

Greenberger-Horne-Zeilinger-Like Symmetry in Four Qubit System

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Abstract

Similar to the three-qubit Greenberger-Horne-Zeilinger (GHZ) symmetry we explore the four-qubit Greenberger-Horne-Zeilinger-Like (GHZL) symmetry, which is symmetry under (i) any one-pair and two-pair flips (ii) qubit permutation (iii) qubit rotations about the z -axis. Like the three-qubit GHZ-symmetric states the four-qubit GHZL-symmetric states can be parametrized by two real parameters. While there are two GHZ-symmetric pure states $|\text{GHZ}\rangle_{\pm} = (1/\sqrt{2})(|000\rangle \pm |111\rangle)$, there is no GHZL-symmetric pure state. It is shown that the whole GHZL-symmetric states involve two entanglement classes, fully separable and double bi-separable classes.

Recently, much attention is being paid to quantum technology[1]. This is mainly due to the fact that the technology based on classical mechanics has its own limitations. For example, there is a time limitation when a classical computer performs a huge numerical calculation. Another example of the limitation is a serious eavesdropping problem in the classical network. There is a belief in quantum physics community that such troublesome problems can be resolved if quantum technology is substituted for classical one.

Most important notion in quantum technology is entanglement[2] of given quantum states. As shown for last two decades it plays a crucial role in quantum teleportation[3], superdense coding[4], quantum cloning[5], and quantum cryptography[6]. It is also quantum entanglement, which makes the quantum computer outperform the classical one[7]. Thus, it is believed that entanglement is a genuine physical resource in quantum information processing (QIP).

Usually, most QIP requires higher qubit system. Therefore, it is important to estimate how much entanglement a given higher qubit state has. However, computing the entanglement analytically for higher-qubit or multipartite states is a very difficult task. Another obstacle for developing the quantum technology is the fact that there are many types of entanglement. Thus, classification of the given entanglement is a very important task in quantum information theory.

Generally, the multipartite entanglement is classified by stochastic local operations and classical communication (SLOCC)[8]. For example, SLOCC enables us to understand that the entanglement of the whole three-qubit pure states can be classified by fully separable (S), three bi-separable (B), W, and Greenberger-Horne-Zeilinger (GHZ) types[9]. Moreover, one can identify the entanglement type for any given pure state by computing the three-tangle[10] and concurrences[11] for its reduced states. Similarly, the entanglement of whole three-qubit mixed states consists of S, B, W, and GHZ types[12]. Furthermore, these classes satisfy a linear hierarchy $S \subset B \subset W \subset \text{GHZ}$. However, unlike the pure states, it is highly nontrivial task to identify the entanglement types of given mixed state except few rare cases. This is mainly due to the fact that analytical computation of the three-tangle for mixed states is impossible so far except few comparatively simple cases (for analytical computation see [13]).

Recently, identification of the entanglement classes for three-qubit mixed states has been significantly progressed. In Ref.[14] the GHZ symmetry was examined in the three-qubit

system. This symmetry is a symmetry the GHZ states $|\text{GHZ}\rangle_{\pm} = (1/\sqrt{2})(|000\rangle \pm |111\rangle)$ have and is expressed as a symmetry under (i) qubit permutation, (ii) simultaneous flips, (iii) qubit rotation about the z -axis. The whole GHZ-symmetric states can be parametrized by two real parameters, *say* x and y . Authors in Ref. [14] succeeded in classifying the entanglement of the GHZ-symmetric states completely. Subsequently, they showed how to compute the three-tangle analytically for the GHZ-symmetric states[15] and how to construct the class-specific optimal witnesses[16]. More recently, the extended GHZ symmetry was discussed[17], which is parametrized by four real parameters.

The purpose of this paper is to extend the analysis of Ref.[14] to four-qubit system. The entanglement classes of the four-qubit pure states were explored in Ref.[18–24]. However, their results can be confusing and seemingly contradictory[24]. Furthermore, as far as we know, the entanglement classes of the four-qubit mixed states have not been explored yet. Thus, we do not know whether or not the entanglement of the four-qubit system obeys certain hierarchies.

Although we do not know the full hierarchies in four-qubit system, it is possible to realize the hierarchies partially unless the genuine four-way entangled states are considered. If ρ is not a four-way entangled state, it should be one of fully separable (S), bi-separable (two separable qubit plus two entangled qubit) (B), double bi-separable (two entangled qubit plus two entangled qubit) (BB), W (one separable qubit plus W state), and GHZ (one separable qubit plus GHZ state). Then, from a background of the lower qubit system it is known that there are two independent hierarchies, $S \subset B \subset W \subset \text{GHZ}$ and $S \subset B \subset \text{BB}$.

It is straightforward to generalize the GHZ symmetry to higher-qubit system. For example, the direct generalization to four-qubit system can be written as a symmetry under (i) simultaneous flips (ii) qubit permutation (iii) qubit rotations about the z -axis of the form

$$U(\phi_1, \phi_2, \phi_3) = e^{i\phi_1\sigma_z} \otimes e^{i\phi_2\sigma_z} \otimes e^{i\phi_3\sigma_z} \otimes e^{-i(\phi_1+\phi_2+\phi_3)\sigma_z}. \quad (1)$$

However, the difference from the three-qubit case arises when the symmetric states are constructed. The general form of the four-qubit states invariant under the transformations (i), (ii), and (iii) is

$$\rho_4^S = \beta [|0000\rangle\langle 1111| + |1111\rangle\langle 0000|] \quad (2)$$

$$+ \text{diag}(\alpha_1, \alpha_2, \alpha_2, \alpha_3, \alpha_2, \alpha_3, \alpha_3, \alpha_2, \alpha_2, \alpha_3, \alpha_3, \alpha_2, \alpha_3, \alpha_2, \alpha_2, \alpha_1)$$

where β , α_1 , α_2 and α_3 are real numbers satisfying $\alpha_1 + 4\alpha_2 + 3\alpha_3 = \frac{1}{2}$. Therefore, unlike the three-qubit case, the symmetric states are represented by three real parameters. Moreover, the number of parameters increase with increasing the number of qubit. This can lead to difficulty in analysis of the entanglement classification in the parameter space, because the corresponding parameter space becomes higher-dimensional.

In order to escape this difficulty our four-qubit generalization of the three-qubit GHZ symmetry is slightly modified as symmetry under (i) any one-pair and two-pair flips, and same (ii) and (iii). We will call this symmetry as GHZ-like (GHZL) symmetry. Then, it is easy to show that the general form of the GHZL-symmetric states is represented by two real parameters x and y in a form

$$\rho_{\text{GHZL}} = x [|0000\rangle\langle 1111| + |1111\rangle\langle 0000|] \quad (3)$$

$$+ \text{diag}(\alpha_1, \alpha_2, \alpha_2, \alpha_1, \alpha_2, \alpha_1, \alpha_1, \alpha_2, \alpha_2, \alpha_1, \alpha_1, \alpha_2, \alpha_1, \alpha_2, \alpha_1)$$

with $\alpha_1 = \frac{1}{16} + \frac{y}{2\sqrt{2}}$ and $\alpha_2 = \frac{1}{16} - \frac{y}{2\sqrt{2}}$. The parameters x and y are chosen such that the Euclidean metric in the (x, y) plane coincides with the Hilbert-Schmidt metric $d^2(A, B) = \frac{1}{2}\text{tr}(A - B)^\dagger(A - B)$. It is worthwhile noting that the sign of x does not change the entanglement because $\rho_{\text{GHZL}}(-x, y) = u\rho_{\text{GHZL}}(x, y)u^\dagger$, where $u = i\sigma_x \otimes \sigma_y \otimes \sigma_y \otimes \sigma_y$. Since ρ_{GHZL} is a quantum state, it should be a positive operator, which restricts the parameters as $y \geq \pm 2\sqrt{2}x - \frac{\sqrt{2}}{8}$ and $|x| \leq \frac{1}{8}$. This can be represented in Fig. 1 as a triangle in the (x, y) space.

Although the four-qubit GHZL symmetry looks similar to the three-qubit GHZ symmetry, there are two crucially different points. First point is that while there is no pure GHZL-symmetric states, there are two GHZ-symmetric pure states $|\text{GHZ}\rangle_{\pm}$. Another different point arises in the reduced states. While all three-qubit reduced states of ρ_{GHZL} reduce to the completely mixed state $\frac{1}{8}\mathbb{1}_8$, the two-qubit reduced states of GHZ-symmetric states do not reduce to the completely mixed state even though they are diagonal.

Like a three-qubit GHZ symmetry there is a correspondence between four-qubit pure states and GHZL-symmetric states. Let $|\psi\rangle$ be a four-qubit pure state. Then, the corresponding GHZL-symmetric state $\rho_{\text{GHZL}}(\psi)$ can be written as

$$\rho_{\text{GHZL}}(\psi) = \int U|\psi\rangle\langle\psi|U^\dagger, \quad (4)$$

where the integral is understood to cover the entire GHZL symmetry group, i.e., unitaries $U(\phi_1, \phi_2, \phi_3)$ in Eq. (1) and averaging over the discrete symmetries. For example, if $|\psi\rangle =$

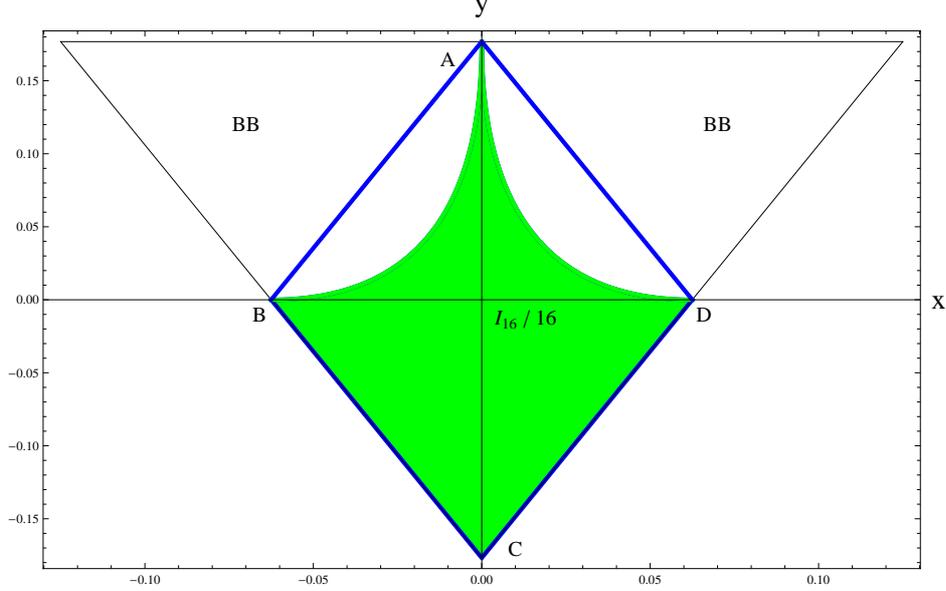


FIG. 1: (Color online) The SLOCC classes of four-qubit GHZL-symmetric states ρ_{GHZL} . The fully separable states reside in the polygon ABCD. Theorem 2 implies that there is no one-qubit tensor product three-qubit entangled states in the GHZL-symmetric states. The remaining states except the polygon ABCD turn out to be the double bi-separable states. Hence, there is no genuine four-way entangled state in the GHZL-symmetric states.

$\sum_{i,j,k,l=0}^1 \psi_{ijkl} |ijkl\rangle$, $\rho_{\text{GHZL}}(\psi)$ becomes Eq. (3) with

$$x = \frac{1}{4} \text{Re} \left[\psi_{0000} \psi_{1111}^* + \psi_{0011} \psi_{1100}^* + \psi_{0101} \psi_{1010}^* + \psi_{0110} \psi_{1001}^* \right] \quad (5)$$

$$\alpha_1 \equiv \frac{1}{16} + \frac{y}{2\sqrt{2}} = \frac{1}{8} \left[|\psi_{0000}|^2 + |\psi_{1111}|^2 + |\psi_{0011}|^2 + |\psi_{0101}|^2 \right. \\ \left. + |\psi_{0110}|^2 + |\psi_{1001}|^2 + |\psi_{1010}|^2 + |\psi_{1100}|^2 \right].$$

Now, we want to discuss the entanglement classes of the GHZL-symmetric states.

Theorem 1. *The fully separable GHZL-symmetric states reside in the polygon ABCD in Fig. 1.*

Proof. Let $|\psi^{\text{sep}}\rangle = (u_1 \otimes u_2 \otimes u_3 \otimes u_4)|0000\rangle$, where

$$u_j = \begin{pmatrix} A_j & -C_j^* \\ C_j & A_j^* \end{pmatrix} \quad \text{with } |A_j|^2 + |C_j|^2 = 1. \quad (6)$$

Then, it is easy to derive the parameters x and y of $\rho_{\text{GHZL}}(\psi^{\text{sep}})$ easily using Eq. (5). Our

method for proof is as follows. Applying the Lagrange multiplier method we maximize x with given y . Then, it is possible to derive a region in the parameter space. If this region is convex, this is the region where the GHZL-symmetric fully separable states reside. If it is not convex, we have to choose the convex hull of it.

From the symmetry it is evident that the maximum of x occurs when $A_1 = A_2 = A_3 = A_4 \equiv A$, Then the constraint of y yields $A^2 = \frac{1}{2} (1 \pm 2^{5/8} y^{1/4})$, which gives

$$x_{max} = \frac{1}{16} \left(1 - 2^{\frac{5}{4}} y^{\frac{1}{2}} \right)^2. \quad (7)$$

Since the sign of x_{max} does not change the entanglement class, the region represented by green color in Fig. 1 is derived. Since it is not convex, we have to choose a convex hull, which is a polygon ABCD in Fig. 1. This completes the proof.

Now, we show that there is no one-qubit product three-qubit entangled state in the GHZL-symmetric states. This fact can be inferred from following theorem.

Theorem 2. *There is no one-qubit product GHZ state in the GHZL-symmetric states.*

Proof. Let $|\psi^{GHZ}\rangle = (G_1 \otimes G_2 \otimes G_3 \otimes G_4)|0\rangle \otimes (|000\rangle + |111\rangle)$, where

$$G_j = \begin{pmatrix} A_j & B_j \\ C_j & D_j \end{pmatrix}. \quad (8)$$

Then, it is easy to compute x and y of $\rho_{GHZL}(\psi^{GHZ})$ easily using Eq. (5). Now, we want to maximize x with given y and $\langle \psi^{GHZ} | \psi^{GHZ} \rangle = 1$. From the symmetry it is evident that the maximum of x occurs when $A_2 = A_3 = A_4 = B_2 = B_3 = B_4 \equiv a$ and $C_2 = C_3 = C_4 = D_2 = D_3 = D_4 \equiv c$. Then, we define $x^\Lambda = x + \Lambda_0 \Theta_0 + \Lambda_1 \Theta_1$, where Λ_0 and Λ_1 are Lagrange multiplier constants, and

$$x = 4A_1 C_1 a^3 c^3 \quad (9)$$

$$\Theta_0 = 4(A_1^2 + C_1^2)(a^2 + c^2)^2 - 1$$

$$\Theta_1 = 4 [A_1^2 a^2 (a^4 + 3c^4) + C_1^2 c^2 (3a^4 + c^4) - 2\alpha_1].$$

Now, we want to maximize x under the constraints $\Theta_0 = \Theta_1 = 0$.

First, we solve the two constraints, whose solutions are

$$A_1^2 = \frac{8\alpha_1(u_1 + u_2) - u_2}{u_1^2 - u_2^2} \quad C_1^2 = \frac{u_1 - 8\alpha_1(u_1 + u_2)}{u_1^2 - u_2^2}, \quad (10)$$

where $u_1 = 4a^2(a^4 + 3c^4)$ and $u_2 = 4c^2(3a^4 + c^4)$. From $\frac{\partial x^\Lambda}{\partial A_1} = \frac{\partial x^\Lambda}{\partial C_1} = 0$ one can express the Lagrange multiplier constants as

$$\Lambda_0 = -\frac{A_1^2 u_1 - C_1^2 u_2}{A_1 C_1} \frac{2a^3 c^3}{u_1^2 - u_2^2} \quad \Lambda_1 = \frac{A_1^2 - C_1^2}{A_1 C_1} \frac{2a^3 c^3}{u_1 - u_2}. \quad (11)$$

Combining Eqs. (10), (11), and $\frac{\partial x^\Lambda}{\partial a} = \frac{\partial x^\Lambda}{\partial c} = 0$, one arrives at

$$8\alpha_1(z^2 + 1)^4 = z^8 + 6z^4 + 1, \quad (12)$$

where $z = \frac{a}{c}$. Then, the maximum of x with given y becomes

$$x_{max} = \frac{z^3 \sqrt{8\alpha_1(1 - 8\alpha_1)(1 + z^2)^6 - z^2(3 + z^4)(1 + 3z^4)}}{(z^4 - 1)^3}. \quad (13)$$

Using Eq. (12) and performing long and tedious calculation, one can show that the right-hand side of Eq. (13) reduces to $\frac{1}{16} (1 - \sqrt{16\alpha_1 - 1})^2$, which results in the identical equation with Eq. (7). This implies that there is no one-qubit product three-qubit GHZ state in the GHZL-symmetric states. This completes the proof.

Since it is well-known that the three-qubit states consist of fully separable (S), bi-separable (B), W, and GHZ states, and they satisfy a linear hierarchy $S \subset B \subset W \subset GHZ$, theorem 2 also implies that there is no one-qubit product three-qubit B (or W) state in the GHZL-symmetric states.

Now, we want to discuss the entanglement classes of remaining GHZL-symmetric states. In order to conjecture the classes quickly, let us consider the following double bi-separable state

$$|\psi^{BB}\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle) \otimes \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle). \quad (14)$$

Then, Eq. (5) shows that the parameters of $\rho_{GHZL}(\psi^{BB})$ are $x = 1/8$ and $y = \sqrt{2}/8$, which correspond to the right-upper corner of the triangle in Fig. 1. Since mixing can result only in the same or a lower entanglement class, the entanglement class of this corner state should be BB or its sub-classes. However, the sub-class of BB is fully separable class, and those states are confined in $ABCD$. Therefore, the corner should be BB-class state. This fact strongly suggests that all remaining states in Fig. 1 are BB-class. The following theorem shows that our conjecture is correct.

Theorem 3. *All remaining GHZL-symmetric states in Fig. 1 are BB-class.*

Proof. Let $|\psi^{BB}\rangle = (G_1 \otimes G_2 \otimes G_3 \otimes G_4)(|00\rangle + |11\rangle) \otimes (|00\rangle + |11\rangle)$, where G_j is given in Eq. (8). Then, it is easy to compute the parameters x and y of $\rho_{GHZL}(\psi^{BB})$ using Eq. (5).

Similar to the previous theorems we want to maximize x with given y . From the symmetry it is evident that the maximum of x occurs when

$$\begin{aligned}
A_1 = A_2 &\equiv a_1 & A_3 = A_4 &\equiv a_3 \\
B_1 = B_2 &\equiv b_1 & B_3 = B_4 &\equiv b_3 \\
C_1 = C_2 &\equiv c_1 & C_3 = C_4 &\equiv c_3 \\
D_1 = D_2 &\equiv d_1 & D_3 = D_4 &\equiv d_3.
\end{aligned} \tag{15}$$

For later convenience we define $\mu_1 = a_1^2 + b_1^2$, $\mu_2 = a_3^2 + b_3^2$, $\mu_3 = c_1^2 + d_1^2$, $\mu_4 = c_3^2 + d_3^2$, $\nu_1 = a_1c_1 + b_1d_1$, and $\nu_2 = a_3c_3 + b_3d_3$.

In order to apply the Lagrange multiplier method we define $x^\Lambda = x + \Lambda_0\Theta_0 + \Lambda_1\Theta_1$, where

$$\begin{aligned}
x &= \frac{1}{2} (\mu_1\mu_2\mu_3\mu_4 + \nu_1^2\nu_2^2) \\
\Theta_0 &= (\mu_1^2 + 2\nu_1^2 + \mu_3^2)(\mu_2^2 + 2\nu_2^2 + \mu_4^2) - 1 \\
\Theta_1 &= (\mu_1^2 + \mu_3^2)(\mu_2^2 + \mu_4^2) + 4\nu_1^2\nu_2^2 - 8\alpha_1.
\end{aligned} \tag{16}$$

The constraints $\Theta_0 = 0$ and $\Theta_1 = 0$ come from $\langle \psi^{BB} | \psi^{BB} \rangle = 1$ and Eq. (5), respectively.

Now, we have eight equations $\frac{\partial x^\Lambda}{\partial \mu_i} = 0$ ($i = 1, 2, 3, 4$), $\frac{\partial x^\Lambda}{\partial \nu_i} = 0$ ($i = 1, 2$), and $\Theta_0 = \Theta_1 = 0$. Analyzing those equations, one can show that the maximum of x occurs when $\mu_1 = \mu_3$ and $\mu_2 = \mu_4$. Then, the constraint $\Theta_1 = 0$ implies

$$x_{max} = \frac{1}{16} + \frac{y}{2\sqrt{2}}, \tag{17}$$

which corresponds to the right side of the triangle in Fig. 1. This fact implies that the whole GHZL-symmetric states are BB or its sub-class. Since the fully separable states are confined in $ABCD$, the remaining states should be BB-class, which completes the proof.

In this paper we considered the GHZL symmetry in four-qubit system. It is shown that the whole GHZL-symmetric states involve only two SLOCC classes, S and BB. Thus, there is no genuine four-way entangled state in the whole set of the GHZL-symmetric states. Following Ref. [16] we can use our result to construct the optimal witness $\mathcal{W}_{\text{BB}\setminus\text{S}}$, which can detect the BB class optimally from a set of S plus BB states.

As remarked earlier if we choose ‘simultaneous flips’ in the first symmetry transformation, the symmetric states are represented by three real parameters as Eq. (2) shows. Probably, these symmetric states involve all kinds of the four-qubit SLOCC classes. The SLOCC classification of Eq. (2) will be reported elsewhere.

Another interesting extension of present paper is to generalize our analysis to any $2n$ -qubit system. Then, our modification of first symmetry transformation should be changed into ‘any one-pair, two-pair, \dots , and n -pair flips’. This would drastically reduce the number of free parameters. The SLOCC classification with this strong symmetry restriction will be explored in the future.

Acknowledgement: This research was supported by the Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education, Science and Technology(2011-0011971).

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