

Computational complexity of curved space-time

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Abstract

String/M theory, a fundamental theory unifying quantum mechanics, gravity, and the other forces of nature, has a combinatorially large number of vacua (solutions) which are candidates to describe the physics of our universe.

In 2006 with Frederik Denef, we showed that the problem of finding the vacua which could describe our universe in detail is very similar to standard NP complete problems such as the subset sum problem, giving strong reasons to think it is NP hard. This raises the possibility that, even were we to find evidence for string theory and measure all of the relevant observables, we may never know which vacuum we live in.

This also raises the question of how early cosmology searches through these many possibilities. In this talk, we suggest a general definition of complexity classes associated to curved space-times, which might be useful for studying this question.

1 Introduction

The mathematical theory of computation began as an idealization of aspects of human thought, such as mathematical calculation, logical deduction and proof. Its primary questions include: What is a mathematical proof? Can proof be reduced to symbolic manipulation? Given a system of axioms, which mathematical statements are provable in a finite amount of time? Are the answers to these questions universal, or contingent on other definitions or axioms? Fundamental concepts of the theory include Gödel's theorems, and Church-Turing universality.

As people began to build real computers, the theory grew to incorporate idealized aspects of physical law, such as time and space. More precise questions were asked, such as to understand the minimum time and space needed to do a computation, as a function of the size of the input data. Gradually, this question was boiled down into more abstract forms, which capture the most important distinctions while leaving out complicated details. These

are the primary questions of computational complexity theory, of which the most famous example is the conjecture that $P \neq NP$.

Although almost all of the details of real computing are abstracted away in this theory, one real world assumption is nearly always made. This is that space and time have roughly the geometry that we see around us, the geometry of Minkowski space-time. This is a very weak assumption, and an entirely reasonable one, which holds in every part of the universe we can directly observe. But it is by no means the only possibility. Einstein's theory allows for very different space-time geometries, and there is very good evidence that the geometry of our universe at very early and at very late times is not Minkowski space-time, but some other geometry.

The theory of geometry of space-time is explained in textbooks on general relativity. We cannot provide much of an introduction here, but some of its concepts are by now common knowledge. For example, one can have clocks which run at different speeds in different parts of space. This is clearly relevant for a theory of computation, as we can imagine communicating with computers which run much faster or slower than our own. As it turns out, one naturally encounters geometries with exponential ratios of clock speeds, so even distinctions as coarse as P versus EXP can change in this setting.

Length scales also undergo exponential variation in curved space-time, and even in curved space. A simple example is a direct product of ordinary time (so, all clocks run at the same speed), with hyperbolic space. In hyperbolic space, the volume contained in a ball of radius R grows exponentially with R , and thus in this space-time, the number of computers we can communicate with in a time T will grow exponentially in T . This motivates a computational model allowing exponential parallelism.

One can embed an infinite tree in which each node has a fixed valence $k > 2$, into hyperbolic space. Such an embedding is often illustrated by one of Escher's "Circle Limit" series of prints, in which figures such as angels and demons are used to tile a disk. Whereas in the Euclidean metric, the figures decrease in size as one approaches the boundary, when measured in the hyperbolic metric they all have the same size, allowing each node of the tree to take up the same spatial volume. Escher learned about the existence of such tilings from the mathematician Coxeter [1], and was very excited to discover a way to portray the infinite, within a finite image.

Let us suppose we can place a computer at each node of such a tree, and that neighboring nodes can communicate in a single time step. The hyperbolic computing class P_h is then defined as follows. We supply input data at a particular distinguished node, allow arbitrary processing and communication, and ask what outputs can be obtained at the distinguished node within a time which can grow at most polynomially in the size of the input.

To see that P_h includes NP , we can start from the definition of NP as the class of problems which can be solved by a 'nondeterministic' computer in polynomial time, *i.e.* one which at each time step can make an unmotivated choice out of a finite set of possibilities. This definition makes it clear that one can solve such problems on a conventional computer by building a search tree, in which each choice corresponds to a node of the tree. The idea is then simply to embed the search tree into our tree of computers. If a choice must be made

from a set of more than k possibilities, this can be handled by grouping several nodes.

Of course, there are many larger complexity classes. Although hyperbolic space allows exponential parallelism, it is not immediately evident that this can be used freely; one might worry that the tree structure leads to communication bottlenecks. But, as I found out after the lecture, computation in hyperbolic space has been studied by Margenstern and collaborators [2], who have shown that P_h is equivalent to the standard complexity class PSPACE, the problems which can be solved using polynomial space.

What I am suggesting here, is to generalize the definition of the class P_h , first to the networks of computers which could be embedded in a more general Riemannian geometry, and then to the case in which clock speeds can vary throughout the network, in other words to networks which can be embedded into curved space-time. To make this concept precise, we might associate each computing element to a timelike curve, and require distinct computing elements to be separated by a distance ϵ .¹ Communication would then be allowed along any null or timelike curve. This is more or less the definition of a group of ‘observers’ in general relativity. In a general space-time, without the large symmetry group of hyperbolic space, one must also specify the distinguished point at which the input and output is provided.

Along with Minkowski space-time, there are two other simplest model space-times, called the de Sitter and anti-de Sitter space-times. Each of these geometries is maximally symmetric, so they are easy to describe, and has a constant curvature tensor, so that they solve the Einstein equation with a nonzero cosmological constant. The corresponding models of computation, in addition to their intrinsic theoretical interest, might be helpful for understanding cosmology.

Given a computational model m , we define the corresponding complexity class P_m of problems which can be solved by m in time growing polynomially with the size of the input. By the usual arguments using polynomial reductions, most details of the model will drop out, and many different space-time geometries will correspond to the same complexity class.

There is some analogy between the association of space-time geometries to a complexity class, and the concept of causal structure [3].² The causal structure of a spacetime is a relation between a pair of points (a, b) , which can be ‘to the future of,’ ‘to the past of,’ or ‘spacelike separated’ (so that no communication is possible). While in Minkowski spacetime, a pair of observers (inextendible timelike curves) are always in communication, in more general spacetimes this is no longer true, and one speaks of causally disconnected regions. Besides the familiar idea of event horizon, another common situation, encountered in de Sitter space-time, is that time evolution gives rise to a large or even infinite number of distinct causally disconnected regions.

In this situation, one must think more carefully about the definition of ‘solving a problem.’ One might allow for multiple inputs provided by causally disconnected observers, and one must decide what type of output counts as a solution – must all of the final causal regions

¹More precisely, each element occupies a ‘tube’ defined to be the set of points within distance ϵ of the curve; then the tubes must be disjoint.

²Since the scales of space and time depend on the conformal factor, it is clear that the complexity class of a geometry is a finer invariant than causal structure.

receive the answer, or only one? If only one, is there any condition on which one? One version of this, in which any observer can receive the answer, leads to the idea of ‘postselection’ or ‘anthropic computing’ as discussed by Aaronson [4].

In the talk, we briefly summarized the properties of the de Sitter and anti-de Sitter spacetimes, and abstracted the features which enter into a corresponding theory of computation. The rules of this game are somewhat flexible; for example time dilatation goes along with an inverse rescaling of energy, and one needs to decide whether this should somehow be incorporated, or perhaps other physically motivated constraints placed on the nodes and their communication. I hope that interested readers will take this up, and formulate rules which lead to interesting complexity theoretic results. One important question is whether the standard classes between P and PSPACE, such as NP, co-NP, Σ_2 and the like, have any realizations of this type.

The eventual application I have in mind is to the study of very early cosmology, and the problem of vacuum selection. String theory is believed to admit a large number of solutions, many of which are *a priori* candidates to describe our universe. Some of the famous problems of theoretical physics, most notably the cosmological constant problem, have found solutions in this context. The combination of Einstein’s theory and general principles of quantum mechanics lead to a picture of cosmology known as the ‘multiverse,’ in which different regions of the universe are described by different solutions and thus have different effective laws of physics. The dynamics which gives rise to these different regions is called ‘eternal inflation’ and is not well understood, but in general it involves exponentially large times and volumes of space. It might well be useful to idealize this dynamics in terms of a computational model.

In the talk at the workshop, I gave an introduction to this discussion within string theory. Almost all of this material appeared in my paper [5] with Frederik Denef, and thus I will not repeat it here.

References

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