

Analytic solution for entangled two-qubit in a cavity field

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An exact solution of the time-dependent master equation that describes the evolution of two two-level qubits (or atoms) within a perfect cavity for the case of multi-photon transition and in the presence of both the Stark shift and phase shift, is obtained. Employing this solution, the significant features of the entanglement when a second qubit is allowed to interact with cavity mode and becomes entangled with the first qubit, is investigated in the context of the measure defined by negative eigenvalues for the partial transposition of the density operator. The effects of Stark shift, distance between the two qubits and an instantaneous phase shift experienced by the second qubit, on the entanglement and probability amplitudes are indicated. It has been shown that, the entanglement as well as the intensity are markedly affected by different parameters when nonlinear two-photon process is involved. Moreover, the quasiprobability distribution function is investigated before and after the sudden phase shift experienced by the second qubit. We believe that this may throw some light on the question of the entanglement of multi-qubit systems.

1 Overview

Investigations into the emerging science of quantum information has led to the widespread belief that entanglement in states shared between two systems can be used as a resource in nonclassical applications [1-3]. The theory of quantum entanglement has occupied a central place in modern research because of its promise of enormous utility in quantum computing, cryptography, etc [4-8]. A major thrust of current research is to find a quantitative measure of entanglement for general states. One of the most intriguing problems of quantum mechanics is the interpretation of the measurement process (for an overview of fundamental problems in quantum measurement, see for example Ref. [9-11]). The reason for this central role of the measurement process is the absence of fundamental, elements of reality, that would simultaneously characterize both the

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dynamics and the measurement results. At this end, in quantum information, the maximally entangled states have a special significance.

For the experiments in the newest fields of physics, quantum computing, quantum communication, and quantum cryptography [4-8] the quantitative analysis of the multi-qubit or ion is of substantial interest [12]. The dipole-dipole interaction between two atoms can be understood through the exchange of virtual photons and depends on the transition dipole moment of the levels involved. It can be characterized by complex coupling constants, or by their real and imaginary parts, where the former affect decay constants and the latter lead to level shifts [13]. There is an inherent interest in analytical and non-perturbative solutions of multi-atom interacting with the cavity field problems, all the more considering quantum systems with more than one particle. One examples of such kind is the system of two two-level qubits in an electromagnetic field [14-20]. Entanglement of identical particles is a property dependent on which single-particle basis is chosen, as any operation should act on each identical particle in the same way. Indeed, individual particles are excitations of a quantum field, and the single-particle basis defines which set of particles are used in representing the many-particle state [21].

One aim of the present paper is to extend the previous models to a much more general model. To be more precise, we assume that two two-level atoms (two qubits) share a bipartite system, taking into account the multi-photon transition and the presence of Stark shift with the second qubit undergoing phase shift. The phase shift could be realized, by a short pulse $\exp(i\phi)$, taking the qubit, which undergoes a phase shift, out of resonance for a short period of time. Another principal aim is to elucidate the extent to which mixed entangled states can affect the entanglement. The emphasis being put on the investigation of the entanglement in a more general situation in which the two atoms (qubits) share a mixed state, rather than a pure state. The issue of attributing objective properties to the constituents of a quantum system composed of identical atoms, does not turn out to be a straightforward generalization of the just analyzed case involving distinguishable atoms, and the problem of entanglement has to be reconsidered. Entangled mixed states may arise when one or both atoms (qubits) of an initially pure entangled state interact, intentionally or inadvertently, with other quantum degrees of freedom resulting in a non-unitary evolution of the pure state into a mixed state. In general it is known that there are also cases when entangled states are mixed with other entangled states and where the sum is separable.

The outline of this paper is arranged as follows: in section 2, we give notations and definitions of the model and its analytical solution to be used in the rest of the paper. The entanglement measure calculation is presented in section 3. By a numerical computation, we examine the influence of distance between the qubits, Stark shift and phase shift on the evolution of the measure of entanglement which will be defined in terms of the negative eigenvalues of the partial

transposition. Finally, section 4 has a few concluding remarks, and a few avenues for further investigations are indicated.

2 Two qubits model

In this section a theoretical model for describing the time evolution of a two two-level qubits (or atoms) interacting with a cavity field. The model differs from the standard micromaser set-up in that instead of a single qubit we have assumed a pair of qubits interacting with a single mode of the cavity field. The position of the first qubit in the cavity is fixed and the second qubit is at some distance L from it. This distance will be a variable parameter of the problem. After a certain interaction time, the second qubit experiences a phase shift with respect to the field [9-11]. Our interest lies in the case where the Stark shift and phase shift are included. In the dipole and rotating wave approximation, we can write ($\hbar = 1$)

$$\begin{aligned} \hat{H} = & \hat{a}^\dagger \hat{a} (\beta_1 S_{ge}^{(1)} S_{eg}^{(1)} + \beta_2 S_{eg}^{(1)} S_{ge}^{(1)}) + \hat{a}^\dagger \hat{a} (\beta_1 S_{ge}^{(2)} S_{eg}^{(2)} + \beta_2 S_{eg}^{(2)} S_{ge}^{(2)}) \\ & + \omega \hat{a}^\dagger \hat{a} + \frac{m\omega}{2} (S_{ee}^{(1)} - S_{gg}^{(1)}) + \frac{\omega}{2} (S_{ee}^{(2)} - S_{gg}^{(2)}) + \gamma_1 S_{eg}^{(1)} \hat{a}^m \\ & + \gamma_1^* S_{ge}^{(1)} \hat{a}^{\dagger m} + \gamma_2 e^{i\phi} S_{eg}^{(2)} \hat{a}^m + \gamma_2^* e^{-i\phi} \hat{a}^{\dagger m} S_{ge}^{(2)}. \end{aligned} \quad (1)$$

We denote by β_1 and β_2 the intensity-dependent Stark shifts, that are due to the virtual transitions to the intermediate relay level. $S_{lm}^{(i)}$ are atomic operators for the i^{th} qubit. \hat{a} and \hat{a}^\dagger are field operators corresponding to annihilation and creation of photons in the cavity mode. We denote by γ_i the coupling constant for the i^{th} qubit.

We consider initially the two qubits with composite states represented by density matrices ρ_1, ρ_2 that operate on the Hilbert space $C^n \otimes C^n$. For a given entangled state ρ_1 , it has been shown that [22] there exist $\rho_2 \in S$, such as $b\rho_1 + (1-b)\rho_2 \in S$, where $b \in [0, 1]$, and S denotes the set of separable states [23]. The usual interpretation of mixed states, is that their creation involves irreversibly destroying information. This has interesting consequences concerning entanglement theory, since there, the irreversibility is often associated with the fact that one is dealing with mixed states. In this paper, we assume different situation in which the two qubits are initially in the following mixed state,

$$\rho = r |e_1, g_2\rangle \langle e_1, g_2| + (1-r) |g_1, e_2\rangle \langle g_1, e_2| \in \mathbf{S}_A. \quad (2)$$

We may write the initial state of the field in vacuum state as,

$$\varpi = |0\rangle \langle 0| \in \mathbf{S}_F. \quad (3)$$

The continuous map \mathcal{E}_t^* describing the time evolution between the qubits and the field is defined by the unitary evolution operator generated by \hat{H} such that

$$\mathcal{E}_t^* : \mathbf{S}_A \longrightarrow \mathbf{S}_A \otimes \mathbf{S}_F,$$

$$\mathcal{E}_t^* \rho = \widehat{U}_t (\rho \otimes \varpi) \widehat{U}_t^*. \quad (4)$$

The interaction Hamiltonian, in this case, leads to an exactly solvable time evolution operator. Resuming our analysis, the time evolution operator can be written as

$$\widehat{U}_t \equiv \exp \left(-\frac{i}{\hbar} \int_0^t \widehat{H}(t') dt' \right), \quad (5)$$

i.e. \widehat{U}_t satisfies the interaction picture Schrödinger equation. Most of the authors who have treated multi-qubits systems interacting with cavity fields have dealt with the case in which the Stark shift has been ignored [9-11]. However, in reality it cannot be ignored. Our interest in the current article lies in looking for a time dependent analytical solution even when the Stark shift and phase shift are non-zero. Using the above equations, in the interaction picture and after some algebraic manipulations we find that the final state $\mathcal{E}_t^* \rho$ at any time $t > 0$ is given by

$$\mathcal{E}_t^* \rho = \sum_{i=1}^3 \sum_{j=1}^3 U_{ij}(t) |\psi_i\rangle \langle \psi_j|, \quad (6)$$

where

$$\begin{aligned} U_{11}(t) &= rA_t A_t^* + (1-r)B_t B_t^*, & U_{12}(t) &= rA_t C_t^* + (1-r)B_t C_t^*, \\ U_{13}(t) &= rA_t B_t^* + (1-r)B_t A_t^*, & U_{22}(t) &= C_t C_t^*, \\ U_{23}(t) &= rC_t B_t^* + (1-r)C_t A_t^*, & U_{33}(t) &= rB_t B_t^* + (1-r)A_t A_t^*, \end{aligned} \quad (7)$$

$\rho_{ij}(t) = \rho_{ji}^*(t)$, and

$$\begin{aligned} A_t &= -\frac{\gamma_2^2}{\mu_1} - \frac{\gamma_1^2 \exp(-im\beta_1 t)}{\mu_1} \left(\cos \mu t + im\beta_1 \frac{\sin \mu t}{\mu} \right), \\ B_t &= -\frac{\gamma_1^* \gamma_2 \exp(i\phi) \exp(-im\beta_1 t)}{\mu_1} \left(\cos \mu t + m\beta_1 \frac{\sin \mu t}{\mu} \right) + \frac{\gamma_1^* \gamma_2 \exp(i\phi)}{\mu_1}, \\ C_t &= -i\gamma_1^* \sqrt{m!} \exp(-im\beta_1 t) \frac{\sin \mu t}{\mu}, \end{aligned} \quad (8)$$

and

$$\begin{aligned} \mu &= \sqrt{m^2 \beta_1^2 + m!(\gamma_1^2 + \gamma_2^2)}, & \mu_1 &= (\gamma_1^2 + \gamma_2^2), \\ \psi_1 &= |e_1, g_2, 0\rangle, & \psi_2 &= |g_1, e_2, 0\rangle, & \psi_3 &= |g_1, g_2, m\rangle. \end{aligned} \quad (9)$$

We have therefore obtained an analytical solution of the final state of the system for this general model. Having obtained the explicit form of the final state of the system under consideration, we are therefore in a position to discuss the statistical properties of the system.

3 Entanglement

The characterization and classification of entanglement in quantum mechanics is one of the cornerstones of the emerging field of quantum information theory. Although an entangled two-qubit state $\mathcal{E}_i^* \rho$ is not equal to the product $\mathcal{E}_i^* \rho_1$ and $\mathcal{E}_i^* \rho_2$ of the two single-qubit states contained in it, it may very well be a convex sum of such products. In general it is known that microscopic entangled states are found that to be very stable, for example electron-sharing in atomic bonding and two-qubit entangled photon states generated by parametric down conversion. Entanglement as one of the most nonclassical features of quantum mechanics is usually arisen from quantum correlations between separated subsystems which can not be created by local actions on each subsystem. By definition, a mixed state of a bipartite system is said to be non-entangled if it can be written as a convex combination of pure product states. Although, in the case of pure states of bipartite systems it is easy to check whether a given state is entangled or not, the question is yet an open problem in the case of mixed states. There is also an increasing attention in quantifying entanglement, particularly for mixed states of a bipartite system. Some of these measures are (i) The entanglement of formation which provides a natural quantitative measure of entanglement with a clear physical motivation [24] for a mixed state ρ . It is defined as the minimal average number of maximally entangled pure states consumed in order to realize the ensemble described by ρ , i.e.,

$$E_F(\rho) = \min_{\{p_i, \psi_i\} \in D} \sum_i p_i S(\rho_i^F), \quad (10)$$

where D is a set that includes all the possible decompositions of pure states $\rho = \sum_i p_i \rho^i$. A slight modification has been suggested in Ref. [25] where, it has been proposed that the total entanglement of formation (entanglement cost) can be written as

$$E_c^{tot}(\rho) = \lim_{n \rightarrow \infty} \frac{E_F(\rho^{\otimes n})}{n}, \quad (11)$$

where $\rho^{\otimes n}$ is the n -fold tensor product of ρ . This is the asymptotic value of the average entanglement of formation. (ii) The relative entropy of entanglement [26-28] which is defined by

$$\begin{aligned} E_R(\rho) &= \min_{\rho^R \in R} S(\rho || \rho^R) \\ &= -S(\rho) + \min_{\rho^R \in R} Tr(-\rho \log \sigma). \end{aligned} \quad (12)$$

A method using quantum relative entropy to measure the degree of entanglement (DEM) in the time development of the Jaynes-Cummings model has been adopted in [29]. The DEM in the time development of a generalized two-level atom has also been formulated [30,31]. Accordingly,

it is generally seen as a desirable property of the axiomatic characterization of entanglement measures that it allows only for entanglement measures which coincide with the von Neumann reduced entropy on pure states [32-35]. (iii) The measure defined by negative eigenvalues for the partial transposition of the density operator [36,37]. It was proved that the negativity is an entanglement monotone [38], hence, the negativity is a good entanglement measure.

In this paper we take the measure of negative eigenvalues for the partial transposition of the density operator. According to the Peres and Horodecki's condition for separability [39,40], a two-qubit state for the given set of parameter values is entangled if and only if its partial transpose is negative. The measure of entanglement can be defined in terms of the negative eigenvalues of the partial transposition in the following form

$$I_{\mathcal{E}_t^* \rho}(t) = 2 \max(0, -\lambda_{neg}) \quad (13)$$

where λ_{neg} is the sum of the negative eigenvalues of the partial transposition of the time-dependent reduced atomic density matrix ρ^a , which can be obtained by tracing out the field variables

$$\rho^a = Tr_f(\mathcal{E}_t^* \rho). \quad (14)$$

In the two qubit system ($C^2 \otimes C^2$) it can be shown that the partial transpose of the density matrix can have at most one negative eigenvalue [40]. The partial transposition of ρ^{aT} has four eigenvalues one of which is negative, then the entanglement is given by

$$I_{\mathcal{E}_t^* \rho}(t) = \sqrt{U_{33}^2(t) + 4U_{21}(t)U_{12}(t) - U_{33}(t)}.$$

The entanglement measure then ensures the scale between 0 and 1 and monotonously increases as entanglement grows. An important situation is that, when $I_{\mathcal{E}_t^* \rho}(t) = 0$ the two qubits are separable and $I_{\mathcal{E}_t^* \rho}(t) = 1$ indicates maximum entanglement between the two qubits. It was proved [38] that the negativity is an entanglement monotone, and hence is a good entanglement measure.

An interesting question is whether or not the entanglement is affected by the different parameters of the present system with the initial state in which one of the qubits is prepared in its excited state and the other in the ground state. In particular, the mixed state parameter r , the Stark shift parameter β_1 , the distance between the qubits L , and the phase shift ϕ . A numeric evaluation of the entanglement measure leads to the plot in figure 1a. We consider the coupling of the first qubit is taken to be constant $\gamma_1 = \gamma$. Although we allow the second qubit to be at a distance L away and experience a variable qubit field coupling $\gamma_2 = \gamma \cos kL$, (where we set $\beta_1 = 0$, $L = 0$, $\phi = 0$, and $r = 1$). It is shown that the two qubits are entangled in this case, with the maximum value $I_{\mathcal{E}_t^* \rho}(t) \approx 0.2$ for $r = 1$, $I_{\mathcal{E}_t^* \rho}(t) \approx 0.4$ for $r = 0.6$ and $I_{\mathcal{E}_t^* \rho}(t) \approx 0.7$ for $r = 0.2$. In all these cases we see that for some interaction times the entanglement is equal

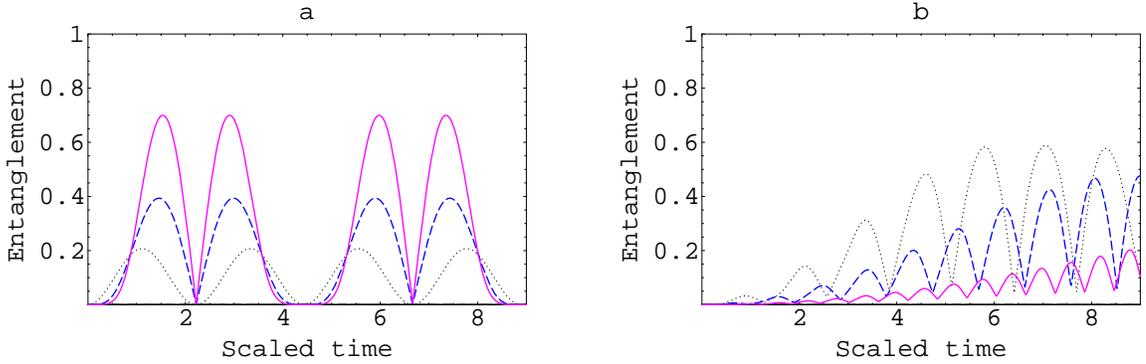


Figure 1: This figure presents the results of a numerical calculation of the time evolution of the negativity as a measure of entanglement. The parameters $\phi = 0, L = 0$ and the number of quanta $m = 2$, where (a) $r = 0.2$ (solid curve), $r = 0.6$ (sashed curve) and $r = 1$ (dotted curve) and (b) $r = 0.2, \beta_1 = 5\gamma$ (solid curve), $\beta_2 = 3\gamma$ (dashed curve) and $\beta_1 = 2\gamma$ (dotted curve).

to zero, this period is increased with decreasing the parameter r . We now consider the two qubits interacting with the cavity field in the presence of Stark shift parameter β_1 , where we set three different values of β_1 for the sake of comparison (see figure 1b). It is remarkable to see that with the value of the Stark shift parameter, $\beta_1 = 2$ the entanglement is nearly zero for the initial period of the interaction time. This period increases with increasing the Stark shift. Also, the maximum value of the entanglement is decreased with increasing β_1 . In this case we can say that, when the system is allowed to evolve without applying a phase shift, the entanglement degree is a periodic function of time. This is particularly because of the nonlinear nature of the coupling in this case (two-photon process).

We now pause to touch on certain entanglement features when a phase shift is applied to the second qubit at the time τ . To this end we consider the same values of the other parameters similar to figure 1a. Three important values for the timing of the phase shift, namely $\tau = \pi(J \pm 0.25), 3\pi$ and $J\pi$, (here $J = 3$) corresponding to the maximum and minimum values of the probability of emission of photon pair. This is illustrated in figure 2, where we have shown the time evolution of the entanglement for different values of the timing of the phase shift and $\phi = \frac{\pi}{2}$. First of all, we note that the regular behavior has been seen only for the case when $\tau = 3\pi$. While for the other two values i.e for $\tau = \pi(J \pm 0.25)$, we see that there are two maximum values of the entanglement $I_{\mathcal{E}^*_\rho}(t) \approx 0.5$ and 0.97 . This situation is quit different from the maximum values of the entanglement at $\tau = 3\pi$, where the two peaks have the same maximum value at $I_{\mathcal{E}^*_\rho}(t) \approx 0.85$. It should be noted that the entanglement vanish for some period of the interaction time only when $\tau = 3\pi$. These properties show that the role played by the timing of the phase shift on the entanglement is essential. Interestingly, when

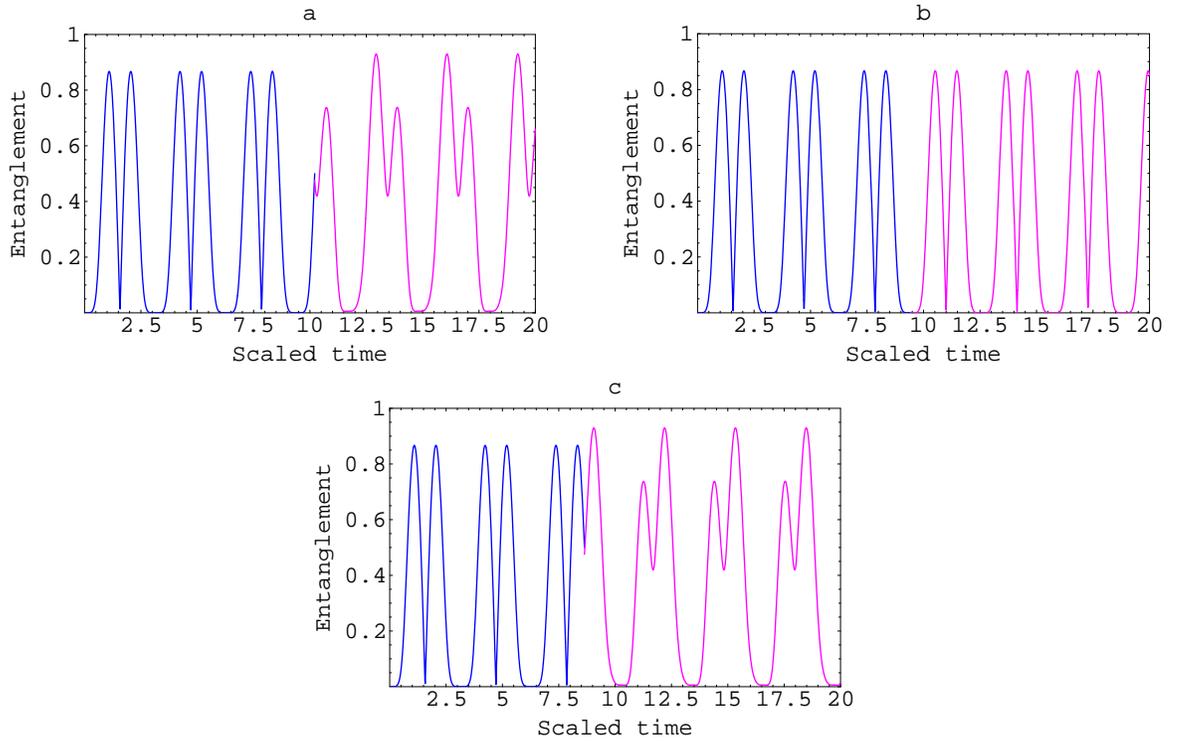


Figure 2: This figure presents the results of a numerical calculation of the time evolution of the negativity as a measure of entanglement. $\phi = \pi/4, \beta_1 = 0, L = 0, m = 2$ and for different values of the timing of an applied phase step, τ , where (a) $\tau = 13\pi/4$, (b) $\tau = 4\pi$ and (c) $\tau = 11\pi/4$

r is taken to be non zero, ($r = 0.2$), the values of the maximum entanglement are decreased, indicating that the mixed state setting leads to a decreasing of the qubit-qubit entanglement. While the qubit-qubit entanglement has the same feature for different values of the mixed state parameter r , the change only occurs on the amplitude of the oscillations (see figure 2).

In the previous discussion, we have discussed different excitation processes which can prepare two qubits in the asymmetric state and we have assumed that both the qubits in the same position i.e $L = 0$. This assumption is only valid if the coupling parameters $\gamma_1 = \gamma_2$. The analysis involved single mode cavities, but ignored spontaneous emission from the qubits and the cavity damping. Here, we will extend this analysis to consider that two qubits separated by an arbitrary distance L , and we would like to highlight briefly why the qubit-qubit entanglement measure might have different features in the presence of the a distance between the qubits. Figure 3a, shows the basic features of the behavior of the qubit-qubit entanglement with different values of the distance between the qubits L . We remark that the entanglement

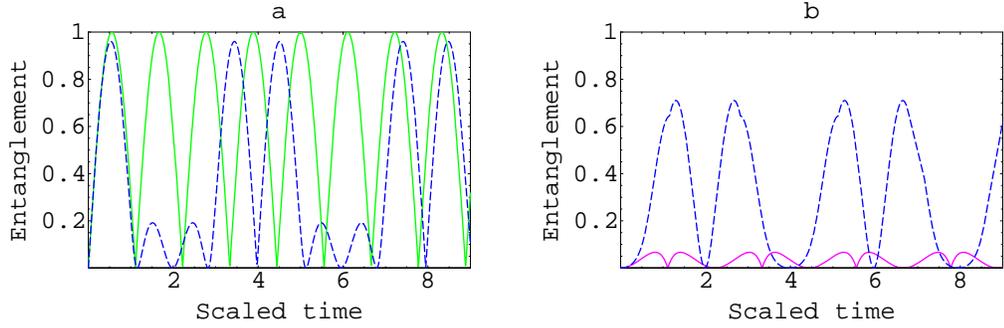


Figure 3: Plot of the negativity as a measure of entanglement against the scaled time γt , the parameters $\beta_1 = 0, \phi = 0, r = 1$ and the number of quanta $m = 2$, where (a) $L = \frac{\pi}{2}$ (solid curve), and $L = \frac{\pi}{3}$ (dotted curve). (b) the same as (a) but $r = 0.2$.

has some kind of periodicity (see figure 3a) when $L = \frac{\pi}{2k}$. It is important to note here that, the minimum value of the entanglement is achieved i.e $I_{\mathcal{E}_t^* \rho}(t) \approx 0$ which means that the two qubits are separable and also the maximum values $I_{\mathcal{E}_t^* \rho}(t) = 1$ is reached which indicates maximum entanglement between the two qubits. Therefore, one can say that an appropriate choice of distance between the two qubits leads to, on one hand, a complete separability between the qubits at other values of the interaction time and on the other hand maximum entanglement in other intervals of the interaction time. Meanwhile, the general feature of the entanglement in the case $L = \frac{\pi}{3k}$ is dramatically changed (see figure 3a). This behavior is affected once the mixed state parameter r is decreased (see figure 3b). It is noticed that the amount of entanglement is strongly decreased due to setting $L = \frac{\pi}{2k}$ and $r = 0.2$, while it increased again when $L = \frac{\pi}{3k}$. Generally speaking, because of the influence of mixed state parameter on entanglement, the

amplitude of local maxima and minima decrease with increasing the deviation of r from the unity. However, as r takes values close to the unity we return to the same behavior in the initial pure state setting i.e $\rho = |e_1, g_2\rangle \otimes \langle e_1, g_2|$. However a slight change in r therefore, dramatically alters the entanglement. This is remarkable as the entanglement is strongly dependent on the initial state, which can be entangled or unentangled. Here, we clarify that it can be done by

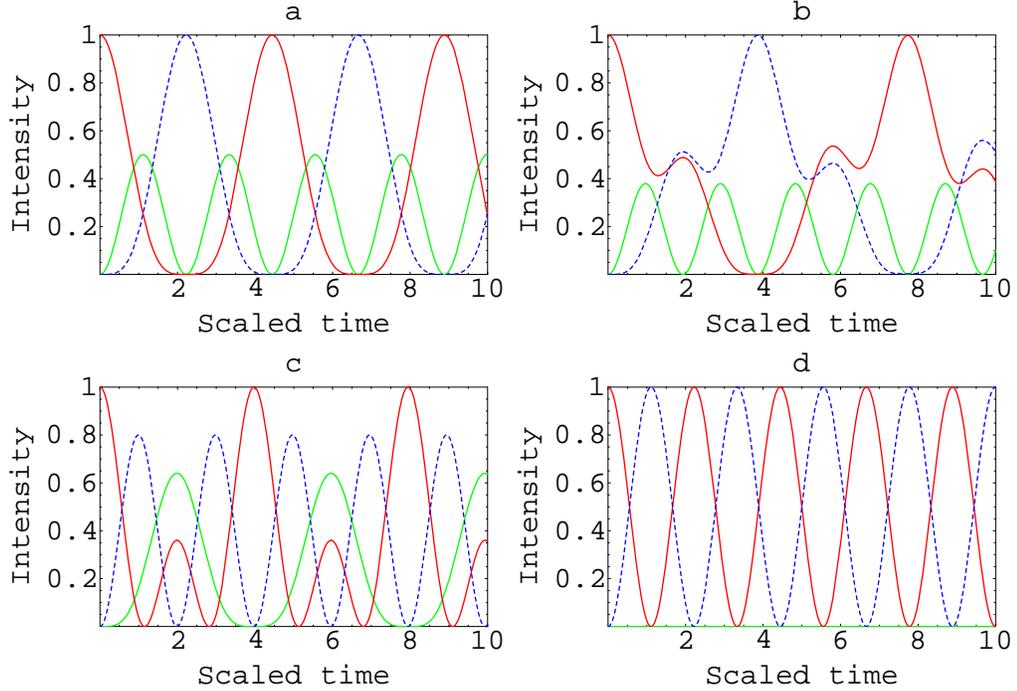


Figure 4: This figure presents the results of the numerical calculations of the time evolution of the probability of the first qubit in excited state (solid line), probability of the second qubit in excited state (dotted line) and probability of emission of photon pair (dashed line), where (a) $\phi = 0, \beta_1 = 0, L = 0, m = 2, r = 1$, (b) $\beta_1 = 0.8\gamma$ (c) $\phi = 0, \beta_1 = 0, L = \frac{\pi}{3}, m = 2, r = 1$, (d) $\phi = 0, \beta_1 = 0, L = \frac{\pi}{2}, m = 2, r = 1$.

using the different initial state, which is strongly affected by the qubit number representation. This naturally leads to the use of occupation numbers of different single-qubit basis states in quantifying identical-qubits entanglement even when the number of qubits is conserved. The occupation-numbers of different modes have already been used in quantum computing [41].

To this end, we devote the discussion in figure 4 to consider the effect of these different parameters on the probability of the first qubit (or atom) in excited state, the probability of the second qubit in excited state and probability of emission of photon pair. We would like to remark that when qubit 2 is initially prepared in either $|g_2\rangle$, ($r = 1$) or $|e_2\rangle$, ($r = 0$), which are stationary states for qubit 2, there will be substantial changes to the evolution of qubit 1 due

to phase shifts introduced in the field through the dispersive interaction. The subsystem qubit 2 plays the role of a single qubit reservoir [42], in the sense that it will induce modifications in the subsystem qubit 1 without having its state changed. If we increase the value of Stark shift parameter $\beta_1 = 0.8$, we have the situation shown in figure 4b. We note a stronger modulation in the oscillations and a clear departure from ordinary Rabi oscillations is verified (see figure 4). The populations clearly exhibit the characteristic features observed in the entanglement behavior and provide us with information about discrete nature of the quantized atom-field interaction.

It is important to refer here to the work in Ref. [43] in which experimentally a superposition state of the ground state and a non-maximally entangled antisymmetric state in two trapped ions has been realized. In the experiment two trapped barium ions were sideband cooled to their motional ground states. Transitions between the states of the ions were induced by Raman pulses using co-propagating lasers. The non-maximally entangled state was used [44] to demonstrate the intrinsic difference between quantum and classical information transfers. The difference arises from the different ways in which the probabilities occur and is particularly clear in terms of entangled states.

There exists a neat explanation from the phase space point of view. Next, we will compute the relevant field quasi probability in phase space. The Q-function computed for the field reduced density matrix $\rho^f(t)$ in the following form

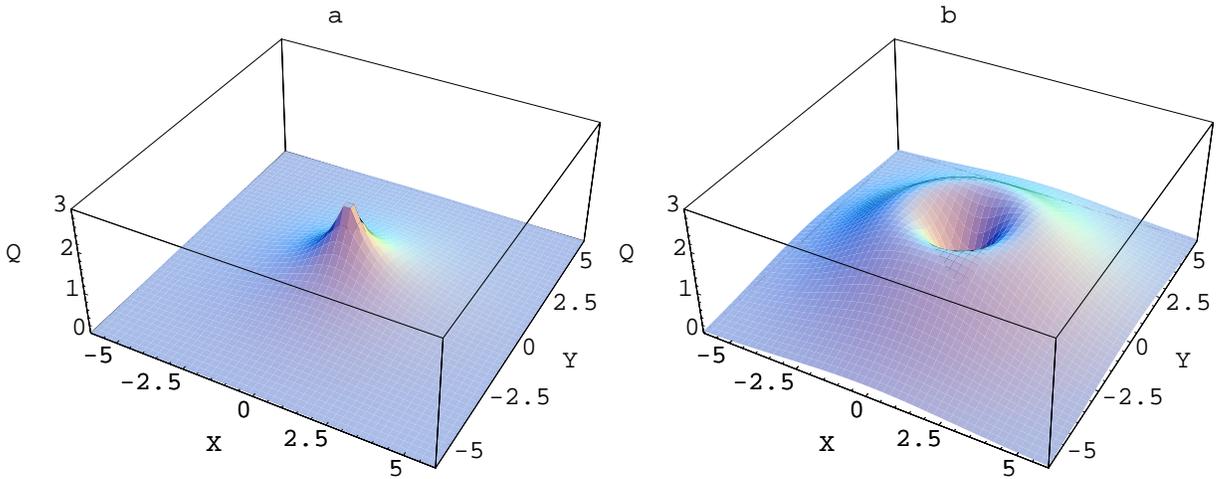


Figure 5: Plot of quasi probability function (Q-function) as a function of $X = Re(\alpha)$ and $Y = Im(\alpha)$, where the coherent state here is given by $|\alpha\rangle = \exp[\frac{|\alpha|^2}{2}] \frac{\alpha^n}{\sqrt{n!}} |n\rangle$, $\alpha = X + iY$. The scaled time $\gamma t = \frac{\pi}{2}$, $L = 0$, $\beta_1 = 0$, $r = 1$. Figure 4a corresponds to the case where the phase shift $\phi = \frac{\pi}{2}$, the timing of phase shift $\tau = 13\pi/4$ and (b) the same as in (a) but $\tau = 3\pi$.

$$Q(\alpha) = \frac{1}{\pi} \langle \alpha | \rho^f(t) | \alpha \rangle,$$

where $|\alpha\rangle$ is a coherent state. The quasiprobability will be obtained in terms of probability amplitudes and photon occupation amplitudes. We now attempt to identify regions in the three dimensional space spanned by the quasi probability function that is inhabited by physical states i.e. characterized by legitimate density matrices. Since the timing of an applied phase step has a large influence on the result, therefore to complete our work we shall consider in this discussion the quasi probability function and how it is affected by the phase step in two different cases. For this purpose we have plotted figures (5) taking into consideration the same values of all parameters as in the above figures. For instance we have depicted the Q-function in figure (5a) for $\beta_1 = 0$, $L = 0$, $r = 1$ and $\gamma t = \pi/2$, when the timing of an applied phase step $\tau = \frac{11\pi}{4}$ and $\phi = \frac{\pi}{2}$. We observe in general there is no change in the figure shape i.e. there is no influence on the Q-function. The Q-function feature is exactly similar to that observed in the absence of the phase shift. As soon as we increase the values of τ such that $\tau = 3\pi$, which means that the phase step is applied at the moment of maximum probability of pair photon emission, then we can observe a drastic change occurring in the function behavior. Therefore the shape of the quasi probability is very sensitive to the choice of the application time of the phase step. Meanwhile, the general feature of the quasi probability distribution function in the case $\tau = \frac{13\pi}{4}$, is almost identical to that in the previous case in which the timing of an applied phase step $\tau = \frac{11\pi}{4}$.

Before we conclude, it is necessary to give a brief discussion on the experimental realization of the present model. It was reported that the cavity can have a photon storage time of $T = 1$ ms (corresponding to $Q = 3 \times 10^8$). The radiative time of the Rydberg atoms with the principle quantum numbers 49, 50 and 51 is about 2×10^{-4} s. The coupling constant of the atoms to the cavity field is $2\pi \times 24$ kHz [46].

4 Conclusion

We have investigated the entanglement in the context of an ensembles of two identical qubits (or atoms) coupled to a cavity field which can become entangled with one another, even when they don't interact directly with each other. We have treated the more general case where initial states of the two qubits can be mixed with any state of the field. We have obtained an exact solution of the density operator taking into account the presence of Stark shift and an instantaneous phase shift experienced by one of the atoms that can be easily interpreted physically, and thus provides insight into the behavior of more complicated multi-qubit systems. It is found that entangled pure states for this generalized case did not have the same entanglement as the mixed state case. Entanglement is measured via the negativity, currently defined only

for an arbitrary system of two qubits, but similar analysis can in principle be applied to other systems such as a bipartite system with arbitrary dimensions. The influences of the Stark shift and the distance between the two-qubits have been presented. We have extended our studies by giving a detailed analysis and explanation of the predicted entanglement, intensity and quasi probability phenomena with nonzero values of the phase shift. The effect of phase shift is discussed when we apply a phase step at the moment of the maximum or minimum probability of photon pair emission. Finally, we have noted that the inclusion of the Stark shift in the present model, under suitable conditions, could lead to zero values of the entanglement, an effect that may have important consequences in other nonlinear processes. Finally, in addition to quantum computing implementations involving identical particles, the result here is also useful for many-body physics.

References

- [1] For an overview of fundamental problems in quantum information, see for example, J. Math. Phys. (September 2002 Special Issue on Quantum Information Theory).
- [2] G. Alber, T. Beth, M. Horodecki, P. Horodecki, R. Horodecki, M. Rötteler, H. Weinfurter, R. Werner and A. Zeilinger, Quantum Information: An Introduction to Basic Theoretical Concepts and Experiments, Springer Tracts in Modern Physics 173 (Springer-Verlag, Berlin, 2001).
- [3] M. A. Nielsen and I. L. Chuang, Quantum Computation and Quantum Information, Cambridge University Press, Cambridge, 2000.
- [4] O. Rudolph, J. Math. Phys. 42, 5306 (2001); C. P. Williams, and S. H. Clearwater, Explorations in Quantum Computing (1998) (New York: Telos, Springer-Verlag).
- [5] M. Abdel-Aty, J. Math. Phys. 44, 1457 (2003) and Virtual J. Quant. Infor. April 2003, Volume 3, Issue 4; A. Sorensen and K. Molmer, Phys. Rev. A 62, 022311 (2000); M. Horodecki, Phys. Rev. A, 57, 3364 (1998); C. H. Bennett, Phys. Today 48, 24 (1995).
- [6] B. M. Terhal, M. Horodecki, D. W. Leung, D. P. DiVincenzo, J. Math. Phys. 43, 4286 (2002).
- [7] O. Rudolph, J. Math. Phys. 42, 5306 (2001); M. J. Donald, M. Horodecki, O. Rudolph, J. Math. Phys. 43, 4252 (2002).
- [8] A. Barenco, C. H. Bennett, R. Cleve, D. P. DiVincenzo, N. Margolus, P. Shor, T. Sleator, J. A. Smolin and H. Weinfurter, Phys. Rev. A 52, 3457 (1995).

- [9] J. A. Wheeler and W. H. Zurek, *Quantum Theory and Measurement*, (Princeton University Press, Princeton 1983).
- [10] S. Goldstein, *Physics Today* 51(3), 42 and 51(4),38 (1998).
- [11] M. Beller, *Physics Today*, 51(9), 29 (1998).
- [12] T. Iwai and T. Hirose, *J. Math. Phys.* 43, 2907 (2002); Z. Ficek and R. Tanas, *Phys. Rep.* 372, 369 (2002).
- [13] G. S. Agarwal, *Quantum Optics*, Springer Tracts in Modern Physics Vol. 70 (Springer-Verlag, Berlin 1974); P. Milman, and R. Mosseri, arXiv: quant-ph/0302202.
- [14] A.-S. F. Obada and Z. M. Omar, *J. Egypt. Math. Soc.* 1, 63 (1993); I. K. Kudryavtsev, A. Lambrecht, H. Moya-Cessa and P. L. Knight, *J. Mod. Opt.* 40, 1605, (1993).
- [15] I. Jex, *Quantum Optics* 2, 433 (1990); *J. Mod. Opt.* 39, 835 (1990).
- [16] T. Q. Song, J. Feng, W. Z. Wang and J. Z. Xu, *Phys. Rev. A* 51, 2648 (1995).
- [17] I. K. Kudryavtsev and P. L. Knight, *J. Mod. Opt.* 40, 1673, (1993).
- [18] I. Tittonen, S. Stenhlpm and I. Jex, *Opt. Commun.* 124, 271 (1996).
- [19] M. Abdel-Aty, *J. Opt. B: Quantum Semiclass. Opt.* (2003) preprint; M. M. Ashraf, *Opt. Commun.* 166, 49 (1999).
- [20] I. Ashraf and A. H. Toor, *J. Opt. B: Quantum Semiclass. Opt.* 2, 772 (2000); M. M. Ashraf, *J. Opt. B: Quantum Semiclass. Opt.* 3, 39 (2001).
- [21] Yu Shi, arXiv: quant-ph/0205069 Feb 2003
- [22] G. Vidal and R. Tarrach, *Phys. Rev. A* 59 (1999).
- [23] P.W. Shor, *J. Math. Phys.* 43, 4334 (2002)
- [24] C. H. Bennett, H. J. Bernstein, S. Popescu, and B. Schumacher, *Phys. Rev. A* 53, 2046 (1996).
- [25] P. Horodecki, R. Horodecki, M. Horodecki, *Acta Phys. Slov.* 48, 141 (1998).
- [26] V. Vedral, M. B. Plenio, K. Jacobs, and P. L. Knight, *Phys. Rev. A* 56, 4452 (1997).
- [27] V. Vedral, M. B. Plenio, M. A. Rippin, and P. L. Knight, *Phys. Rev. Lett.* 78, 2275 (1997).
- [28] V. Vedral and M. B. Plenio, *Phys. Rev. A* 57, 1619(1998).

- [29] S. Furuichi and M. Ohya, *Lett. Math. Phys.* **49** , 279 (1999).
- [30] S. Furuichi and M. Abdel-Aty, *J. Phys. A: Math. & Gen.* **34**, 6851 (2001).
- [31] M. Abdel-Aty, S. Furuichi and A.-S. F. Obada, *J. Opt. B: Quant. Semiclass. Opt.* **4**, 37 (2002)
- [32] S. J. D. Phoenix and P. L. Knight, *Ann. Phys. (N. Y)* **186**, 381 (1988).
- [33] S. J. D. Phoenix and P. L. Knight, *Phys. Rev. A* **44**, 6023 (1991); *Phys. Rev. Lett.* **66**, 2833 (1991).
- [34] M. Abdel-Aty, *J. Phys. B: At. Mol. Opt. Phys.* **33**, 2665 (2000).
- [35] M. Abdel-Aty, and A.-S. F. Obada, *Eur. Phys. J. D* **23**, 155 (2003).
- [36] J. Lee and M. S. Kim, *Phys. Rev. Lett.* **84**, 4236 (2000).
- [37] J. Lee, M. S. Kim, Y. J. Park and S. Lee, *J. Mod. Opt.* **47**, 2151 (2000).
- [38] G. Vidal and R. F. Werner, *Phys. Rev. A* **65**, 032314 (2002).
- [39] A. Peres, *Phys. Rev. Lett.* **77**, 1413 (1996).
- [40] M. Horodecki, P. Horodecki, and R. Horodecki, *Phys. Lett. A* **223**, 1 (1996)
- [41] E. Knill, R. Laflamme and G. J. Milburn, *Nature (London)* **409** , 46 (2001)
- [42] J.A. Roversi, A. Vidiella-Barranco and H. Moya-Cessa, *Mod. Phys. Lett. B* **17**, 219 (2003)
- [43] Q.A. Turchette, C.S. Wood, B.E. King, C.J. Myatt, D. Leibfried, W.M. Itano, C. Monroe, and D.J. Wineland, *Phys. Rev. Lett.* **81**, 3631 (1998).
- [44] S. Franke, G. Huyet, and S.M. Barnett, *J. Mod. Opt.* **47**, 145 (2000).
- [45] W. K. Wootters, *Phys. Rev. Lett.* **80** , 2245 (1998).
- [46] M. Brune, E. Hagley, J. Dreyer, X. Maitre, A. Maali, C. Wunderlich, J. M. Raimond, and S. Haroche, *Phys. Rev. Lett.* **77**, 4887 (1996).