p_μ) (usually simply written as (x, p)) on the cotangent bundle $M = T^*\mathbb{R}^n$ ($\mu = 1, \dots, n$), so that (x, p) stands for the one-form $p_\mu dx^\mu \in T_x^*\mathbb{R}^n$. In mechanics a key role is played by the Poisson bracket

$$\{f, g\} = \frac{\partial f}{\partial x^\mu} \frac{\partial g}{\partial p_\mu} - \frac{\partial f}{\partial p_\mu} \frac{\partial g}{\partial x^\mu}, \quad (2.1)$$

where $f_1, f_2 \in C^\infty(M)$. Here $C^\infty(M) \equiv A_0$ stands for the real vector space of real-valued smooth functions on M . Its elements are classical observables. Apart from the Poisson bracket, there is another bilinear map from $A_0 \otimes_{\mathbb{R}} A_0 \rightarrow A_0$, namely the ordinary (pointwise) multiplication \cdot . Let us write $f\sigma g$ for fg ($\equiv f \cdot g$), and $f\alpha g$ for $\{f, g\}$. The algebraic operations σ and α satisfy the following properties:

1. $f\sigma g = g\sigma f$ (symmetry);

2. $f\alpha g = -g\alpha f$ (anti-symmetry);
3. $(f\alpha g)\alpha h + (h\alpha f)\alpha g + (g\alpha h)\alpha f = 0$ (Jacobi identity);
4. $(f\sigma g)\alpha h = f\sigma(g\alpha h) + g\sigma(f\alpha h)$ (Leibniz rule);
5. $(f\sigma g)\sigma h = f\sigma(g\sigma h)$ (associativity).

The meaning of α and σ is as follows. To start with the latter, we remark that the spectrum $\text{spec}(f)$ of a function $f \in C^\infty(M)$ is the set of values it takes (that is, the possible values that the observable f may have). If f is concretely given (i.e., we know “ $f(m_1) = a_1, f(m_2) = a_2 \dots$ ” then we obviously know the spectrum immediately. However, f may be regarded as an abstract element of the algebra A_0 . The point is now that $\text{spec}(f)$ is completely determined by its location in A_0 , equipped with the product σ (forgetting the Poisson bracket). Namely, if $a \in \text{spec}(f)$ then $f - a1$ (where 1 is the function on M which is identically equal to 1) fails to have an inverse in A_0 , whereas, conversely, $(f - a1)^{-1}$ is a well-defined element of A_0 , satisfying $(f - a1)^{-1}\sigma(f - a1) = 1$ if $a \notin \text{spec}(f)$. Hence we may *define* $\text{spec}(f)$ as the set of real numbers a for which $f - a1$ fails to have an inverse in A_0 . A closely related point is that σ allows one to define functions of observables (starting from $f^2 = f\sigma f$); this is related to the previous point via the spectral calculus.

The Poisson bracket α determines the role any observable plays as the generator of a flow on the space M of pure states on A_0 . To explain this, we need to introduce the concept of the state space of an algebra. The state space $\mathcal{S}(A)$ of a real algebra A may be defined as the space of normalized positive functionals on A , i.e., the linear maps $\omega : A \rightarrow \mathbb{R}$ which satisfy $\omega(f^2) \geq 0$ for all $f \in A$, and $\omega(1) = 1$. If ω_1 and ω_2 are states then $\lambda\omega_1 + (1 - \lambda)\omega_2$ is a state if $\lambda \in [0, 1]$. A state is defined to be pure if it does not allow such a decomposition unless $\lambda = 0, 1$; otherwise, it is called mixed. The physical interpretation of $\omega(f)$ is that this number equals the expectation value of the observable f in the state ω . Any point m of M defines a pure state on A_0 by $m(f) = f(m)$, and these in fact exhaust the set of pure states. This statement holds equally well if we had taken A_0 to be $C_c^\infty(M)$ or $C_0^\infty(M)$ (the smooth functions with compact support, and those vanishing at infinity, respectively), but the pure

state space of $C_b^\infty(M)$ (the bounded smooth functions) is the so-called Cech-Stone compactification of M .

Back to the Poisson bracket, each $f \in A_0$ defines a so-called Hamiltonian vector field X_f on M by

$$X_f g = \{g, f\}, \quad (2.2)$$

and this generates a Hamiltonian flow ϕ_t^f on M (as the solution of the differential equation $d\phi_t^f/dt = X_f(\phi_t^f)$), cf. [1, 29]. That X_f is indeed a vector field (i.e., a derivation of $C^\infty(M)$) is a consequence of the Jacobi identity on α .

The example $M = T^*\mathbb{R}^n$ has the following feature: any two points of M can be connected by a (piecewisely) smooth Hamiltonian flow. This property is equivalent to the following: $\{X_f(m) | f \in A_0\} = T_m M$ for all $m \in M$. That is, the Hamiltonian vector fields span the tangent space at any point of M .

To sum up, observables take values, and one may define functions of them, which two properties are determined by the product σ ; moreover, they generate flows of the pure state space, which are determined by the Poisson bracket α .

2.2 Poisson algebras and their representations

Definition 1 *A Poisson algebra is a vector space A over the real numbers, equipped with two bilinear maps $\alpha, \sigma : A \otimes_{\mathbb{R}} A \rightarrow A$ which satisfy the five conditions in the preceding subsection.*

The examples of Poisson algebras we will consider are of the type $A = C^\infty(M)$ for some manifold M , which has a Poisson structure, in the sense that α is some Poisson bracket and σ is multiplication. In that case, M together with the Poisson structure is called a Poisson manifold. If M has the special feature discussed after (2.2) that any two points can be joined by a piecewisely smooth Hamiltonian curve, then M is called symplectic. If not, we can impose an equivalence relation [47] \sim on M , under which $x \sim y$ iff x and y can be joined by a piecewisely smooth Hamiltonian curve. The equivalence class L_x of any point can be shown to be a manifold, which is embedded in M . If i is the embedding map then the relation $\{i^*f, i^*g\}_{L_x} = i^*\{f, g\}_M$ defines a Poisson structure $\{, \}_{L_x}$ on L_x , which is obviously

symplectic, and we call L_x a symplectic leaf of M . More advanced considerations show that any Poisson manifold is foliated by its symplectic leaves [29].

If $M = S$ is symplectic then the Poisson bracket can be derived from a symplectic form on S [1, 29]. The corresponding $A = C^\infty(S)$ are in some sense the ‘canonical models’ of Poisson algebras. This motivates the following

Definition 2 *A representation of a Poisson algebra A is a map $\pi_c^S : A \rightarrow C^\infty(S)$, where S is a symplectic manifold, satisfying the following conditions for all $f, g \in A$:*

1. $\pi_c^S(\lambda f + \mu g) = \lambda \pi_c^S(f) + \mu \pi_c^S(g)$, for all $\lambda, \mu \in \mathbb{R}$ (linearity);
2. $\pi_c^S(fg) = \pi_c^S(f)\pi_c^S(g)$ (preserves σ);
3. $\pi_c^S(\{f, g\}_M) = \{\pi_c^S(f), \pi_c^S(g)\}_S$ (preserves α);
4. The vector field $X_{\pi_c^S(f)}$ is complete if X_f is (self-adjointness).

The c in π_c^S stands for ‘classical’, and the above defines a ‘classical’ representation (as opposed to a ‘quantum’ representation of algebraic objects by operators on a Hilbert space; as we shall see later, the distinction between classical and quantum is actually blurred). A vector field is called complete if its flow exists for all times. If f had compact support then its flow is automatically complete [1]. Condition 4 excludes situations of the following type. Take $M = T^*\mathbb{R}$ with the usual Poisson structure (2.1), and take S any open set in M . If i is the embedding of S into M , with the Poisson structure borrowed from M by restriction, then $\pi_c^S(f) = i^*f$ satisfies 1-3 but not 4 (unless $S = M$).

The following theorem shows that all representations are actually of the type $\pi_c^S = J^*$, where $J : S \rightarrow M$ is a Poisson morphism.

Theorem 1 *Let M be a finite-dimensional Poisson manifold, $A = C^\infty(M)$ the corresponding Poisson algebra, and let $\pi_c^S : A \rightarrow C^\infty(S)$ be a representation of A . Then there exists a map $J : S \rightarrow M$ such that $\pi_c^S = J^*$.*

Proof. For the elementary C^* -algebra theory used in the proof, cf. e.g. [33, 43, 9]. Take $s \in S$, and define a linear functional $\tilde{J}(s)$ on $C_0^\infty(M)$ by putting $\langle \tilde{J}(s), f \rangle =$

$(\pi_c^S(f))(s)$ for $f \in C_0^\infty(M)$. By property 2 of a representation, $\tilde{J}(s)$ is multiplicative, hence positive and continuous, so it extends to a pure state on the commutative C^* -algebra $C_0(M)$ (which is the complexification of the norm-closure of $C_0^\infty(M)$). Hence by the Gel'fand isomorphism $\tilde{J}(s)$ corresponds to a point $J(s)$ of M . Hence we have found the required map $J : S \rightarrow M$. \square

For reasons to emerge in subsect. 2.3 below, we will refer to J as the *generalized moment map*. Property 3 of a representation implies that J is what is called a Poisson morphism. Such maps have been studied extensively in the literature [47, 29]. The self-adjointness condition 4 translates into a condition on J , which is called *completeness* by A. Weinstein. Examples suggest that it is actually a classical analogue of the condition on representations of real operator algebras on Hilbert spaces that these preserve self-adjointness (a special case of which is the familiar requirement that group representations be unitary). However, this self-adjointness condition is actually a completeness condition, too, for it guarantees that the unitary flow on Hilbert space generated by the self-adjoint representative of a given operator can be defined for all times (also cf. sect. 3 below). Further conditions on π_c^S could be imposed to guarantee that J is smooth, but as far as we can see we can develop the theory without those.

The following proposition (which is well known, cf. [47, 29]) is crucial for the analysis of irreducible representations (to be defined shortly). Here J_* denotes the push-forward of J [1].

Proposition 1 *Let $J : S \rightarrow M$ be the Poisson morphism corresponding to a representation π_c^S of the Poisson algebra $C^\infty(M)$. Then for any $f \in C^\infty(M)$*

$$J_*X_{\pi_c^S(f)} = X_f, \tag{2.3}$$

where X_f is the Hamiltonian vector field defined by f (etc.). Moreover, the image of the flow of $X_{\pi_c^S(f)}$ under J is the flow of X_f .

Proof. Take $g \in C^\infty(M)$ arbitrary. By definition of π_c^S and J , we have

$$\{\pi_c^S(f), \pi_c^S(g)\}_S(s) = \{f, g\}_M(J(s)) \tag{2.4}$$

Upon use of (2.2), this leads to the identity $(J_*X_{\pi_c^S(f)}g)(J(s)) = (X_fg)(J(s))$, whence the result. \square

Since S is symplectic, the symplectic form ω provides an isomorphism $\tilde{\omega} : T_s^*S \rightarrow T_sS$ for any $s \in S$. This is given by $\tilde{\omega}(df) = X_f$, or $\tilde{\omega}^{-1}(X) = i_X\omega$ (evaluated at any point s). Now let \tilde{T}_sS denote the subspace of T_sS which is spanned by Hamiltonian vectors (i.e., of the form $X_{\pi_c^S(f)}$, $f \in C^\infty(M)$, taken at s). Then

$$\tilde{T}_sS = \tilde{\omega} \circ J^*(T_{J(s)}^*M), \quad (2.5)$$

and $\tilde{\omega}$ is a bijection between \tilde{T}_sS and $J^*(T_{J(s)}^*M)$, where J^* is the pull-back of J (to 1-forms, in this case). This follows rapidly from the preceding proposition.

Definition 3 *A representation π_c^S of a Poisson algebra $C^\infty(M)$ is called irreducible if*

$$\{X_{\pi_c^S(f)}(s) | f \in C^\infty(M)\} = T_sS \quad \forall s \in S. \quad (2.6)$$

As mentioned before in a different variant, this condition guarantees that any two points in S can be joined by a piecewisely smooth curve, whose tangent vector field is of the form $X_{\pi_c^S(f)}$. Of course, since S is symplectic any two points can be joined by such a curve with tangent vectors X_g , $g \in C^\infty(S)$, even if π_c^S is not irreducible, but one may not be able to take $g = \pi_c^S(f)$. Note, that we could have broadened our definition of a representation by allowing S to be a Poisson manifold; in that case, however, the irreducibility condition would force S to be symplectic anyway. In the literature [47, 29] people appear to be mainly interested in the opposite situation, where a Poisson morphism $J : S \rightarrow M$ (S symplectic) is called *full* if (in our language) the corresponding representation $\pi_c^S = J^*$ is faithful. As the following result shows, this is indeed quite opposite to an irreducible representation, which has a large kernel unless M is symplectic itself.

Theorem 2 *If a representation $\pi_c^S : C^\infty(M) \rightarrow C^\infty(S)$ of a Poisson algebra is irreducible then S is symplectomorphic to a covering space of a symplectic leaf of M .*

Proof. We first show that $J : S \rightarrow M$ is an immersion. Namely, if $J_*X = 0$ for some $X \in T_s S$ then $\langle J_*X, \theta \rangle_{J(s)} = \langle X, J^*\theta \rangle_s = 0$, but by (2.5) and (2.6) any $\theta' \in T^*S$ may be written as $\theta' = J^*\theta$ for some $\theta \in T^*_{J(s)}M$. Hence $X = 0$, and J is an immersion. Since J is a Poisson morphism, it follows that S is locally symplectomorphic to $J(S) \subset M$.

Next, $J(S)$ must actually be a symplectic leaf of M . For suppose that there is a proper inclusion $J(S) \subset L$, where L is a symplectic leaf of M . It follows from the Darboux-Weinstein theorem [1, 29] that any point x in a symplectic space has a neighbourhood U_x such that any two points in U may be connected by a smooth Hamiltonian curve. If we take x to lie on the boundary of $J(S)$ in L , then we find that there exist $m_1 \in J(S)$ and $m_2 \in L$ which can be connected by a smooth curve γ with tangent vector field X_f , for some $f \in C^\infty(M)$. Let $m_1 = J(s_1)$, and consider the flow $\tilde{\gamma}$ of $X_{\pi_c^S(f)}$ starting at s_1 . By the proposition above, $J \circ \tilde{\gamma} = \gamma$. However, since $m_2 \notin J(S)$, the flow $\tilde{\gamma}$ must suddenly stop, which contradicts the self-adjointness (completeness) property 4 of a representation. Hence to avoid a contradiction we must have $J(S) = L$.

A similar argument shows that $J : S \rightarrow J(S)$ must be a covering projection. For J not to be a covering projection, there must exist a point $m \in M$, a neighbourhood V_m of m , and a connected component $J_i^{-1}(V_m)$ of $J^{-1}(V_m)$, so that $J(J_i^{-1}(V_m)) \subset V_m$ is a proper inclusion. But in that case we could choose points $s_1 \in J(J_i^{-1}(V_m))$ and $s_2 \in V_m$ which can be connected by a smooth hamiltonian curve, and arrive at a contradiction to the self-adjointness property of π_c^S . \square

2.3 The Lie-Kirillov-Kostant-Souriau Poisson structure

We obtain a basis class of Poisson algebras by taking $M = \mathfrak{g}^*$, which is the dual of the Lie algebra \mathfrak{g} of some Lie group G . We may regard $X \in \mathfrak{g}$ as an element of $C^\infty(\mathfrak{g}^*)$, by $X(\theta) = \langle \theta, X \rangle$, and the Poisson structure of \mathfrak{g}^* is completely determined by putting

$$\{X, Y\} = [X, Y] \tag{2.7}$$

(cf. [1, 29] for more information). The classical algebra of observables $C^\infty(\mathfrak{g}^*)$ describes a particle which doesn't move, but only has an internal degree of freedom (e.g., spin if $G = SU(2)$).

Let $\pi_c^S : C^\infty(\mathfrak{g}^*) \rightarrow C^\infty(S)$ be a representation of $C^\infty(M)$, with S connected. For each $X \in \mathfrak{g}$ we define a function f_X on S by

$$f_X = \pi_c^S(X). \quad (2.8)$$

By definition of a representation

$$\{f_X, f_Y\}_S = f_{[X, Y]}. \quad (2.9)$$

If \tilde{X} is the Hamiltonian vector field defined by f_X (so that $\tilde{X}g = \{g, f_X\}_S$) then (2.9) and the Jacobi identity imply that $[\tilde{X}, \tilde{Y}] = -[\tilde{X}, \tilde{Y}]$ (where the first bracket is the commutator of vector fields and the second one is the Lie bracket on \mathfrak{g}). By self-adjointness, the flow φ_t^X of \tilde{X} is defined for all t , and this leads to an action $\tilde{\pi}_c^S$ of $\exp X \in G$ on S by $\tilde{\pi}_c^S(\exp X)s = \varphi_1^X(s)$. If G is simply connected this eventually defines a proper symplectic action of G on S .

Conversely, let G act on a symplectic manifold S so as to preserve the symplectic form ω . We may then define a vector field \tilde{X} for each $X \in \mathfrak{g}$ by

$$(\tilde{X}f)(s) = \frac{d}{dt}f(e^{tX}s)|_{t=0}, \quad (2.10)$$

where we have written the action of $x \in G$ on $s \in S$ simply as xs . The action is called Hamiltonian $i_{\tilde{X}}\omega = df_X$ for some $f_X \in C^\infty(S)$ (this is guaranteed if $H^1(S, \mathbb{R}) = 0$), and strongly Hamiltonian if (2.9) is satisfied on top of that. If the former condition is met, one can define a map $J : S \rightarrow \mathfrak{g}^*$ by means of

$$\langle J(s), X \rangle = f_X(s), \quad (2.11)$$

with pull-back $J^* : C^\infty(\mathfrak{g}^*) \rightarrow C^\infty(S)$. In that case we clearly see from (2.8) that the map J defined by (2.11) is a special case of the generalized moment map constructed in Theorem 1. Indeed, J in (2.11) is called the moment(um) map in the literature [20, 1, 29]. (Note the varying sign conventions. We follow [1] in putting $i_{\tilde{X}}\omega = df_X$,

$X_f g = \{g, f\}$, and $\{f_X, f_Y\}_S = f_{[X, Y]}$, but the alternative convention $i_{\tilde{X}}\omega = -df_X$, $X_f g = \{f, g\}$, and $\{f_X, f_Y\}_S = -f_{[X, Y]}$ occurs as well.)

If the symplectic G -action on S is Hamiltonian but not strongly so, the right-hand side of (2.9) acquires an extra term, and this situation may be analyzed in terms of Lie algebra cohomology [20, 1, 29]. The result is that the Poisson bracket (2.7) can be modified, so that $\pi_c^S = J^*$ defines a representation of $C^\infty(\mathfrak{g}^*)$, equipped with the modified Poisson structure.

In the strongly Hamiltonian case J^* produces a representation $\pi_c^S \equiv J^*$ of $C^\infty(\mathfrak{g}^*)$ equipped with the Lie-Kirillov-Kostant-Souriau Poisson structure (2.7). The fact that J is a Poisson morphism may be found in [1, 29, 20], and it remains to check the self-adjointness condition. We observe that vector fields on S of the type $X_{\pi_c^S(f)}$ ($f \in C^\infty(\mathfrak{g}^*)$) are tangent to a G -orbit, so that their flow $\gamma_t^{\pi_c^S(f)}$ cannot map a point of S into a different orbit. This reduces the situation to the case where G acts transitively on S . In that case, the vector fields $\{\tilde{X} | X \in \mathfrak{g}\}$ span the tangent space of S at any point, so that π_c^S is irreducible. By Theorem 2, the image of J must be a symplectic leaf of \mathfrak{g}^* , hence a co-adjoint orbit (this shows, incidentally, that the famous Kostant-Souriau theorem which asserts that any symplectic space which allows a transitive strongly Hamiltonian action of a Lie group G is symplectomorphic to a covering space of a co-adjoint orbit of G [20, 29] is a special case of our Theorem 2). Now take $f \in C^\infty(\mathfrak{g}^*)$ with Hamiltonian vector field X_f and flow γ_t^f . Since (by definition) G acts transitively on any co-adjoint orbit in \mathfrak{g}^* , we may write $\gamma_t^f(\theta) = \pi_{\text{co}}(x_t)\theta$ for some curve x_t in G (not uniquely defined, and dependent on the argument $\theta \in \mathfrak{g}^*$); here π_{co} is the co-adjoint representation of G on \mathfrak{g}^* . We now use Proposition 1 and the equivariance of J (that is, $J \circ x = \pi_{\text{co}}(x) \circ J$ [20, 1, 29]) to derive $J \circ x_t(s) = J \circ \gamma_t^{\pi_c^S(f)}(s)$ for any $s \in S$; here x_t depends on $J(s)$. Since J is an immersion this implies $\gamma_t^{\pi_c^S(f)}(s) = x_t(s)$, hence $\gamma_t^{\pi_c^S(f)}$ is defined whenever x_t is; in particular, if γ_t^f is complete then $\gamma_t^{\pi_c^S(f)}$ is, so that the representation π_c^S is self-adjoint.

In passing, we have observed that the irreducible representations of $C^\infty(\mathfrak{g}^*)$ are given by the co-adjoint orbits in \mathfrak{g}^* (and their covering spaces).

3 Quantum mechanics and Jordan-Lie algebras

3.1 Weyl quantization on flat space

To introduce some relevant mathematical structures in a familiar context we briefly review the Weyl quantization procedure of a particle moving on $Q = \mathbb{R}^n$, with phase space $M = T^*\mathbb{R}^n$ (as in subsect. 2.1). It is convenient to introduce a partial Fourier transform of $f \in C^\infty(M)$ by

$$\check{f}(x, \dot{x}) = \int \frac{d^n p}{(2\pi)^n} e^{ip\dot{x}} f(x, p); \quad (3.1)$$

this makes \check{f} a function on the tangent bundle $T\mathbb{R}^n$, where we use canonical coordinates $(x, \dot{x}) \equiv \dot{x}^\mu \partial / \partial x^\mu \in T_x \mathbb{R}^n$. For (3.1) to make sense we must have that f is integrable in the fiber direction (i.e., over p). Let f be such that $\check{f} \in C_c^\infty(T\mathbb{R}^n)$; we refer to this class of functions as $\overline{\mathfrak{A}}_0$. We then define an operator $Q_\hbar(f)$ on the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^n)$ by

$$(Q_\hbar(f)\psi)(x) = \int d^n y \tilde{Q}_\hbar(f)(x, y)\psi(y), \quad (3.2)$$

with kernel

$$\tilde{Q}_\hbar(f)(x, y) = \hbar^{-n} \check{f}\left(\frac{x+y}{2}, \frac{x-y}{\hbar}\right). \quad (3.3)$$

This operator is compact (it is even Hilbert-Schmidt, since the kernel is in $C_c^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, and thus it is bounded. (The norm of an operator T on a Hilbert space \mathcal{H} is defined by $\|T\| = \sup_\psi (T\psi, T\psi)^{1/2}$, where the supremum is over all vectors ψ of unit length. An operator T is called bounded if this norm is finite. An operator is called compact if it may be approximated in norm by operators with a finite-dimensional range [36]. Compact operators behave to some extent like finite-dimensional matrices).

A crucial property of $Q_\hbar(f)$ is that it is self-adjoint (since f is real-valued). This means, that Q_\hbar may be regarded as a map from $\overline{\mathfrak{A}}_0$ into $\mathfrak{A} = \mathcal{K}(L^2(\mathbb{R}^n))_{\text{sa}}$ (the set of self-adjoint compact operators on $\mathcal{H} = L^2$). As a real subspace of $\mathcal{B}(\mathcal{H})$, \mathfrak{A} is itself a normed space, which is, in fact, complete (because $\mathcal{K}(\mathcal{H})$ is). We can make $\overline{\mathfrak{A}}_0$ into a real Banach space, too, by equipping it with the norm

$$\|f\|_0 = \sup_{m \in M} |f(m)|. \quad (3.4)$$

The completion of $\overline{\mathfrak{A}_0}$ under this norm is $\mathfrak{A}_0 = C_0(M, \mathbb{R})$ (the space of real-valued continuous functions on M which vanish at infinity).

We interpret $Q_{\hbar}(f)$ as the quantum observable corresponding to the classical observable f . Accordingly, we call \mathfrak{A} the (quantum) algebra of observables (of a particle on \mathbb{R}^n). As in the classical case, we may identify two algebraic operations on \mathfrak{A} (that is, bilinear maps $\mathfrak{A} \otimes_{\mathbb{R}} \mathfrak{A} \rightarrow \mathfrak{A}$). They are

$$A\sigma_{\hbar}B = \frac{1}{2}(AB + BA); \quad A\alpha_{\hbar}B = \frac{1}{i\hbar}(AB - BA). \quad (3.5)$$

The latter depends on \hbar , so we will rename \mathfrak{A} , equipped with σ_{\hbar} and α_{\hbar} , as \mathcal{A}_{\hbar} (the norm $\| \cdot \|$ does not depend on \hbar). One may verify the following properties:

1. $A\sigma_{\hbar}B = B\sigma_{\hbar}A$ (symmetry);
2. $A\alpha_{\hbar}B = -B\alpha_{\hbar}A$ (anti-symmetry);
3. $(A\alpha_{\hbar}B)\alpha_{\hbar}C + (C\alpha_{\hbar}A)\alpha_{\hbar}B + (B\alpha_{\hbar}C)\alpha_{\hbar}A = 0$ (Jacobi identity);
4. $(A\sigma_{\hbar}B)\alpha_{\hbar}C = A\sigma_{\hbar}(B\alpha_{\hbar}C) + B\sigma_{\hbar}(A\alpha_{\hbar}C)$ (Leibniz rule);
5. $(A\sigma_{\hbar}B)\sigma_{\hbar}C - A\sigma_{\hbar}(B\sigma_{\hbar}C) = \frac{\hbar^2}{4}(A\alpha_{\hbar}C)\alpha_{\hbar}B$ (weak associativity);
6. $\| A\sigma_{\hbar}B \| \leq \| A \| \| B \|$ (submultiplicativity of the norm);
7. $\| A^2 \| \leq \| A + B \|^2$ (spectral property of the norm).

We see that 1-4 are identical to the corresponding properties of a Poisson algebra, and 5 implies that we are now dealing with a deformation of the latter in a non-associative direction, in that the symmetric product σ_{\hbar} is now non-associative. A weak form of associativity does hold, this is the so-called associator identity

$$(A^2\sigma_{\hbar}B)\sigma_{\hbar}A = A^2\sigma_{\hbar}(B\sigma_{\hbar}A), \quad (3.6)$$

which can be derived from 1-5. The last two properties imply $\| A^2 \| = \| A \|^2$ [45], which leads to the usual spectral calculus.

Before commenting on the general structure we have found, let us find the meaning of the products σ_{\hbar} and α_{\hbar} (cf. subsect. 2.1). We start with σ_{\hbar} . In classical

and quantum mechanics alike, the spectrum of a self-adjoint operator is identified with the values the corresponding observable may assume. We have seen that the spectrum of a classical observable is determined by the symmetric product σ . In standard Hilbert space theory (which is applicable, as we have realized \mathfrak{A} as a set of operators acting on $\mathcal{H} = L^2$) the spectrum of a self-adjoint operator A is defined as the set of values of z for which the resolvent $(A - z)^{-1}$ fails to exist as an element of $\mathcal{B}(\mathcal{H})$ [36]. More abstractly, the spectrum of an element A of a C^* -algebra \mathcal{B} is defined by replacing $\mathcal{B}(\mathcal{H})$ by \mathcal{B} in the above [43, 9]. In fact, this definition only uses the anti-commutator (rather than the associative operator product, which combines the anti-commutator and the commutator), so that we conclude that the symmetric product on the algebra of observables determines the spectral content. This observation is originally due to Segal [40] (and was undoubtedly known to von Neumann, who introduced the anti-commutator), and a quick way to see this is that the spectrum of A is determined by the C^* -algebra $C^*(A)$ it generates; this is a *commutative* sub-algebra of \mathcal{B} (or, $\mathcal{K}(\mathcal{H})$ in our example above) which clearly only sees the anti-commutator σ_{\hbar} , which coincides with the associative product on $C^*(A)$ (cf. [22, 3.2]). This argument is closely related to the fact that the Jordan product σ_{\hbar} allows one to define functions of an observable, starting with $A^2 \equiv A\sigma_{\hbar}A$. Conversely, one could start with a squaring operation, and define the Jordan product by $A\sigma_{\hbar}B = 1/2((A + B)^2 - A^2 - B^2)$, cf. the Introduction of [9]. The connection between spectra and functions of observables is provided by the spectral calculus.

Next, we wish to relate the commutator α_{\hbar} to the role observables play as generators of transformations of the space of pure states. As explained prior to (2.2), we may introduce states of an algebra of observables as normalized positive linear functionals ω on \mathfrak{A} ; positivity here means that $\omega(A^2) \geq 0$ for all $A \in \mathfrak{A}$ (and $A^2 = A\sigma_{\hbar}A$ as before), and normalized means that $\|\omega\| = 1$ (which is equivalent to the property $\omega(1) = 1$ if \mathfrak{A} has a unit 1, which is not the case for $\mathfrak{A} = \mathcal{K}(\mathcal{H})$). The state space of $\mathcal{K}(\mathcal{H})$ may be shown to be the space of all density matrices on \mathcal{H} (i.e., the positive trace-class operators [36] with unit trace). Pure states are as defined before, and we may consider the weak*-closure $P(\mathfrak{A})$ of the set of all pure states of \mathfrak{A} . In our

example, any unit vector $\psi \in L^2$ defines a pure state ω_ψ by $\omega_\psi(A) = (A\psi, \psi)$, and, conversely, any pure state is obtained in this way. Noting that the space of pure states thus obtained is already weakly closed, we find that $P(\mathcal{K}(\mathcal{H}))$ is equal to the projective Hilbert space $P\mathcal{H}$ (which by definition is the set of equivalence classes $[\psi]$ of vectors of unit length, under the equivalence relation $\psi_1 \sim \psi_2$ if $\psi_1 = \exp(i\alpha)\psi_2$ for some $\alpha \in \mathbb{R}$). For example, $P\mathbb{C}^2 = S^2$ (the two-sphere) is the pure state space of the algebra of hermitian 2×2 matrices. More generally, $P\mathcal{H}$ is a Hilbert manifold modeled on the orthoplement of an arbitrary vector in \mathcal{H} . Hence $P\mathbb{C}^n$ is modeled on \mathbb{C}^{n-1} . To see this, take an arbitrary vector $\chi \in \mathcal{H}$ (normalized to unity), and define a chart on the open set $O_\chi \equiv \{\psi \in \mathcal{H} | (\psi, \chi) \neq 0\}$ by putting $\Phi_\chi : O_\chi \rightarrow \chi^\perp$ equal to $\Phi_\chi(\psi) = (\psi/(\psi, \chi)) - \chi$. (We assume the inner product to be linear in the first entry.)

The fundamental point is that $P\mathcal{H}$ has a Poisson structure [1]. To explain this, note first that $T_\psi\mathcal{H} \simeq \mathcal{H}$, since \mathcal{H} is a linear space; a vector $\varphi \in \mathcal{H}$ determines a tangent vector $\varphi_\psi \in T_\psi\mathcal{H}$ by its action on any $f \in C^\infty(\mathcal{H})$

$$(\varphi_\psi f)(\psi) = \frac{d}{dt}f(\psi + t\varphi)|_{t=0}. \quad (3.7)$$

The symplectic form ω on \mathcal{H} is then defined by

$$\omega(\varphi_\psi, \varphi'_\psi) = -2\hbar \text{Im}(\varphi, \varphi'). \quad (3.8)$$

We now regard $A \in \mathfrak{a}$ not as an operator on \mathcal{H} , but as a function \tilde{f}_A on \mathcal{H} , defined on $\psi \neq 0$ by

$$\tilde{f}_A(\psi) = \frac{(A\psi, \psi)}{(\psi, \psi)}. \quad (3.9)$$

(The value at $\psi = 0$ is irrelevant). The point is that this definition quotients to $P\mathcal{H}$, so that $A \in \mathfrak{a}$ defines a function f_A on $P\mathcal{H}$ in the obvious way. Also, the symplectic structure quotients down to $P\mathcal{H}$ (the professional way of seeing this [1] is that $U(1)$ acts on \mathcal{H} by $\psi \rightarrow \exp(i\alpha)\psi$, this action is strongly Hamiltonian and leads to a moment map $J : \mathcal{H} \rightarrow \mathbb{R}$ given by $J(\psi) = (\psi, \psi)$, and $P\mathcal{H}$ is the Marsden-Weinstein reduction $J^{-1}(1)/U(1)$), and this leads to the Poisson bracket

$$\{f_A, f_B\} = f_{A\alpha_n B}, \quad (3.10)$$

with α_{\hbar} defined in (3.5). An analogous equation determines the Poisson bracket on \mathcal{H} itself. As explained in (2.2) and below, the function \tilde{f}_A (hence A) defines a vector field \tilde{X}_A on \mathcal{H} , whose value at the point ψ is found to be

$$\tilde{X}_A(\psi) = -\frac{i}{\hbar}A\psi. \quad (3.11)$$

The flow $\tilde{\varphi}_t^A$ of this vector field is clearly

$$\tilde{\varphi}_t^A(\psi) = e^{-itA/\hbar}\psi. \quad (3.12)$$

Since this flow consists of unitary transformations of \mathcal{H} , it quotients to a flow φ_t^A on $P\mathcal{H}$, which is generated by a vector field X_A which is just the projection of \tilde{X}_A to the quotient space. This, in turn, is the vector field canonically related to $f_A \in C^\infty(P\mathcal{H})$ via the Poisson structure (3.10).

Parallel to the discussion following (2.2), we remark that that \mathfrak{A} acts on \mathcal{H} irreducibly, in the sense that any two points in (a dense subset of) \mathcal{H} may be connected by some flow generated by an element of \mathfrak{A} . By projection, a similar statement holds for flows on $P(\mathfrak{A}) = P\mathcal{H}$. By (3.11), this is equivalent to the property that the collection $\{A\psi | A \in \mathfrak{A}\}$ is dense in \mathcal{H} for each fixed ψ , and this, in turn, by (3.11) is exactly the irreducibility condition used in Definition 3 for Poisson algebras.

To sum up, we have shown that the product α_{\hbar} indeed leads to the desired connection between observables and flows on the pure state space of \mathcal{A}_{\hbar} (note that all the \mathcal{A}_{\hbar} are isomorphic to \mathfrak{A} for $\hbar \neq 0$), just as in the classical case.

Remarkably, the Jordan product σ_{\hbar} has a geometric expression in terms of the functions f_A on $P\mathcal{H}$, too [11]. Let g be the Kähler metric on \mathcal{H} , which is defined by (cf. (3.8))

$$g(\varphi_{\psi}, \varphi'_{\psi}) = \hbar \operatorname{Re}(\varphi, \varphi'). \quad (3.13)$$

Then a calculation shows that

$$f_{A\sigma_{\hbar}B} = \hbar g(\tilde{X}_A, \tilde{X}_B) + f_A f_B; \quad (3.14)$$

this should be compared with (3.10), which for this purpose may be rewritten as

$$f_{A\alpha_{\hbar}B} = \omega(\tilde{X}_A, \tilde{X}_B) + 0. \quad (3.15)$$

We see that the entire Jordan-Lie algebraic structure of \mathfrak{A} is encoded in the Kähler structure of $P\mathcal{H}$, which is given by hermitian metric Ω defined by the inner product:

$$\Omega(\varphi_\psi, \varphi'_\psi) = \hbar(\varphi, \varphi'). \quad (3.16)$$

Clearly, $\Omega = g - \frac{1}{2}i\omega$.

3.2 Jordan-Lie algebras

We now generalize some of these considerations.

Definition 4 *A real Banach space \mathfrak{A} equipped with two bilinear maps $\sigma_\hbar, \alpha_\hbar : \mathfrak{A} \otimes_{\mathbb{R}} \mathfrak{A} \rightarrow \mathfrak{A}$, which satisfy properties 1-7 in the preceding subsection, is called a Jordan-Lie algebra. If $\hbar \neq 0$ \mathfrak{A} is called non-associative, and if $\hbar = 0$ \mathfrak{A} is called associative. In the latter case the operation α_0 is only required to be densely defined.*

The Jordan-Lie structure of von Neumann's choice of $\mathcal{B}(\mathcal{H})$ as the algebra of observables in quantum mechanics was emphasized in [17]. We here propose that Jordan-Lie algebras are the correct choice to take as algebras of observables in quantum mechanics; allowing more possibilities than $\mathcal{B}(\mathcal{H})_{\text{sa}}$ or $\mathcal{K}(\mathcal{H})_{\text{sa}}$ allows the incorporation of superselection rules, and the quantization of systems on topologically non-trivial phase spaces [26]. The example above already illustrates the remarkable fact that *conventional quantum mechanics may be described without the use of complex numbers*. The reader may object that a factor i appears in (3.5), but the resulting product α_\hbar maps two self-adjoint operators into a self-adjoint operator, and it is the algebraic structure on \mathfrak{A} (given by σ_\hbar and α_\hbar), a real vector space, which determines all physical properties. Also, the (pure) state space is a real convex space and all observable numbers in quantum mechanics are of the form $\omega(A)$, where ω is a state and A an observable.

A first major advantage of starting from Jordan-Lie algebras is that Poisson algebras are a special case (in which the symmetric product is associative), obtained by putting $\hbar = 0$ in property 5. Hence classical and quantum mechanics are described by the same underlying algebraic structure (of which the former represents a limiting

case), a point not at all obvious in the usual description in terms of either symplectic manifolds or Hilbert spaces.

A second comment is that the axioms imply that \mathfrak{A} must be the self-adjoint part of a C^* -algebra, so that we recover a mathematical structure that has proved to be exceptionally fruitful in the study of quantum mechanics [40, 26], quantum field theory [21], statistical mechanics [10, 21], and pure mathematics [14, 13, 32, 42, 43, 33]. Indeed, we may define an associative multiplication on $\mathfrak{A}_{\mathbb{C}} = \mathfrak{A} \otimes_{\mathbb{R}} \mathbb{C}$ by means of

$$AB = A\sigma_{\hbar}B + \frac{1}{2}i\hbar A\alpha_{\hbar}B; \quad (3.17)$$

the associativity follows from the axioms, cf. [17]. The involution in $\mathfrak{A}_{\mathbb{C}}$ is simply given by the extension of $A^* = A$ for $A \in \mathfrak{A}$. The norm axioms imply that $\mathfrak{A}_{\mathbb{C}}$ thus obtained is a C^* -algebra.

The meaning of σ_{\hbar} and α_{\hbar} is the same as in the example of the compact operators. To explain this, it is convenient to use ‘Kadison’s function representation’ [25] of the self-adjoint part of any C^* -algebra (hence of any Jordan-Lie algebra). Let K be the state space of \mathfrak{A} (equipped with the weak*-topology); this space is compact if \mathfrak{A} has a unit, which we shall assume (if not, one can adjoin one in a canonical way without any loss of information [43, 22]). Then \mathfrak{A} is isometrically isomorphic with the space $A(K)$ of all affine real-valued continuous functions on K (with norm given by the supremum); since K is a convex subspace of the linear space of all continuous linear functionals on \mathfrak{A} , convex combinations $\lambda(\omega_1) + (1 - \lambda)\omega_2$ ($\lambda \in [0, 1]$) of states are well-defined, and a function f on K is called affine if $f(\lambda(\omega_1) + (1 - \lambda)\omega_2) = \lambda f(\omega_1) + (1 - \lambda)f(\omega_2)$ for all $\omega_i \in K$ and all $\lambda \in [0, 1]$ (cf. [43, III.6] for detailed information on such spaces). The isomorphism between $A \in \mathfrak{A}$ and $\tilde{A} \in A(K)$ is simply given by $\tilde{A}(\omega) = \omega(A)$. The spectral theory of \mathfrak{A} , which, as we have seen in the case $\mathfrak{A} = \mathcal{K}(\mathcal{H})_{\text{sa}}$, is governed by the symmetric product σ_{\hbar} (using an argument which extends to the general case), translates into a spectral theory for such affine functions [4]. Conversely, if one starts from $A(K)$ as the basic structure, one may set up a spectral calculus, which exploits the very special properties that K has because it is the state space of a C^* -algebra (hence, in particular, of a Jordan algebra). This

spectral theory may then be used to *define* σ_\hbar [4], making the intimate connection between the symmetric product and the spectral calculus even clearer than in the realization of \mathfrak{A} as operators on a Hilbert space.

By the affine property, an element of $A(K)$ is completely determined by its values on the pure state space $P(\mathfrak{A})$ (which is the w^* -closure of the extreme boundary of K [43, 33]). We can define an equivalence relation \sim on $P(\mathfrak{A})$, saying that $\omega_1 \sim \omega_2$ if both states give rise to unitarily equivalent representations (via the GNS construction, which provides a connection between states and representations [9, 43]). Each equivalence class defines a so-called folium of $P(\mathfrak{A})$. Each such folium is a Hilbert manifold, which is diffeomorphic (hence affinely isomorphic) to the pure state space $P(\mathcal{H})$ for some Hilbert space \mathcal{H} (cf. the previous subsection). Therefore, it admits a Poisson structure, which is defined exactly as in the case $\mathfrak{A} = \mathcal{K}(\mathcal{H})$ (the compactness of A and B in (3.10) was not essential). The Poisson structures on the folia can be combined into a Poisson structure on $P(\mathfrak{A})$, which is degenerate iff \mathfrak{A} (unlike the compact operators) admits more than one equivalence class of irreducible representations. This eventually leads us to regard elements of $A(K)$ (hence of \mathfrak{A}) as generators of transformations of $P(\mathfrak{A})$, and we see that the flow of a given operator cannot leave a given folium. This suggests that $P(\mathfrak{A})$ is a Poisson manifold, which is foliated by the symplectic leaves $P(\mathcal{H})$, but much remains to be done before this statement can be made precise, let alone proved (the main problems are to patch the folia together in the weak*-topology on $P(\mathfrak{A})$, and to deal with the states that are not pure but are weak* limits of pure states. In the uniform topology on K and $P(\mathfrak{A})$ things are easy, because $P(\mathfrak{A})$ splits up as a collection of disjoint components, each component being a folium, but this topology is not the relevant one).

Thus the idea is to identify the inequivalent irreducible representations of \mathfrak{A} (that is, its superselection sectors [21]) with the symplectic leaves of the pure state space $P(\mathfrak{A})$, providing a nice parallel with the classical case. The total state space K of \mathfrak{A} may be equipped with a Poisson structure, too, but it is clear that the symplectic leaves of this Poisson space cannot be identified with inequivalent representations. For already in the simplest case where \mathfrak{A} consists of the hermitian $n \times n$ matrices the

state space is foliated by an uncountable number of symplectic leaves, whereas the inequivalent representations are labeled by a positive integer. (To see this, note that K can be embedded in the dual $\mathfrak{u}(n)^*$ of the Lie algebra of $U(n)$, equipped with the canonical Lie-Poisson structure, and this embedding is a Poisson morphism. Hence the symplectic leaves of K are simply given by those leaves of $\mathfrak{u}(n)^*$ which lie in K ; these are generalized flag manifolds, and there are uncountably many even of a given orbit type).

In any case, we see that the role of the antisymmetric product α_{\hbar} as the agent which relates observables to flows on the pure state space survives unscratched for Jordan-Lie algebras. Conversely, we would like to define this product in terms of the Poisson structure on $P(\mathfrak{A})$. This can presumably be done using a result of Shultz [41], who proved that the commutator on the self adjoint part \mathfrak{A} of a C^* -algebra \mathfrak{A} is *abstractly* determined by specifying transition probabilities and an orientation on $P(\mathfrak{A})$. These transition probabilities are the usual ones if one passes from states to their GNS representations (and are zero for disjoint states, that is, states leading to inequivalent representations). Specifying $|(\psi_1, \psi_2)|^2$ plus an orientation is equivalent to specifying $\text{Im}(\psi_1, \psi_2)$, so we see from (3.8) that the theorem in [41] can very simply be understood by saying that the commutator is given by the Poisson bracket (3.10), and that Poisson and Jordan isomorphisms between two state spaces are induced by isomorphisms of the corresponding Jordan-Lie algebras.

We return to the axioms 1-7 on a Jordan-Lie algebra. Especially the norm axioms, but also property 5 look rather arbitrary, and it would be nice to reformulate them in such a way, that the following question may be answered: *which physical postulates of quantum mechanics imply its description in terms of Jordan-Lie algebras and their state spaces?* A similar question concerned with the Hilbert space formulation of conventional quantum mechanics is analyzed in [30, 7]. Since a Jordan-Lie algebra is isomorphic to the self-adjoint part of a C^* -algebra, we can look at the literature for help. It turns out to be fruitful to shift emphasis from the Jordan-Lie algebra \mathfrak{A} to its state space K (from which \mathfrak{A} can be recovered as $A(K)$, as we have reviewed above). The question above may then be reformulated

by asking which properties of a compact convex set K make $A(K)$ isomorphic to a Jordan-Lie algebra, and what the physical meaning of these properties is (as before, we stress the point that by eliminating complex numbers and Hilbert spaces from quantum mechanics through its reformulation in terms of Jordan-Lie algebras and their state spaces, we feel that we have come closer to the physical meaning of this theory).

The latter question has partly been answered in the work of Alfsen and Shutz [4, 3, 41], and others (cf. the reviews [2, 45]). As a consequence of these papers, the origin of the Jordan structure in quantum mechanics (as well as the norm axioms, which only use the Jordan product σ_{\hbar}) is now quite well understood. The key property of K that leads to a Jordan structure and the associated spectral calculus is the existence of sufficiently many projective faces in K ; a projective face plays a role similar to that of a closed subspace of a Hilbert space (or the corresponding projector) and is physically a yes-no question. Projective faces are orthocomplemented, and have other nice properties making them suitable as a basic ingredient of quantum logic [7, 12]. Other properties of K which are necessary to derive the Jordan structure are related to the property that pure states in quantum mechanics can be prepared through filtering procedures, and to the symmetry of transition amplitudes (which reflects the symmetry between pure states and finest detectors [21]).

Further properties of the state space K leading to a Lie bracket on $\mathfrak{A} \simeq A(K)$ are known [3, 2], but their physical meaning is not so clear. We hope to be able to show that these properties are equivalent to $P(\mathfrak{A})$ admitting a Poisson structure which foliates the pure state space in a way consistent with the representation theory of \mathfrak{A} as a Jordan algebra. A crucial property of non-associative Jordan-Lie algebras (i.e., $\hbar \neq 0$) is that the restriction of $A(K)$ to $P(\mathfrak{A})$ does not coincide with the space of all continuous functions on $P(\mathfrak{A})$ (unlike the classical case; the essential point is that not nearly every function on $P(\mathfrak{A})$ extends to an affine function on K , because non-pure elements of K generically have many decompositions as convex sums of pure states [9, 43]). This non-uniqueness constrains the allowed functions on the

extreme boundary $P(\mathfrak{A})$ of K , which do have an affine extension to K , enormously. Such constraints do not arise when every mixed state in K has a unique extremal decomposition, and this happens precisely when \mathfrak{A} is associative, i.e., in the classical case.). Together with the Poisson structure this property should be related to the uncertainty principle (at least in its naive textbook formulation).

3.3 Representation theory of Jordan-Lie algebras

As we have seen in Definition (2), a representation of a Poisson algebra is a map into $C^\infty(S)$ for some symplectic space S , which preserves all the algebraic structures and in addition satisfies a completeness condition. The motivation was that $C^\infty(S)$ for symplectic S is a ‘canonical’ model of a Poisson algebra. More importantly, irreducibility implies that S must be symplectic. Similarly, the canonical model of a Jordan-Lie algebra is the algebra of all bounded self-adjoint operators on a complex Hilbert space \mathcal{H} . The latter is naturally equipped with the Jordan-Lie structure (3.5), and this motivates

Definition 5 *A representation of a non-associative Jordan-Lie algebra \mathfrak{A} is a map $\pi_q^\chi : \mathfrak{A} \rightarrow \mathcal{B}(\mathcal{H}_\chi)$, for some Hilbert space \mathcal{H}_χ , satisfying for all $A, B \in \mathfrak{A}$*

1. $\pi_q^\chi(\lambda A + \mu B) = \lambda \pi_q^\chi(A) + \mu \pi_q^\chi(B)$ (*linearity*);
2. $\pi_q^\chi(A \sigma_{\hbar} B) = \frac{1}{2}(\pi_q^\chi(A) \pi_q^\chi(B) + \pi_q^\chi(B) \pi_q^\chi(A))$ (*preserves Jordan product*);
3. $\pi_q^\chi(A \alpha_{\hbar} B) = \frac{1}{i\hbar}(\pi_q^\chi(A) \pi_q^\chi(B) - \pi_q^\chi(B) \pi_q^\chi(A))$ (*preserves Lie product*);
4. $\pi_q^\chi(A)^* = \pi_q^\chi(A)$ (*self-adjointness*).

These conditions are, of course, equivalent to the usual ones on representations of the C^* -algebra \mathfrak{A}_C (the self-adjointness condition 4 is the requirement that π_q^χ is a $*$ -representation of \mathfrak{A}_C), but we have put them in the given form to make the analogy with the classical Definition 2 clear. In similar vein, the classical irreducibility condition Definition 3 is, as we have seen from the discussion following (3.12), essentially the same as the usual definition of irreducibility for representations of C^* -algebras, which in the present framework reads

Definition 6 A representation π_q^χ of a Jordan-Lie algebra \mathfrak{A} on a Hilbert space \mathcal{H}_χ is called irreducible iff every vector in \mathcal{H}_χ is cyclic for $\pi_q^\chi(\mathfrak{A})$ (that is, the set $\{A\psi | A \in \mathfrak{A}\}$ is dense in \mathcal{H}_χ for each fixed $\psi \in \mathcal{H}_\chi$).

All this may be reformulated in terms of the (pure) state space of \mathcal{H}_χ , and the Jordan and Lie products on $A(K)$ as discussed in the previous subsection, but we leave this to the reader.

There is a decisive difference between the classical case ($\hbar = 0$; Jordan product $\sigma \equiv \sigma_0$ associative) and the quantum case as far as irreducibility is concerned. Irreducible representations of a Poisson algebra $C^\infty(M)$ are highly reducible as representations of the corresponding Jordan algebra (in which the anti-symmetric product α is ignored), whereas irreducible representations of this Poisson algebra (which are just points of M) do not lead to representations of $C^\infty(M)$ at all. In the quantum case, a representation of a non-associative Jordan-Lie algebra is irreducible iff it is irreducible as a representation of the underlying Jordan algebra. This looks curious, because the irreducibility condition above may be formulated in terms of the vector fields (3.11), which are defined using the Lie product (see (3.10)). However, the unitary flow (3.12) is completely defined in terms of the Jordan product (which allows the definition of functions of an operator).

The naive quantum analogue of the generalized moment map J (cf. Theorem 1) is rather trivial: given a representation $\pi_q^\chi(\mathfrak{A})$, we may define a map $\tilde{J} : \mathcal{H}_\chi \rightarrow K$ (where K is the state space of \mathfrak{A}) by specifying the value of the state $\tilde{J}(\psi)$ on arbitrary $A \in \mathfrak{A}$ to be

$$(\tilde{J}(\psi))(A) = \frac{(\pi_q^\chi(A)\psi, \psi)}{(\psi, \psi)}. \quad (3.18)$$

This evidently reduces to a map $J : P\mathcal{H}_\chi \rightarrow K$, which is the naive quantum analogue of the classical generalized moment map. Namely, for π_q^χ irreducible, the image of \tilde{J} is contained in the pure state space $P(\mathfrak{A})$. Thus we see that the quantum moment map just expresses the correspondence between states and vectors in a Hilbert space, which is central to the GNS construction [43, 9], and lies at the heart of operator algebras. A difference between the classical and the naive quantum moment map is that the image of the former is the set of pure states, even if the representation

is reducible, while the image of the latter may well lie among the mixed states (namely if the representation is reducible). Also, the Marsden-Weinstein symplectic reduction construction [1, 29] cannot be ‘quantized’ in terms of \tilde{J} in any obvious way. Hence one needs a deeper quantum analogue of the classical moment map, and this is given by the concept of a Hilbert C^* -module, see [28].

The quantum counterpart of the classical Theorem 2 has yet to be proved (and even properly formulated); this would express that $P(\mathfrak{A})$ is foliated by its symplectic leaves, which, as we have seen in the preceding subsection, should be identified with folia of states leading to equivalent representations.

3.4 The group algebra

For reasons to emerge later, a quantum analogue of the Poisson algebra $C^\infty(\mathfrak{g}^*)$ (cf. subsect. 2.3) is the group algebra $JL(G) = C^*(G)_{\text{sa}}$; it is the quantum algebra of observables of a particle whose only degree of freedom is internal. Here G is any Lie group with Lie algebra \mathfrak{g} . For simplicity, we only define $JL(G)$ for unimodular G (look up $C^*(G)$ in [33] for the general case). The starting point is to construct a dense subalgebra of $C^*(G)$. This is done by defining a product $*$ and involution $*$ on $C_c^\infty(G)$ by

$$(f * g)(x) = \int_G dx f(xy)g(y^{-1}); \quad f^*(x) = \overline{f(x^{-1})}, \quad (3.19)$$

where dx is a Haar measure on G . The norm is defined in [33]; in the special case that G is amenable (this holds, for example, when G is compact) one may put $\|f\| = \|\pi_q^L(f)\|$, where π_q^L is a representation of $C_c^\infty(G)$ (regarded as an associative $*$ -algebra) on $\mathcal{H}_L = L^2(G)$, given by

$$(\pi_q^L(f)\psi)(x) = \int_G dy f(y)(\pi_L(y)\psi)(x), \quad (3.20)$$

with $(\pi_L(y)\psi)(x) = \psi(y^{-1}x)$. The closure of $C_c^\infty(G)$ in this norm is the group algebra $C^*(G)$. The corresponding Jordan-Lie algebra $JL(G)$ is its self-adjoint part, equipped with the products σ_{\hbar} and α_{\hbar} , defined as in (3.5) (with AB replaced by $f * g$, etc.).

The representation theory of $JL(G)$ coincides with that of $C^*(G)$, which is well-known [33]: every (non-degenerate) representation π_q^χ of $JL(G)$ on a Hilbert space \mathcal{H}_χ corresponds to a unitary representation π_χ of G on \mathcal{H}_χ , the passage from $\pi_\chi(G)$ to $\pi_q^\chi(JL(G))$ being accomplished by the analogue of (3.20), with L replaced by χ . In particular, irreducible representations of $JL(G)$ correspond to irreducible unitary representations of G .

In traditional quantization theory (applied to this special case) one tried to associate a Hilbert space and certain operators to a co-adjoint orbit $\mathcal{O} \subset \mathfrak{g}^*$ and the associated Poisson algebra $C^\infty(\mathcal{O})$ (which we look upon as an irreducible representation of $C^\infty(\mathfrak{g}^*)$). This was very successful in special situations, e.g., G nilpotent. In that case there is a one-to-one correspondence between co-adjoint orbits and unitary representations, given by the Dixmier-Kirillov theory [15]. The same strategy was reasonably successful in some other cases, like G compact and semi-simple, when any irreducible unitary representation of G can be brought into correspondence with at least some co-adjoint orbit via the Borel-Weil theory [24]; on the other hand, most co-adjoint orbits do not correspond to any unitary representation of G at all. However, in the general case no correspondence between co-adjoint orbits and irreducible representations exists, and modern research in representation theory looks in different directions [46] (note that this does not undermine the hard fact that the classical irreducible representations of $C^\infty(\mathfrak{g}^*)$ are completely classified by the co-adjoint orbits and their covering spaces).

The natural correspondence between classical and quantum mechanics exists at an algebraic level, namely in their respective Jordan-Lie algebras of observables. The irreducible representations of a classical Poisson algebra are not necessarily related to those of the corresponding quantum Jordan-Lie algebra, and both should be constructed in their own right.

4 Quantization

4.1 The definition of a quantization

We now return to the Weyl quantization on \mathbb{R}^n reviewed in subsect. 3.1. We have seen how we may regard Q_\hbar as a map from the dense subspace $\overline{\mathfrak{A}_0}$ of the commutative Banach algebra $\mathfrak{A}_0 = C_0(T^*\mathbb{R}^n)$ to the space of self-adjoint compact operators $\mathfrak{A} = \mathcal{K}(L^2(\mathbb{R}^n))_{\text{sa}}$. Here \mathfrak{A}_0 also has a densely defined Poisson structure (which is, in particular, defined on $\overline{\mathfrak{A}_0}$), and may be regarded as an associative Jordan-Lie algebra, equipped with the products $\cdot = \sigma \equiv \sigma_0$ and $\{ , \} = \alpha \equiv \alpha_0$. The space \mathfrak{A} may be dressed up with the products σ_\hbar and α_\hbar , defined in (3.5), and thus a family of non-associative Jordan-Lie algebras $\{\mathcal{A}_\hbar\}$ is defined (the norm in \mathcal{A}_\hbar is borrowed from \mathfrak{A} , and is independent of \hbar for $\hbar \neq 0$. The norm on \mathfrak{A}_0 is defined in (3.4)). We define $Q_0 : \mathfrak{A}_0 \rightarrow \mathfrak{A}_0$ as the identity map. It may be shown [27] that the following properties hold for all $f, g \in \overline{\mathfrak{A}_0}$:

1. $\lim_{\hbar \rightarrow 0} \| Q_\hbar(f)\sigma_\hbar Q_\hbar(g) - Q_\hbar(f\sigma_0 g) \| = 0$;
2. $\lim_{\hbar \rightarrow 0} \| Q_\hbar(f)\alpha_\hbar Q_\hbar(g) - Q_\hbar(f\alpha_0 g) \| = 0$;
3. the function $\hbar \rightarrow \| Q_\hbar(f) \|$ is continuous on $I = \mathbb{R}$.

Condition 2 is an analytic reformulation of the correspondence between commutators of operators and Poisson brackets of functions first noticed by Dirac. The first condition is based on the correspondence between anti-commutators of operators and pointwise products of functions, first noticed by von Neumann. The third condition is a precise formulation of (one form) of the correspondence principle due to Bohr. Recalling that $f\sigma_0 g = fg$ and $f\alpha_0 g = \{f, g\}$, note the consistency of the above conditions with (3.14) and (3.10). In the context of C^* -algebras conditions 2 and 3 in their present form were first written down by Rieffel [38] (who did not impose either condition 1 or self-adjointness on a quantization map). The connection between deformations of algebras and quantization theory was analyzed in a different mathematical setting in [8, 6].

The example of a particle on \mathbb{R}^n and the general considerations in sections 2 and 3 motivate the following

Definition 7 *Let \mathfrak{A}_0 be a commutative Jordan algebra with a densely defined Poisson bracket (making \mathfrak{A}_0 into an associative Jordan-Lie algebra, cf. Def. 4), and let $\overline{\mathfrak{A}_0}$ be a dense subalgebra on which the Poisson bracket is defined. A quantization of this structure is a family $\{\mathcal{A}_\hbar\}_{\hbar \in I}$ of non-associative Jordan-Lie algebras (Def. 4), and a family $\{Q_\hbar\}_{\hbar \in I}$ of maps defined on $\overline{\mathfrak{A}_0}$, such that the image of Q_\hbar is in \mathcal{A}_\hbar , and the above conditions 1-3 are satisfied.*

As we have seen, Weyl quantization satisfies this definition. A generalization of Weyl quantization to arbitrary Riemannian manifolds is given in [27]. The axioms above are not quite satisfied by this generalized quantization prescription, in that the range in \hbar for which Q_\hbar is defined depends on its argument. This is easily remedied, however, by constructing cutoff functions in \hbar , cf. the example below. The cutoff, on the other hand, upsets the physical interpretation of $Q_\hbar(f)$ as the quantum observable corresponding to the classical observable f for all $\hbar \in I$, and for that reason in [27] we preferred to leave $Q_\hbar(f)$ undefined whenever it could no longer be interpreted properly. This complication only occurs for manifolds for which the exponential map is not a diffeomorphism on the entire tangent space at each point. A further generalization is to admit internal degrees of freedom, through which the particle can couple to a gauge field. This case is covered in [27], too, and from this general class of examples it has become clear that the definition of quantization given above is satisfied by a number of realistic physical examples.

A non-self-adjoint version of the quantization of $C_0(\mathfrak{g}^*)$ by $C^*(G)$ (cf. subsects. 2.3 and 3.4) was first given by Rieffel [39], and the physically relevant self-adjoint version, i.e., the construction of the maps $Q_\hbar : C_0(\mathfrak{g}^*) \rightarrow JL(G)$ is a special case of the theory in [27] if G is compact (obtained by taking $P = H = G$ in that paper, and exploiting the fact that $(T^*G)/G \simeq \mathfrak{g}^*$ with the usual Poisson structure). We define $\overline{\mathfrak{A}_0} \subset \tilde{C}_0(\mathfrak{g}^*)$ as the space of those functions f on \mathfrak{g}^* whose Fourier transform \check{f} is in $C_c^\infty(\mathfrak{g})$ (since $\mathfrak{g}^* \simeq \mathbb{R}^n$ we can define the Fourier transform as usual, cf. (3.1), omitting the x -dependence). The quantization map is given by

$$(Q_\hbar(f))(e^{-\hbar X}) = \hbar^{-n} \check{f}(X), \quad (4.1)$$

which defines the left-hand side as an element of $C^*(G)_{\text{sa}} = JL(G)$ for those values of \hbar for which \hbar times the support of \check{f} lies in the neighbourhood of $0 \in \mathfrak{g}$ on which the exponential function is a diffeomorphism from \mathfrak{g} to G . Since \check{f} has compact support, the allowed values of \hbar will lie in an interval $(-\hbar_0, \hbar_0)$, where \hbar_0 depends on f . If the group G is exponential (which is the case if G is simply connected and nilpotent [15]) then $\hbar_0 = \infty$. In general, one could extend the quantization to any value of \hbar , without violating the conditions required by Def. 7, by multiplying $Q_\hbar(f)$ by a function h which is 1 in $(-.99\hbar_0, .99\hbar_0)$ (say).

4.2 Positivity and continuity

While the Weyl quantization of subsect. 3.1 (as well as its generalization to Riemannian manifolds) satisfies Def. 7 of a quantization, there are two serious problems with it. The first is lack of positivity; this means that if $f \geq 0$ in $\mathfrak{A}_0 = C_0(T^*\mathbb{R}^n)$ then it is not necessarily true that $Q_\hbar(f) \geq 0$ in \mathfrak{A} (see e.g. [18, 2.6]). From the equality

$$(Q_\hbar(f)\Omega, \Omega) = \int_{T^*\mathbb{R}^n} \frac{d^n x d^n p}{(2\pi)^n} W_\Omega^\hbar(x, p) f(x, p), \quad (4.2)$$

with the Wigner function

$$W_\Omega^\hbar(x, p) = \int_{\mathbb{R}^n} d^n \dot{x} e^{ip\dot{x}} \Omega(x - \frac{1}{2}\hbar\dot{x}) \overline{\Omega(x + \frac{1}{2}\hbar\dot{x})}, \quad (4.3)$$

we see that the potential non-positivity of $Q_\hbar(f)$ is equivalent to the fact that the Wigner distribution function (4.3) is not necessarily positive definite.

The second problem is that Q_\hbar (for fixed $\hbar \neq 0$) is not continuous as a map from $\overline{\mathfrak{A}_0}$ to \mathfrak{A} (both equipped with their respective norm topologies). Hence it cannot be extended to \mathfrak{A}_0 in any natural way. The problem here is that we wish to work in a Banach-algebraic framework; the map Q_\hbar is continuous as an operator from $L^2(T^*\mathbb{R}^n)$ to $HS(L^2(\mathbb{R}^n))$ (if both are regarded as Hilbert spaces, the latter being the space of Hilbert-Schmidt operators on $L^2(\mathbb{R}^n)$, with the inner product $(A, B) = \text{Tr}AB^*$), and also as a map from the Schwartz space $\mathcal{S}'(T^*\mathbb{R}^n)$ to the space of continuous linear maps from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$, cf. [18, 19].

Both problems may be resolved simultaneously if we construct a positive quantization, that is, find a map $Q'_\hbar : \mathfrak{A}_0 \rightarrow \mathfrak{A}$ which is positive. For a positive map

between two C^* -algebras is automatically continuous (see [43], p. 194). Let $\{\Omega_\hbar\}_{\hbar>0}$ be a family of normalized vectors in $L^2(\mathbb{R}^n)$, which satisfy the condition that in the limit $\hbar \rightarrow 0$ the Wigner function $W_{\Omega_\hbar}^\hbar$ is smooth in all variables (including \hbar), vanishes rapidly at infinity, and converges to $(2\pi)^n \delta(x, p)$ in the distributional topology defined by the test function space $\overline{\mathfrak{A}_0}$ (defined after (3.1)). An example is

$$\Omega_\hbar(x) = (\pi\hbar)^{-n/4} e^{-x^2/2\hbar}, \quad (4.4)$$

with Wigner function

$$W_{\Omega_\hbar}^\hbar(x, p) = (2/\hbar)^n e^{-(x^2+p^2)/\hbar}. \quad (4.5)$$

We then define a new quantization map Q_\hbar^Ω by

$$Q_\hbar^\Omega(f) = Q_\hbar(W_{\Omega_\hbar}^\hbar * f), \quad (4.6)$$

with Q_\hbar the Weyl quantization (3.3), $\tilde{W}_{\Omega_\hbar}^\hbar$ defined by $\tilde{W}_{\Omega_\hbar}^\hbar(x, p) = W_{\Omega_\hbar}^\hbar(-x, -p)$, and $*$ being the convolution product in \mathbb{R}^{2n} . It follows from Prop. 1.99 in [18] that Q_\hbar^Ω is a positive map. Since the uniform operator norm is majorized by the Hilbert-Schmidt norm, it follows from the triangle inequality and the first continuity property of Q_\hbar mentioned above that Q_\hbar^Ω defines a quantization if for all $f, g \in \overline{\mathfrak{A}_0}$

$$\begin{aligned} L^2 - \lim_{\hbar \rightarrow 0} \left(\{ \tilde{W}_{\Omega_\hbar}^\hbar * f, \tilde{W}_{\Omega_\hbar}^\hbar * g \} - \tilde{W}_{\Omega_\hbar}^\hbar * \{ f, g \} \right) &= 0; \\ L^2 - \lim_{\hbar \rightarrow 0} \left((\tilde{W}_{\Omega_\hbar}^\hbar * f) \cdot (\tilde{W}_{\Omega_\hbar}^\hbar * g) - \tilde{W}_{\Omega_\hbar}^\hbar * (f \cdot g) \right) &= 0, \end{aligned} \quad (4.7)$$

and if the function $\hbar \rightarrow \| Q_\hbar^\Omega(f) \|$ is continuous for all such f . These conditions are all satisfied if Ω_\hbar is as specified prior to (4.4), and thus Q_\hbar^Ω is indeed a positive definite quantization (note that Q_\hbar^Ω is automatically self-adjoint, since $W_{\Omega_\hbar}^\hbar$ is real-valued). It can be extended to \mathfrak{A}_0 by continuity, but the extension obviously does not satisfy the quantization condition involving the Poisson bracket (which is not a continuous map on \mathfrak{A}_0 in either variable).

This procedure may be extended to arbitrary manifolds Q ; the smearing $f \rightarrow \tilde{W}_{\Omega_\hbar}^\hbar * f$ will in general be replaced by the use of Friedrichs mollifiers. It is clear that this positive definite quantization procedure is not intrinsic: it depends on the choice of the Ω_\hbar . It may be argued that the Weyl quantization procedure is not

intrinsic either, because from a geometric point of view [27] it relies on the choice of a diffeomorphism between a tubular neighbourhood of Q in TQ , and one of ΔQ in $Q \times Q$. In any case, one may argue that points in space should be stochastic objects, with a probability distribution related to Ω_{\hbar} . This point of view is defended, in a quite different context, in [35, 5].

5 Lie groupoids, Lie algebroids, and their Jordan-Lie algebras

The (generalized [27]) Weyl quantization of $C_0(T^*Q)$ by $\mathcal{K}(L^2(Q))_{\text{sa}}$ and the quantization of $C_0(\mathfrak{g}^*)$ by $JL(G) = C^*(G)_{\text{sa}}$ are both special cases of a rather general construction involving Lie groupoids, which are a certain generalization of Lie groups that are of great physical and mathematical relevance (cf. [31, 16] for a comprehensive discussion of these structures, illustrated with many examples).

5.1 Basic definitions

We recall that a category G is a class B of objects together with a collection of arrows. Each arrow x leads from object $s(x)$ (the source of the arrow) to the object $t(x)$ (the target). If $s(x) = t(y)$ then the composition xy is defined as an arrow from $s(y)$ to $t(x)$, and this partial multiplication on G is associative whenever it is defined. Also, each object $b \in B$ comes with an arrow $i(b)$, which serves as the identity map from $s(i(b)) = b$ to $t(i(b)) = b$, so that $xi(b) = x$ (defined when $s(x) = b$) and $i(b)x = x$ (defined when $t(x) = b$). Hence we obtain an inclusion i of B into G . A category is called small if B is a set.

Definition 8 *A groupoid is a small category in which each arrow is invertible.*

Hence for each $x \in G$ the arrow x^{-1} is defined, with $s(x^{-1}) = t(x)$ and $t(x^{-1}) = s(x)$, and one has $i \circ s(x) = x^{-1}x$ and $i \circ t(x) = xx^{-1}$. We may regard G as a fibered space over B , with two projections $S : G \rightarrow B$ and $t : G \rightarrow B$. One may pass to topological groupoids by requiring continuity of the relevant structures, and to Lie groupoids by demanding smoothness:

Definition 9 *A Lie groupoid is a groupoid in which G and B are smooth manifolds (taken to be Hausdorff, paracompact and finite-dimensional), so that the inclusion i is a smooth embedding, the projections s and t are smooth surjective submersions, and the inverse $x \rightarrow x^{-1}$, as well as the partial multiplication $(x, y) \rightarrow xy$ are smooth maps.*

Variations on this definition are possible, cf. [31, 16]; for example, in the former ref. the assumption is added that G is transitive, in the sense that the map $s \times t : G \rightarrow B \times B$ is surjective (that is, any two points in B can be connected by an arrow), but since a corresponding transitivity assumption is not part of the definition of a Lie algebroid (see below) in [31], we follow [16] in dropping it.

We see that a Lie group is a special case of a Lie groupoid, namely a case in which B consists of one point b (and $i(b) = e$ is the identity of G), so that all arrows can be composed. One may generalize the passage from a Lie group to a Lie algebra in the present context. First note that each $x \in G$ defines not only an arrow from $s(x)$ to $t(x)$, but in addition leads to a map $L_x : t^{-1}(s(x)) \rightarrow t^{-1}(t(x))$, defined by $L_x(y) = xy$. Similarly, one has a map $R_x : s^{-1}(t(x)) \rightarrow s^{-1}(s(x))$ given by $R_x(y) = yx$. As in the Lie group case, we would like to define left- and right invariant vector fields on G . Hence we would obtain (say) a left-invariant flow φ_τ on G , satisfying $\varphi_\tau(L_x(y)) = L_x\varphi_\tau(y)$ for $y \in t^{-1}(s(x))$. The problem is that L_x is only a partially defined multiplication, so that the right-hand side is only defined if $t(\varphi_\tau(y)) = s(x)$, that is, the target of $\varphi_\tau(y)$ must not depend on the time τ . Hence $(d/d\tau)t(\varphi_\tau) = 0$, or $t_*X = 0$ if $X = (d/d\tau)(\varphi_\tau)|_{\tau=0}$. In conclusion, we may define a left-invariant vector field ξ_L by the conditions

$$t_*\xi_L = 0; \quad (L_x)_*\xi_L = \xi_L \quad \forall x \in G, \quad (5.1)$$

and a right-invariant vector field ξ_R by the conditions

$$s_*\xi_R = 0; \quad (R_x)_*\xi_R = \xi_R \quad \forall x \in G. \quad (5.2)$$

It is easily shown [31, 16] that the commutator (Lie bracket) of two left (right) invariant vector fields is left (right) invariant. Hence we may define

Definition 10 *The Lie algebroid \mathfrak{g} of a Lie groupoid G is the real vector space of all vector fields on G satisfying (5.1), equipped with the following structures:*

i) a projection $pr : \mathfrak{g} \rightarrow B$ (namely the obvious one, coming from the projections $TG \rightarrow G \xrightarrow{s} B$), which makes \mathfrak{g} a vector bundle over B ;

ii) a projection $q : \mathfrak{g} \rightarrow TB$, given by $q = s_$;*

iii) a Lie bracket on $\Gamma(\mathfrak{g})$ (the space of smooth sections of \mathfrak{g}), which is given by the commutator on $\Gamma(TG)$, and which satisfies

$$q([\xi_L^1, \xi_L^2]) = [q(\xi_L^1), q(\xi_L^2)]; \quad (5.3)$$

$$[\xi_L^1, f\xi_L^2] = f[\xi_L^1, \xi_L^2] + q(\xi_L^1)f \cdot \xi_L^2 \quad \forall f \in C^\infty(B). \quad (5.4)$$

Of course, an equivalent definition is obtained by replacing (5.1) by (5.2), and s and s_* by t and t_* , respectively. One may define a Lie algebroid without reference to Lie groupoids as vector bundle E over B , together with an additional projection $q : E \rightarrow TB$ satisfying (5.3) (the ‘anchor’ of E [31]) and a Lie bracket on $\Gamma(E)$ satisfying the analogue of (5.4). A Lie algebroid is called transitive if q is a surjective; if a Lie groupoid G is transitive then so is its algebroid \mathfrak{g} . A simple example is $E = TQ$ as a vector bundle over Q , with q the identity map.

One may generalize the identification $\mathfrak{g} \simeq T_e G$ for a Lie group G as follows. Since $x = x(x^{-1}x) = L_x(i \circ s(x))$, every left-invariant vector field ξ_L on G is determined by its values at $i(B) \equiv G_0 \subset G$. We have the decomposition $T_{G_0}G = T_{G_0}G_0 \oplus \ker(t_*)[T_{G_0}G$ (where $[$ means ‘restricted to’), so we see that $\mathfrak{g} \simeq T_{G_0}G/T_{G_0}G_0$, which is just the normal bundle N_i of the embedding $i : B \rightarrow G$. Equivalently, if we define $T^t G$ as the vector bundle over G consisting of elements of TG annihilated by t_* (with the canonical projection pr_t onto G borrowed from TG), then $\mathfrak{g} = i^*(T^t G)$, the pull-back bundle over B given by the map $i : B \rightarrow G$. Conversely, $T^t G = s^*(\mathfrak{g})$ as a pull-back bundle [31]. Note that $T^t G$ is itself a Lie algebroid over G , with the anchor $q : T^t G \rightarrow TG$ just given by inclusion.

An interesting property of a Lie algebroid E is that a connection on E allows one to define generalized geodesics on E (hence on the base space B). Namely, one

obtains a vector field ξ on E , whose value at $Y \in E$ is given by the horizontal lift of $q(Y) \in TB$ at Y . The flows of this field are the desired generalized geodesics (for $E = TB$ equipped with the Levi-Civita connection these are the usual geodesics). This leads to the construction of a map $\exp : \mathfrak{g} \rightarrow G$ which generalizes the one for Lie groups [34]. Namely, by the preceding paragraph T^tG regarded as a vector bundle over G inherits the chosen connection A on \mathfrak{g} (considered a vector bundle over B) as the pull-back s^*A , and this implies that one has a generalized geodesic flow γ_τ on T^tG . Now for $X \in \mathfrak{g}$,

$$e^X = pr_t(\gamma_1(X)), \quad (5.5)$$

where on the right-hand side we regard $X \in \mathfrak{g} \subset T^tG$ via the natural embedding of $\mathfrak{g} \equiv \ker(t_*)[T_{G_0}G$ in T^tG . If G is a Lie group then obviously no connection needs to be chosen (all vectors on \mathfrak{g} are vertical, so the geodesic flow on \mathfrak{g} is the identity map), and the map \exp reduces to the usual one.

5.2 Algebras of observables from Lie algebroids and groupoids

Generalizing the Poisson algebra $C^\infty(\mathfrak{g}^*)$ of a Lie algebra \mathfrak{g} (which is a vector bundle over a single point), one may associate a Poisson algebra $C^\infty(E^*)$ to any Lie algebroid E [16]; here E^* is the dual of E as a vector bundle. The Poisson structure is completely determined by specifying the Poisson bracket between arbitrary sections ξ_1, ξ_2 of E and functions f_1, f_2 on B . Here any $\xi \in \Gamma(E)$ defines an element $\tilde{\xi} \in C^\infty(E^*)$ as follows: if pr is the projection in E^* then $\tilde{\xi}(\theta) = \langle \theta, \xi(pr(\theta)) \rangle$. These functions $\tilde{\xi}$ are obviously linear on the fibers of E^* . Furthermore, $f \in C^\infty(B)$ defined $\tilde{f} \in C^\infty(E^*)$ by pull-back. The Poisson brackets are

$$\{\tilde{\xi}_1, \tilde{\xi}_2\} = [\widetilde{\xi_1, \xi_2}]; \quad \{\tilde{f}_1, \tilde{f}_2\} = 0; \quad \{\tilde{\xi}, \tilde{f}\} = \widetilde{q(\xi)f}. \quad (5.6)$$

This bracket may subsequently be extended to a dense subset of $C^\infty(E^*)$ (in a suitable topology) by imposing the Leibniz rule on products of linear functions. On $E = TQ$ this procedure is equivalent to imposing the identities $\{\sigma(\xi_1), \sigma(\xi_2)\} = \sigma([\xi_1, \xi_2])$, $\{\tilde{f}_1, \tilde{f}_2\} = 0$, and $\{\sigma\xi, \tilde{f}\} = \widetilde{\xi(f)}$, where $\sigma(\xi) \in C^\infty(T^*Q)$ is the symbol of the vector field ξ on Q . This leads to the canonical Poisson structure on

$C^\infty(T^*Q)$. In case that $E = \mathfrak{g}$ is the Lie algebroid of a Lie groupoid, a more intrinsic construction of this Poisson structure is given in [16, II.4.2].

In similar spirit, we can construct a non-commutative C^* -algebra (hence a non-associative Jordan-Lie algebra) from a Lie groupoid G (indeed, from almost any topological groupoid [37], but the construction is more canonical in the Lie case, where a natural measure class is singled out, see below). To do so, we need to choose a measure μ_b on each fiber $t^{-1}(b)$ of G , in such a way that the family of measures thus obtained is left-invariant (that is, the map $L_x : t^{-1}(s(x)) \rightarrow t^{-1}(t(x))$ should be measure-preserving for all x). Since the fibers are manifolds, we naturally require that each measure μ_b is equivalent to the Lebesgue measure (on a local chart). The precise choice of the μ_b does not matter very much in that case, as the C^* -algebras corresponding to different such choices will be isomorphic. In both the Lie and the general case, groupoid C^* -algebras are of major mathematical interest, as they provide fascinating examples of non-commutative geometry (cyclic cohomology) and topology (K-homology), cf. [13, 32]. The algebra is constructed starting from $C_c^\infty(G)$, which is equipped with a product

$$f * g(x) = \int_{t^{-1}(s(x))} d\mu_{s(x)}(y) f(xy)g(y^{-1}), \quad (5.7)$$

and an involution

$$f^*(x) = \overline{f(x^{-1})}, \quad (5.8)$$

which are clearly generalizations of (3.19). The construction of the norm is described in [37] (for general groupoids), and the closure of $C_c^\infty(G)$ in this norm is the groupoid algebra $C^*(G)$. Its self-adjoint part, with the multiplications σ_{\hbar} and α_{\hbar} (cf. (3.5)), is the Jordan-Lie algebra $JL(G)$.

For G a Lie group we thus recover the group algebra, whose representation theory is discussed in subsect. 3.4; the opposite case is the so-called coarse groupoid $G = Q \times Q$, where Q is a manifold. This has base space $B = Q$, and source and target projections $s((x, y)) = y$, $t((x, y)) = x$. The inclusion is $i(x) = (x, x)$, the inverse is $(x, y)^{-1} = (y, x)$, and the composition rule is $(x_1, y)(y, x_2) = (x_1, x_2)$. The measures μ_b may all be taken to be identical to a single measure μ on Q , and one easily finds

that $C^*(Q \times Q) = \mathcal{K}(L^2(Q; \mu))$, cf. [13]. Its self-adjoint subspace $JL(Q \times Q)$ is the quantum algebra of observables of a particle moving on Q [27], and it will not come as a surprise that the Poisson algebra of the Lie algebroid TQ of $Q \times Q$ is just $C^\infty(T^*Q)$, the classical algebra of observables of the particle. The quantum algebra $JL(Q \times Q)$ has only one irreducible representation, namely the defining one on $L^2(Q; \mu)$ (up to unitary equivalence). Similarly, the classical algebra $C^\infty(T^*Q)$ has only one classical irreducible representation (up to symplectomorphisms), given by $S = T^*Q$. These are the quantum as well as classical Jordan-Lie analogues of the well-known Stone- von Neumann uniqueness theorem on regular representations of the canonical commutation relations (see e.g. [10, 18, 15] for this theorem in its various settings).

The situation where G is either a Lie group, or the coarse groupoid of some manifold, are both special cases of so-called gauge groupoids [31, 16]. A gauge groupoid is equivalent to a principal fibre bundle (P, Q, H) , where P is the total space, Q is the base space, and H is a Lie group acting on P from the right. The corresponding groupoid is denoted by $P \times_H P$. It is a quotient of the coarse groupoid $P \times P$, obtained by imposing the equivalence relation $(x_1, x_2) \sim (y_1, y_2)$ iff $(x_1, x_2) = (y_1h, y_2h)$ for some $h \in H$; we denote the equivalence class of (x, y) by $[x, y]$. Accordingly, $B = Q = P/H$, the inverse is $[x, y]^{-1} = [y, x]$, the projections are $s([x, y]) = pr_{P \rightarrow Q}(y)$, $t([x, y]) = pr_{P \rightarrow Q}(x)$, the inclusion is $i(q) = [s(q), s(q)]$ (for an arbitrary section s of P), and multiplication $[x_1, y_1] \cdot [y_2, x_2]$ is defined iff $y_2 = y_1h$ for some $h \in H$, and the composition equals $[x_1h, x_2]$ in that case. For $H = \{e\}$ we get the coarse groupoid, and for $P = H = G$ we get a Lie group G . It can be shown that any transitive groupoid is of the form $P \times_H P$ [31].

If H is compact the groupoid C^* -algebra is $C^*(P \times_H P) \simeq C^*(Q \times Q) \otimes C^*(H)$ [27], which is the quantum algebra of observables of a particle moving on Q with an internal degree of freedom, namely a charge coupling to a gauge field defined on the bundle (P, Q, H) . The Lie algebroid of $P \times_H P$ is $(TP)/H$ (where the action of H on TP is the push-forward of its action on P). The corresponding Poisson algebra $C^\infty((T^*P)/H)$ was already known to be the classical algebra of observables

of a particle coupling to a Yang-Mills field [20], and it is satisfying that the quantum algebra $C^*(P \times_H P)_{\text{sa}}$ can be obtained as a deformation of it; the quantization maps Q_{\hbar} are given in [27].

The irreducible representations of the classical algebra of observables $A_0 = C^\infty((T^*P)/H)$ correspond to the symplectic leaves of $(T^*P)/H$ (and their covering spaces), which are discussed in [20]. There is a one-to-one correspondence between the set of these leaves, and the set of co-adjoint orbits in \mathfrak{h}^* (the dual of the Lie algebra of H): each leaf $P_{\mathcal{O}}$ is a fiber bundle over T^*Q whose characteristic fiber is the co-adjoint orbit \mathcal{O} . Hence locally $P_{\mathcal{O}} \simeq T^*Q \times \mathcal{O}$, and the orbit $\mathcal{O} \subset \mathfrak{h}^*$ clearly serves as a classical internal degree of freedom (‘charge’) of the particle. Hence the representation theory of $C^\infty((T^*P)/H)$ is isomorphic to that of $C^\infty(\mathfrak{h}^*)$ with the Lie etc. Poisson structure discussed in subsect. 2.3.

An analogous situation prevails in the quantum case $\mathfrak{A} = JL(P \times_H P)$ [27]. The representation theory of this algebra is isomorphic to that of $JL(H)$ (see subsect. 3.4), hence each irreducible unitary representation π_χ of H on a Hilbert space \mathcal{H}_χ induces an irreducible representation π^χ of \mathfrak{A} , and *vice versa*. The Hilbert space \mathcal{H}^χ carrying the representation $\pi^\chi(\mathfrak{A})$ is naturally realized as $\mathcal{H}^\chi \simeq L^2(Q) \otimes \mathcal{H}_\chi$, so that we see that \mathcal{H}_χ acts as an internal degree of freedom of the particle (a ‘quantum charge’).

To sum up, we see that classical internal degrees of freedom are co-adjoint orbits of a Lie group, whereas the quantum analogue of this is an irreducible unitary representation of the same group, compare with the discussion in subsect. 3.4.

We end in a speculative manner. In [27] one finds a proof of the transitive case of the following

Conjecture 1 *Let G be a Lie groupoid, and \mathfrak{g} its Lie algebroid. Then there exists a quantization relating the Poisson algebra $C^\infty(\mathfrak{g}^*)$ canonically associated to \mathfrak{g} to the Jordan-Lie algebra $JL(G) = C^*(G)_{\text{sa}}$, in the sense of Definition 7.*

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