

Lecture 6.5

Josephson junctions

A Josephson junction is one across which superconducting *pairs* can tunnel. The tunneling current depends on the difference of the order parameter phase θ between the two sides, highlighting the physical significance of θ . One striking consequence is the *Shapiro steps*, in which the DC (I, V) curve shows steps in the current at voltages proportional to the *frequency* of a perturbing rf field. Another one is the interference between two Josephson junctions in parallel as a function of the flux between them, which is of great utility since it is the basis of SQUID magnetometers.

The Josephson effect is not just relevant to artificial devices; imperfect superconductors naturally contain Josephson junctions. In particular, granular superconductors (as in thin deposited metal films) can be modeled as a random array of superconducting islands connected by Josephson links. In high-temperature cuprate superconductors, every twinning grain boundary is a Josephson coupling; while in a perfect *untwinned* crystal, the intrinsic coupling between CuO_2 layers also has the form of Josephson tunneling. *Sec. 6.5 W treats a REGULAR array of junctions.*

The Josephson effect should be distinguished from *single*-electron (or quasiparticle) tunneling. In superconductors, that is a probe of the density of states of quasiparticles, which are modeled only at the BCS level (see Lec. 7.5). The microscopic calculation of I_c itself is also touched in Lec. 7.5 .

Sorry, a more complete overview of this chapter could be written.

6.5 A Josephson equations

Consider two pieces of superconductor 1, 2 with a tunnel junction between them. The respective voltages are V_1 and V_2 and the phases are θ_1 and θ_2 . The Josephson junction equations say

$$\frac{d\theta}{dt} = \frac{e_*}{\hbar} V(t) \quad (6.5.1)$$

$$I_{12} = I_c \sin \theta \quad (6.5.2)$$

where $\theta = \theta_1 - \theta_2$ is the phase difference of the superconducting order parameter on the two sides, $V = V_{12} = V_1 - V_2$ is the voltage, and $I = I_{12}$ is the current from 1 to 2 – the *Josephson current*. It is in fact a supercurrent, since it can flow while the voltage is zero (namely if θ has a constant nonzero value – see (6.5.1)). The maximum value of the supercurrent according to (6.5.1) is I_c , which therefore is called the *critical current*: if you tried to pass a larger current, the junction would go normal (or part of the current would have to follow a parallel, resistive path)

If you work out the differential AC response from (6.5.1) and (6.5.2) you see that the Josephson junction behaves roughly as an inductor. In circuit diagrams it is indicated by the symbol \times . [Look in Fig. 6.5.3.]

Potential energy of a junction

The work done by the current at a rate IV must go into a potential energy $U(\theta)$: we have

$$\frac{d}{dt}U(\theta) = IV = (I_c \sin \theta) \left(\frac{\hbar}{e_*} \frac{d\theta}{dt} \right) \quad (6.5.3)$$

whence the “potential energy”

$$U(\theta) = -J_J \cos \theta \quad (6.5.4)$$

with $J_J = \hbar I_c / e_*$ as the coupling constant. This $\cos(\theta_1 - \theta_2)$ coupling looks just like the exchange coupling of two spins that happen to be confined to the XY plane, if θ_i were the spin orientation. (See Sec. Fig. Ex. 6.5.3 for an array of such junctions as the analog of a lattice of spins).

For the complete Hamiltonian we add the charging energy

$$U_C = \frac{1}{8} C_{12} V_{12}^2. \quad (6.5.5)$$

The JJ equations of motion can be derived from (6.5.4) plus (6.5.5) using the general “Hamilton’s equations” as in Sec. 6.1 B .

In any case, the fact that charge is conjugate to phase, and the uncertainty relation between them, is fundamental.

6.5 B Justification of Josephson equations

Let’s return to the idea of a single one-particle wavefunction ψ occupied by a macroscopic number N_b of (Cooper pair) bosons, each with charge $-e_* \equiv -2e$.¹ The wavefunction is

$$|\psi\rangle = \psi_1|1\rangle + \psi_2|2\rangle \quad (6.5.6)$$

with normalization $|\psi_1|^2 + |\psi_2|^2 = 1$.

The Hamiltonian has the form of a hopping model:

$$\hat{\mathcal{H}}_{\text{hop}} = - \sum_{ij} H_{ij} |i\rangle \langle j| \quad (6.5.7)$$

where

$$H = \begin{pmatrix} -e_* V_1 & -w \\ -w & -e_* V_2 \end{pmatrix} \quad (6.5.8)$$

here V_i are respective external potentials due (say) to gate electrodes, and the hopping amplitude is $-w_{12}$. The latter being negative ensures ψ_1 and ψ_2 have the *same* phase, in the ground state.

This derivation is essentially a rerun of the “derivation” of Ginzburg equations in Lec. 6.1 except the bosons’ motion in continuous space is reduced here to hopping on

¹This section is adapted from the concluding chapter (chapter 21) in *The Feynman Lectures in Physics*, vol. III (R. P. Feynman, R. B. Leighton, and M. Sands, Addison-Wesley, 1965).

just two sites, representing the two pieces of superconductor. We could follow either of two routes, as in chapter (a) derive the Josephson potential energy (6.5.4) by taking the expectation of the Hamiltonian (6.5.7) in the single-particle wavefunction (6.5.6); or (b) convert the Schrödinger equation into an equation for macroscopic quantities, as in deriving the time-dependent Ginzburg-Landau equation in Sec. 6.1 B .

Here I follow (b), although (a) might be simpler. There is also a third approach: if you have an S-N-S junction (superconductors with a sandwich of normal state between them), the order parameter $\Psi(x)$ is nonzero in the normal region due to the proximity effect, decaying exponentially with x . Under these circumstances, the GL equations will reduce to the Josephson equations. Finally, there is a fourth approach from the microscopic BCS theory, which is summarized below.

The Schrödinger equation of motion for Hamiltonian (6.5.7) is

$$\frac{d}{dt} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = -\frac{i}{\hbar} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} - \frac{i}{\hbar} = \begin{pmatrix} -e_*V_1 & -w \\ -w & -e_*V_2 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \quad (6.5.9)$$

The first part of the trick for obtaining the Josephson equations is to substitute in Eqs. (6.5.9) the change of variables

$$\psi_i(t) = e^{-i\theta_i(t)} \sqrt{n_i(t)} \quad (6.5.10)$$

For example

$$\frac{d}{dt} |\psi_2|^2 = -\frac{w}{\hbar} (i\psi_1^* \psi_2 - i\psi_2^* \psi_1) = |\psi_1| |\psi_2| \sin(\theta_1 - \theta_2) \quad (6.5.11)$$

(I am now omitting the time arguments in the formulas.)

The second part of the trick is to reinterpret the variables as representing macroscopic quantities. Thus, the total number on each site is $N_i \equiv N_b |\psi_i|^2$. The number current from 1 to 2 is $-dN_1/dt = dN_2/dt = N_b d|\psi_2|^2/dt$; multiply this by $-e_*$ to get the electric current,

$$I(t) \equiv I_{1 \rightarrow 2} = -e_* \frac{2w}{\hbar} |\psi_1 \psi_2| N_b \sin(\theta_1 - \theta_2) \equiv I_c \sin \theta(t) \quad (6.5.12)$$

which is (6.5.2). (The actual variations in N_i are small compared to N_i , even in a nanoparticle, so we can treat the factor $|\psi_1 \psi_2|$ as a constant.)

From (6.5.9), and substituting from (6.5.10) it also follows that

$$\frac{d\theta_i(t)}{dt} = -\frac{e_*}{\hbar} V_i(t) \quad (6.5.13)$$

– and (6.5.1) follows – provided we define

$$-e_* V_i(t) = \frac{d}{d|\psi_i|^2} \langle \hat{\mathcal{H}}_{\text{hop}} \rangle. \quad (6.5.14)$$

The physical meaning is that what we call the “voltage” V_i is the (electro)chemical potential – the cost of adding or removing a charge. In the (usual) limit of nearly decoupled islands, when $w \ll e_* |V|$, the voltage simply is $-e_* V_i(t)$. In this case, $|1\rangle$ and $|2\rangle$ are practically eigenfrequencies, and then the familiar $e^{-iE_i t/\hbar}$ dependence implies the phase winds according to $d\theta_i/dt = -e_* V_i(t)$, giving (6.5.1) trivially.

Microscopic formula for Josephson critical current

It can be shown (Lec. 7.5) from the microscopic theory² that the Josephson critical current is related to the resistance R_n the same junction would have in the *normal* state, by

$$I_c R_n = \frac{\pi \Delta(0)}{e_*} \quad (6.5.15)$$

where $\Delta(0)$ is the gap at $T = 0$, a property of the material not the geometry. I emphasize this holds for the I_c and R_n of each particular junction.³

The l.h.s. of (6.5.15) is technically pertinent: the frequency $e_* I_c R_n / \hbar \sim \pi \Delta(0) / \hbar$ turns out to be the junction's maximum switching rate. Beyond the BCS theory, the l.h.s. need not equal the gap; e.g., it may be increased by tuning the system near to its metal-insulator transition.⁴

6.5 C Statics and dynamics of a junction

We next study the statics and dynamics resulting from these – classical! – equations of motion. We have a (classical) Hamiltonian in terms of θ_i 's. It turns out the boson pair numbers N_i are canonically conjugate to the θ_i . drives in the point that the phase angle θ_i which turn out to be the pair numbers n_i .

[Do not trust all signs in this section!]

Mechanical analogy of motion

In an isolated pair, the voltage is related to the charge by

$$Q(t) = CV(t) \quad (6.5.16)$$

where C is a capacitance [I should call it C_{12}] between the two superconducting islands; $Q(t)$ and $-Q(t)$ are the deviations of islands 1 and 2 from neutrality.

This dynamical equations from inserting (6.5.16) are

$$\frac{d\theta}{dt} = \frac{e_*}{\hbar C} Q(t) \quad (6.5.17)$$

$$\frac{dQ}{dt} = -I_c \sin \theta = -\frac{e_*}{\hbar} \frac{dU(\theta)}{d\theta} \quad (6.5.18)$$

This is exactly the same as the Hamilton's equations if we added a "kinetic" energy $Q^2/2C$ to (6.5.4). Here θ is proportional to a generalized coordinate, $dU(\theta)/d\theta$ is analogous to a force, and $Q \propto N_1 - N_2$, is the momentum conjugate to θ , so this looks analogous to the KE of a relative coordinate.) (**WARNING: signs may be in**

²V. Ambegaokar and A. Baratoff, Phys. Rev. Lett. 10, 486 (1963); erratum, 11, 104 (1963). *Actually they derived the $T > 0$ dependence; the basic formula was already derived by Josephson and by Anderson, I think.* Here's an elementary-language explanation: I_c is proportional to w , a *pair hopping amplitude*; it consequently is proportional to $|A|^2$, where A is the hopping amplitude for a single electron and the A^* factor is for the other electron. On the other hand, as elucidated e.g. in Lec. 2.1 the normal conductance R_n^{-1} is a *probability* rate for hopping of *one* electron, a process with amplitude A . Then by Fermi's golden rule, $R_n \propto |A|^2$ for that process.

³A first step to seeing the plausibility of a relation of form (6.5.15) is to imagine the junction's surface area is varied while tunnel barrier is kept constant: then I_c scales as the area while R_n scales inversely, so the product indeed would remain constant.

⁴J. M. Freericks et al

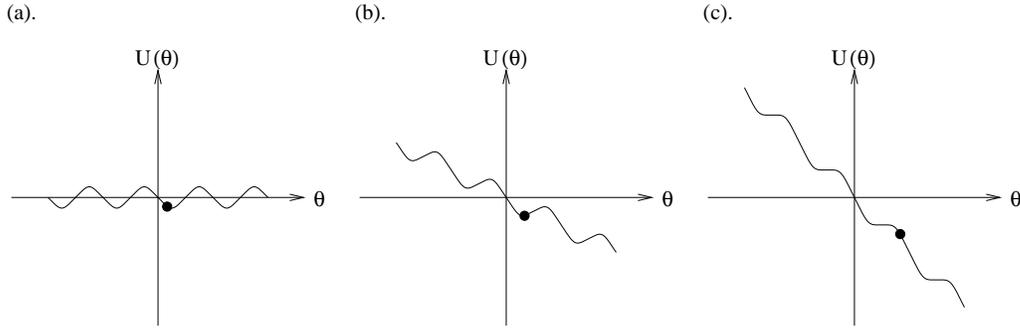


Figure 6.5.1: Washboard potential. (a) zero I_{ext} (b) small I_{ext} (c) large I_{ext} . The dot represents the instantaneous $\theta(t)$.

error.) Indeed, these are exactly Newton's equations for the motion of a pendulum (parametrized so the pendulum shaft points along θ); it equally well describes the classical motion of a particle in a sinusoidal potential. To round out the analogy, $V(t)$ is a velocity, C is a mass, and I_c is like the force of gravity mg .

This is analogous to an LC circuit with $\omega_J^2 = 1/L_{eff}C$, since the Josephson junction behaves like an inductance L_{eff} – but only for small currents, since this is a nonlinear circuit element. That resulting frequency of small oscillations is $\omega_J^2 = e_* I_c / \hbar C$, called the *Josephson plasma frequency*, because the charge is sloshing back and forth as in a plasma mode. That is, the Josephson plasma oscillation is analogous to the bulk plasma oscillation in the same fashion that the Josephson coupling is analogous to the phase stiffness (superfluid density) and the Josephson current is analogous to the G-L supercurrent. Indeed, the plasma frequency is nonzero in a metal on account of the long-range Coulomb interactions, which are expressed here by $1/C$.⁵

Note too that there is usually an Ohmic shunt resistance $R_{||}$ in parallel with the Josephson tunneling, at least when $T > 0$. (Not to be confused with R_n .) It is due to single-electron tunneling across the barrier (which in turn comes from Bogoliubov quasiparticle excitations found at $T > 0$). This serves as viscous damping of the “pendulum” motion. [I should write it in the equation. It would also be nice to have a figure of a pendulum – perhaps in water, to represent the damping – add this to the chart of analogies.]

Analogy to spin resonance

We can make an analogy of Josephson effect to spin resonance: $\theta(t)$ is analogous to the angle of the spin around the field axis, and $V(t)$ is analogous to the constant field.

The Josephson coupling is quite analogous to the exchange coupling of two spins

$$U_{exch} = -J_{12} \mathbf{s}_1 \cdot \mathbf{s}_j \quad (6.5.19)$$

(see Lec. 4.0 (?)). When the spins are confined to the XY plane (in spin space), so $\vec{s}_i = (\cos \theta_i, \sin \theta_i, 0)$, then $U_{exch} = -J_{12} \cos(\theta_1 - \theta_2)$, the same functional form as the Josephson effect. The closest analogy would be the exchange coupling of two ferromagnetic particles in contact, which has form (6.5.19). Indeed, we can push the spin

⁵Do not confuse the “Josephson plasma frequency” with the “Josephson frequency.” The former means $\theta(t)$ oscillates back and forth in the bottom of a well in Fig. 6.5.1, dependent on the effective inductance L_{eff} and some capacitance. The latter means $\theta(t)$ is sliding forever (with some damping) down the modulated slope of the washboard potential.

analogy further: when the respective moments are not collinear, there is necessarily a spin current from 1 to 2, proportional (it turns out) to $\mathbf{s}_1 \times \mathbf{s}_2$ ⁶ In the continuum description of any bulk magnet, a twist (gradient) in the order parameter direction similarly corresponds to a spin current, analogous to the GL supercurrent.

6.5 D The Josephson effects

Attach leads to the two islands. The current is given by the Josephson formula, but the charge is no longer confined to sloshing back and forth between the two islands: now steady states are possible, as well as additional oscillating states.

SORRY: in the explanations below, I did not sort out the role of resistance.

DC Josephson effect

The equations allow solutions with nonzero I_{DC} , but only with zero V_{DC} ; the phase adjusts itself. If you drive with a current bias, there's a solution only for $|I_{DC}| < I_c$, but not beyond.

Basic AC Josephson effect

First consider the voltage-driven case. If you apply a nonzero V_{DC} , the Josephson equation says you get out I_{AC} at a frequency ($f = \omega/2\pi$) proportional to voltage:

$$f \equiv \omega/2\pi = 0.48 \frac{\text{GHz}/\mu\text{V}}{V_{DC}}. \quad (6.5.20)$$

This, combined with the DC result just before, means the I-V curve is weird. For all applied V_{DC} , the resulting I_{DC} is zero, except at $V_{DC} = 0$ where any $|I_{DC}| < I_c$ is possible.

The frequency is so high, it's hard to detect! Therefore, the current driven case (leading to a nonzero average voltage) is the most commonly observed. We can imagine driving the system with a constant current source I_{ext} so the equation becomes

$$\frac{dQ}{dt} = I_{ext} - I_c \sin \theta \quad (6.5.21)$$

This is the equation of motion that would be obtained from an effective potential modified as follows:

$$U(\theta) \rightarrow U(\theta) - \frac{\hbar}{e_*} I_{ext} \theta \quad (6.5.22)$$

i.e., a net downwards tilt is added to the potential; the result is called a “washboard potential” (Fig. 6.5.1(b,c)). The damping – which I omitted from the equations of motion – ensures that for small I_{ext} the system finds steady-state equilibrium in one of the minima of Fig. 6.5.1(b), which means there is zero voltage drop. For $I_{ext} > I_c$, there is no minimum (Fig. 6.5.1(c)), which means the system rolls “downhill” at a fixed mean drift rate (owing to damping) $\omega_0 \equiv \overline{d\theta/dt}$. Now recall that $d\theta/dt$ is the voltage; hence this state has a nonzero mean voltage, which by (6.5.1) is simply

$$V_{DC} = \frac{\hbar\omega_0}{e_*}. \quad (6.5.23)$$

⁶This is pertinent to SO(5) theory of high- T_c superconductivity, see Lec. 8.4 .