

Geometrical Representation of Entanglement of a Three-Qubit $|W\rangle$ State

Pedro Felipe Gardeazábal Rodríguez* and Jagdish Rai Luthra†

*Departamento de Física, Universidad de los Andes, Bogotá, Colombia, p-gardea@uniandes.edu.co

†Departamento de Física, Universidad de los Andes, Bogotá, Colombia, jluthra@uniandes.edu.co

Abstract. In this paper we study the quantum entanglement properties of a tripartite system of qubits. We use the concept of two-tangle and three-tangle to understand the nature of entanglement of $|W\rangle$ and $|GHZ\rangle$ states. We construct simple geometrical representations for graphical interpretations of the entanglement for these states.

INTRODUCTION

The study of quantum entanglement is of great current interest. There exist several measures of entanglement which include reduced entropy [1], concurrence [2], entanglement of formation [3], entanglement witness [4], distances from the closest separable state [4] and the tangle [5]. Entanglement is well understood only for a bipartite system in pure states as it can always be given Schmidt decomposition. For a system in a mixed state, entanglement is well understood only for a bipartite system of two qubits and a bipartite system of a qubit and a qutrit. For a multipartite system, the concept of entanglement is rather complicated. In this paper we study a tripartite system of qubits and try to develop measures of entanglement. Of particular interest to us is the concept of the tangle, a measure introduced by Wootters [6]. We provide a geometrical representation for this measure and apply it to the understanding of entanglement of a tripartite system. In particular we study the $|GHZ\rangle$ and the $|W\rangle$ state and using the concept of tangle we analyze the level and distribution of entanglement among the three qubits in these two states.

$|W\rangle$ AND $|GHZ\rangle$ STATES

The $|W\rangle$ state of a tripartite system of a qubits is defined as

$$|W\rangle = \alpha|001\rangle + \beta|010\rangle + \gamma|100\rangle. \quad (1)$$

The generalized $|W\rangle$ state of a three qubit system shows interesting entanglement properties. It may appear to be less entangled than the $|GHZ\rangle$ state but it is more robust in entanglement. For example, if we trace the $|GHZ\rangle$ state over one of the qubits, the resulting state is completely unentangled, but the $|W\rangle$ state retains its entanglement even after tracing one of the particles. If we use concurrence as a measure of entanglement, the $|W\rangle$ state is found to have a zero concurrence while the $|GHZ\rangle$ states has a concurrence of one. As the $|W\rangle$ state is obviously entangled, we need to develop alternative measures of entanglement.

In this work we study the properties of entanglement and, in particular, of the $|W\rangle$ and the $|GHZ\rangle$ states. A general $|GHZ\rangle$ state can be represented using one parameter as

$$|GHZ\rangle = \alpha|000\rangle + \beta|111\rangle. \quad (2)$$

In Fig.1 we have given a geometrical representation of the $|GHZ\rangle$ state. The curves show the degree of entanglement as obtained from the reduced entropy and concurrence. There are four lobes corresponding to the maximum entanglement of the $|GHZ\rangle$ state. This is very similar to the entanglement of the four Bell states. However for the entanglement of the $|W\rangle$ state no simple one parameter description is possible. In the next section we use the concept of tangle to understand the entanglement of $|W\rangle$ states.

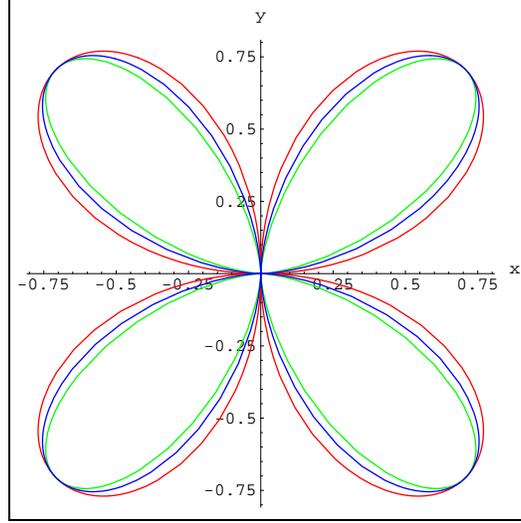


Figure 1. Geometrical representation of $|GHZ\rangle$ three-qubit state: concurrence (outer), reduced entropy (middle), tangle (inner).

TANGLE AND ITS GEOMETRICAL REPRESENTATION

The tangle for two particles is defined as [5]:

$$\tau_{AB} = \left(\max[\sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}; 0] \right)^2, \quad (3)$$

where the λ_i are the eigenvalues of the matrix $\rho_{AB}\tilde{\rho}_{AB}$, ρ_{AB} is the density matrix

$$\rho_{AB} = |\psi\rangle\langle\psi| \quad (4)$$

and

$$\tilde{\rho}_{AB} = \sigma_{yA}\sigma_{yB}\rho_{AB}^*\sigma_{yA}\sigma_{yB}. \quad (5)$$

The density matrix ρ_{AB} is a positive definite matrix because $|\psi\rangle$ is a pure state. Henceforth, there is some bases in which $\rho_{AB}\tilde{\rho}_{AB}$ has only one eigenvalue $\lambda_1 \neq 0$. Therefore the tangle, for a bipartite pure state, is

$$\tau_{AB} = \lambda_1. \quad (6)$$

For pure states, the concurrence, another measure of entanglement, is the square root of the tangle. For a tripartite system the concept of two-tangle has been extended. For an estimate of the entanglement between the three qubits we use the two- and three-tangles. The general relation satisfied by the two- and three-tangles is given by [5]:

$$\tau_{A(BC)} = \tau_{AB} + \tau_{AC} + \tau_{ABC}. \quad (7)$$

For the generalized $|W\rangle$ state in spherical coordinates:

$$|\psi_W\rangle = \sin\theta\cos\varphi|001\rangle + \sin\theta\sin\varphi|010\rangle + \cos\theta|100\rangle, \quad (8)$$

the various two-tangles are given by

$$\tau_{AB} = \sin^2(2\theta)\sin^2(\varphi) \quad (9)$$

$$\tau_{AC} = \sin^2(2\theta)\cos^2(\varphi) \quad (10)$$

$$\tau_{BC} = \sin^4(\theta)\sin^2(2\varphi). \quad (11)$$

In figure 2 we show a typical two-tangle τ_{AC} . For this state the three-tangle is always zero as expected, as there is no three way entanglement present in the generalized $|W\rangle$ state. For a normal $|W\rangle$ state the two-tangles are all constant

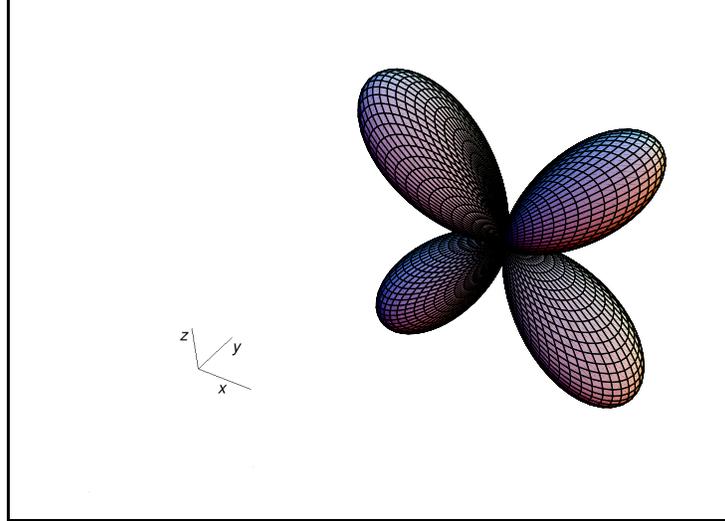


Figure 2. Two-tangle in the $|W\rangle$ state

and are equal to $4/9$. For generalized $|W\rangle$ states the two-tangle varies from zero to one. The maximum entanglement is achieved when the generalized $|W\rangle$ state reduces to a Bell state. The two-tangles τ_{AB} and τ_{AC} are related by a rotation about the z -axis by $\pi/2$.

We have also studied the entanglement of one particle with respect to the other two. We found that the two-tangle of A with (BC) is the sum of the two-tangle τ_{AB} and τ_{AC} . This implies that the entanglement is freely distributed among the qubits. This is not true for all pure states of a tripartite system. However, for the generalized $|GHZ\rangle$ we found that the three-tangle varies between zero and one. The maximum value of one is achieved for Bell-like normal $|GHZ\rangle$ states. The two-tangles for the $|GHZ\rangle$ states are all zero showing no two-way entanglement.

CONCLUSIONS

Two kind of tripartite qubit entangled states have been studied: The $|W\rangle$ and the $|GHZ\rangle$. It has been geometrically shown that $|GHZ\rangle$ has only three-particle entanglement, but not pairwise entanglement. In contrast, the $|W\rangle$ state has no three-particle entanglement but it does have pairwise entanglement.

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