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Collège de France abroad Lectures

Quantum information with real or
artificial atoms and photons in cavities

Lecture 5:

An introduction to Circuit QED describing
Josephson junctions as qubits and LC circuits
as quantum oscillators



Introduction to superconducting qubits

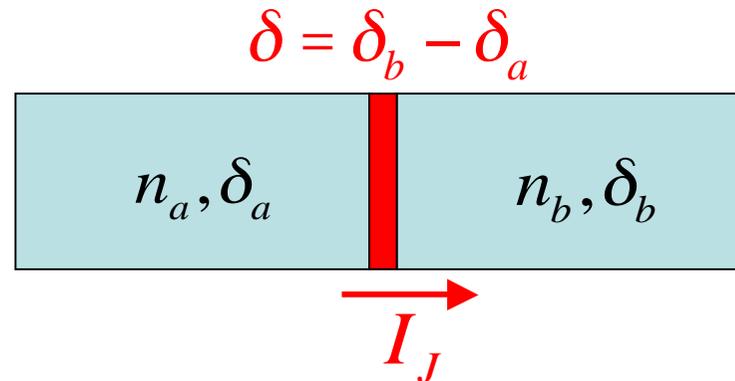
The physics of superconducting qubits has made tremendous progresses during the last years. Circuits made of Josephson junctions have been turned into artificial atoms which can be manipulated and measured by methods similar to the ones previously developed in ion trap or Cavity QED physics. These systems are very promising for quantum information. In this course, I will focus on one aspect of this physics, namely the use of Josephson qubits to prepare and reconstruct non classical fields of radiofrequency resonators (circuit QED) and I will compare this physics with Cavity QED. Before describing experiments in Lecture 6, I will introduce in this Lecture the main properties of the Josephson effects in mesoscopic circuits, whose understanding is essential to describe the workings of a superconducting qubit.

I will describe a superconductor at T close to $0K$ as a metal in which a fraction of the conduction band electrons have formed loosely bound pairs (Cooper pairs) behaving in first approximation as a gas of composite bosons. A superconducting junction will be described as the combination of two superconducting metals separated by an insulating barrier and we will show that this system behaves as a non-linear induction coupled to a capacity, a system whose properties will be analyzed first classically, then by quantum physics (&V-A). We will then show that by coupling this elementary system to convenient circuits, one can build a device fulfilling all the requirements of an useful qubit (&V-B). We will restrict ourselves here to the description of one kind of device, *the phase qubit*, which is the one used in the experiments which will be analyzed in lecture 6.

V-A

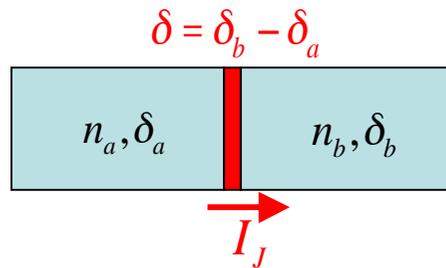
The dynamics of a Josephson junction

Description of a superconducting junction



A Josephson junction (JJ) is realized with two pieces of superconducting metal separated by a few nanometer wide insulating barrier through which Cooper pairs tunnel. When the system is isolated, we can define the number of Cooper pairs n_a, n_b and the quantum phases δ_a, δ_b of the Cooper pair wave functions on the two sides of the JJ. The charge and phase differences $n_a - n_b$ and $\delta = \delta_b - \delta_a$ are the essential parameters describing the properties of the JJ. B. Josephson has shown in 1962 that a current $I_J = I_0 \sin \delta$ circulates across the JJ without applied voltage if δ is constant and that an ac current of frequency $\nu = 2eV/h$ oscillates between the two sides if a voltage V is imposed on the JJ. These fundamental results can be derived either from the heuristic Ginsburg-Landau model of superconductivity, or from the more fundamental Bardeen Cooper Shrieffer (BCS) theory. They can also be understood by a simple model describing the ensemble of Cooper pairs as a gas of composite bosons tunneling between two potential wells separated by a barrier (similar Josephson effects have been recently demonstrated in Bose-Einstein condensates made of cold atoms).

Simple description of an isolated junction



The classical parameters characterizing the JJ are the differences of pair numbers $2p = n_a - n_b$ and of quantum phases $\delta = \delta_a - \delta_b$ between the two sides. These quantities will play the role of conjugate variables when quantizing the model. We can intuitively understand this canonical conjugation from an analogy with quantum optics where photon numbers and field phases are quasi conjugate variables (we will not discuss here the difficulties in defining the phase as an observable).

Let us call C the capacitance of the JJ (C , in practice $\sim 10^{-12}\text{F}$, is proportional to the gap area and inversely proportional to its width). The electric potential drop across the junction for a pair imbalance p (Cooper pair charges $\pm Q = \pm 2ep$) is:

$$V = \frac{Q}{C} = \frac{2ep}{C}$$

Let us combine this relation with the Josephson equations. The dc Josephson effects tells that a dc current (time derivative of p) equal to $I_0 \sin \delta$ flows across the JJ if δ is constant (I_0 , of the order of $10\mu\text{A}$ for the typical junctions we will consider, is the junction critical current, above which the metal transits to a normal non-superconducting phase):

$$I_J = -2e \frac{dp}{dt} = I_0 \sin \delta$$

The ac Josephson effect states that the time derivative of δ is proportional to the voltage applied to the JJ: a dc voltage induces a linear variation of δ with time and makes the current across the junction to oscillate (this superconducting regime is sustained until a critical voltage is reached). Combined with the first equation, this leads to:

$$\frac{d\delta}{dt} = \frac{2eV}{\hbar} = \frac{4e^2 p}{\hbar C}$$

We obtain two equations coupling p and δ which derive from an effective Hamiltonian.

The open circuit JJ Hamiltonian

Expressing the 2 Josephson relations as canonical eqs deriving from an Hamiltonian H , we get:

$$\frac{dp}{dt} = -\frac{I_0}{2e} \sin \delta = -\frac{1}{\hbar} \frac{\partial H}{\partial \delta} \quad ; \quad \frac{d\delta}{dt} = \frac{4e^2 p}{\hbar C} = \frac{1}{\hbar} \frac{\partial H}{\partial p} \quad \rightarrow \quad H = \frac{2e^2}{C} p^2 - \frac{\hbar I_0}{2e} \cos \delta$$

$$H = E_C p^2 - E_J \cos \delta \quad ; \quad E_C = \frac{2e^2}{C} \quad , \quad E_J = \frac{\hbar I_0}{2e}$$

H rules the dynamics of a non-linear oscillator whose 'momentum' and 'position' are p and δ . The combined dc and ac Josephson effects induce an oscillation: a phase difference δ induces a current (dc effect). This current produces a charge imbalance which creates, by capacitive effect, a potential across the JJ. This voltage induces in turn, via the ac Josephson effect, a variation of δ . This produces a coupled oscillation of the phase and the charge. For small δ values, such that $\cos \delta \sim 1 - \delta^2/2$, the oscillator behaves linearly, its "linearized" Hamiltonian H_l being (within an irrelevant constant):

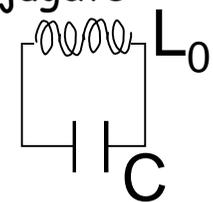
$$H_l = E_C p^2 + E_J \frac{\delta^2}{2} = \frac{Q^2}{2C} + \frac{L_0 I_{Jl}^2}{2} \quad (Q = 2ep, L_0 = \frac{\hbar}{2eI_0}, I_{Jl} = I_0 \delta)$$

This is the Hamiltonian of a classical "lump circuit" of capacitance C and inductance $L_0 = \hbar/2eI_0$, with charge $Q=2ep$ and current $I_{Jl}=I_0\delta$. The electric and magnetic energies of this system play the roles of the kinetic and potential energies of a mechanical oscillator. Making a notation change to introduce the magnetic flux Φ_{Jl} across the inductance, we can rewrite H_l under a symmetrical form, where Q and Φ_{Jl} , proportional to p and δ , are also conjugate variables:

$$H_l = \frac{Q^2}{2C} + \frac{\Phi_{Jl}^2}{2L_0} \quad ; \quad (\Phi_{Jl} = L_0 I_{Jl} = \frac{\hbar \delta}{2e})$$

The frequency of this linearized oscillator is:

$$\omega_{Jl} = \frac{1}{\sqrt{L_0 C}} = \sqrt{\frac{2eI_0}{\hbar C}}$$



The quantized JJ realizes a qubit

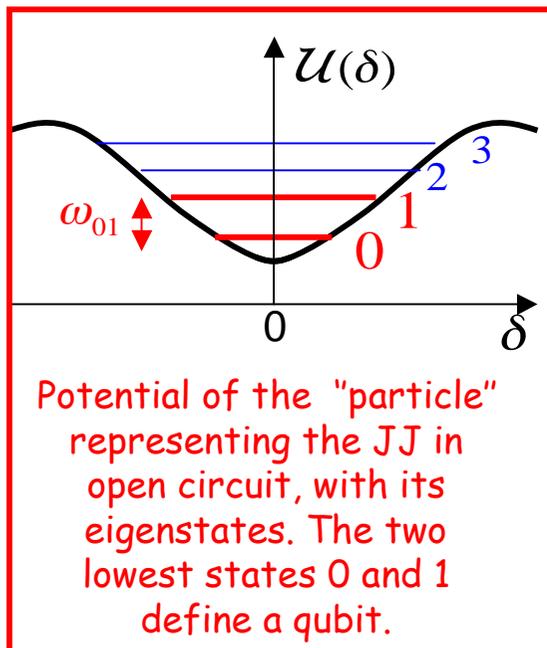
When the conjugate variables Q and Φ are quantized, they become non-commuting operators. The product $Q\Phi$ has the dimension of an action (energy \times time), with the commutation rule:

$$[Q, \Phi_{\text{Jl}}] = i\hbar I$$

which becomes equivalently for the dimensionless conjugate p and δ variables:

$$[p, \delta] = iI$$

The linear L_0C quantum oscillator has a ladder of equidistant levels separated by the energy $\hbar\omega$. This equidistance is broken in the JJ system, due to the departure of the actual «potential» $\cos\delta$ from the δ^2 parabolic law: the open-circuit JJ is a non-linear oscillator.



Due to the breaking of the transitions degeneracy, it is possible to manipulate with a microwave the two lowest states without exciting upper levels. We thus define a qubit whose frequency ω_{01} , close to ω_{Jl} , falls in the radiofrequency (several GHz) domain (make an order of magnitude estimate of ω with the values of e , I_0 and C given above).

In fact, this open circuit qubit is not practical because it is not controllable. We will see that by coupling it through wires or inductances to external circuits, one can turn it into a manipulable device. Before describing practical circuits, we will estimate the p and δ fluctuations in this very simple system.

Fluctuations of the open circuit

Let us evaluate as a function of E_J and E_C the JJ frequency and the p and δ variances in the open circuit ground state. We express the uncertainty relation between p and δ and by identifying to $\hbar\omega_{01}/4$ the mean value of the two terms of H_J (equipartition of kinetic and potential energy in the quasi harmonic well):

$$\langle p^2 \rangle \langle \delta^2 \rangle = \frac{1}{4} \quad ; \quad E_C \langle p^2 \rangle = \frac{E_J}{2} \langle \delta^2 \rangle = \frac{\hbar\omega_{01}}{4} \quad \rightarrow \quad \frac{\langle p^2 \rangle}{\langle \delta^2 \rangle} = \frac{E_J}{2E_C}$$
$$\rightarrow \langle \delta^2 \rangle = \sqrt{\frac{E_C}{2E_J}} \quad ; \quad \langle p^2 \rangle = \sqrt{\frac{E_J}{8E_C}} \quad ; \quad \hbar\omega_{01} \approx \sqrt{2E_J E_C}$$

Depending upon E_J/E_C , we distinguish two limiting cases:

If $E_J/E_C \gg 1$ (inductive energy of the JJ dominant): The charge variance $4e^2 \langle p^2 \rangle$ is big and the phase variance small: the phase is well defined and the charge fuzzy.

If $E_J/E_C \ll 1$ (capacitive energy of the JJ dominant): the phase has large fluctuations and the charge difference between the two sides of the junction is well defined.

The same distinction applies to practical qubits: « phase qubits » have a relatively well-defined phase and a fuzzy charge which can be considered as "continuous", while charge qubits have well-defined charge whose discreteness is a characteristic feature. We will only consider here phase qubits (see below).

Magnetic effects: flux quantization

The phase of the Cooper pairs wave function is constant in a small wire only in zero magnetic field. In a field B deriving from a vector potential $A(\mathbf{r})$, the phase of the wave function acquires a gauge term taking the form:

$$\psi \propto \exp\left[-2ie\vec{A}(\vec{r})\cdot\vec{r} / \hbar\right]$$

The phase variation between two points becomes:

$$\delta_1 - \delta_2 = \frac{2e}{\hbar} \int_1 \vec{A}\cdot d\vec{l}$$

In a bulk metal, this condition, combined with phase unicity requires that the curviline integral of A , proportional to the flux of B across the contour (Stokes law), must be zero: the magnetic field is excluded (Meissner effect). An interesting situation arises for a superconducting ring. The circulation of the vector potential around the ring should be a multiple of 2π , ensuring the unicity of the phase modulo 2π :

$$\frac{2e}{\hbar} \oint A dl = 2q\pi \quad (q \text{ integer})$$

This implies flux quantization in the ring:

$$\Phi = q\Phi_0 \quad ; \quad \Phi = \oint \vec{A}\cdot d\vec{l} = \iint \vec{B}\cdot d\vec{\Sigma} = \text{magnetic flux across ring} \quad ;$$

$$\Phi_0 = \frac{h}{2e} = \text{flux quantum} = 2,067 \cdot 10^{-15} \text{ Weber}$$

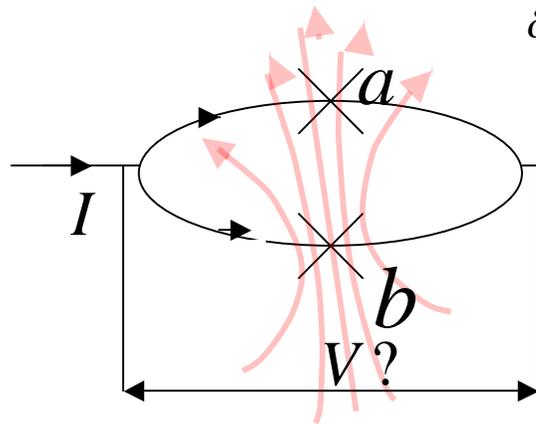
The superconducting current inductively generated adjusts itself to screen partially the incident flux so that the total flux satisfies the quantization condition. If the ring includes a junction, the quantization fixes the phase jump at the JJ:

$$\frac{2e}{\hbar} \oint \vec{A}\cdot d\vec{l} = 2\pi \frac{\Phi}{\Phi_0} = \delta$$

The SQUID: superconducting interferometer detecting weak magnetic fluxes

A superconducting loop with two JJs (a) and (b) is crossed by a magnetic flux Φ to be measured. The phase must recover the same value (modulo 2π) after one turn. Calling δ_a and δ_b the phase jumps of the two JJ's and assuming that the self-inductance of the circuit is negligible, we get:

$$\delta_a - \delta_b + 2\pi \frac{\Phi}{\Phi_0} = 0 \quad (2\pi) \rightarrow \delta_a = \delta_0 + \pi \frac{\Phi}{\Phi_0} \quad ; \quad \delta_b = \delta_0 - \pi \frac{\Phi}{\Phi_0}$$



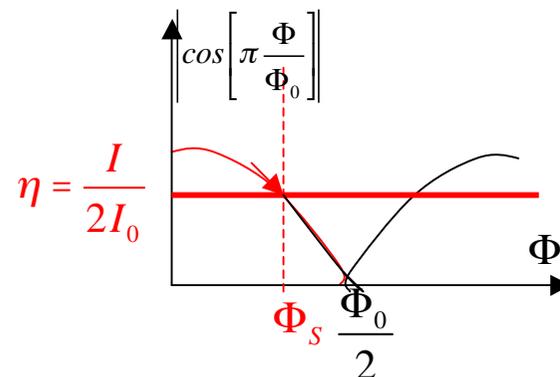
When it is fed by a constant current I , the circuit must satisfy the equation describing the interference between two Josephson currents:

$$I = I_a + I_b = I_0 \sin\left(\delta_0 + \pi \frac{\Phi}{\Phi_0}\right) + I_0 \sin\left(\delta_0 - \pi \frac{\Phi}{\Phi_0}\right) = 2I_0 \sin\delta_0 \cos\pi \frac{\Phi}{\Phi_0}$$

Let us fix I to the value $I=2\eta I_0$ with $\eta < 1$. The device operates in the superconducting regime if there is a value of δ_0 satisfying the above equation, which implies $|\cos(\pi\Phi/\Phi_0)| > \eta$. As soon as Φ exceeds the threshold:

$$\Phi_s = \frac{\Phi_0}{\pi} \text{Arc cos } \eta$$

the SQUID transits to the normal state and a voltage V develops between the two ports of the circuit. This voltage is used to detect small variations of fluxes, much smaller than Φ_0 if the working point of the device is well adjusted.



Conditions to realize a superconducting qubit

We have described a JJ as a quantum system. Its variables p and δ , defined as macroscopic quantities pertaining to a system made of a large number of particles, are non-commuting operators, obeying to an evolution equation ruled by a quantum Hamiltonian. The phase defined modulo 2π is consistent with the discreteness of p which must assume integer values. In the systems described below (phase qubits), p will present large fluctuations and its discrete character will not be essential (this is different in charge qubits, not considered here). We will thus describe p and δ as continuous variables. The states of this quantum system have discrete energies and it is possible to isolate the transition between the ground state $|0\rangle$ and the first excited state $|1\rangle$, which is non-degenerate with other transitions because of the JJ non linearity. Restricting ourselves to exciting the $0 \rightarrow 1$ transition with microwave pulses, we can force the system to evolve in the 0-1 subspace, thus realizing a qubit.

To get an operational system, we must include one (or several JJ's) in an electrical circuit in order to realize the following operations:

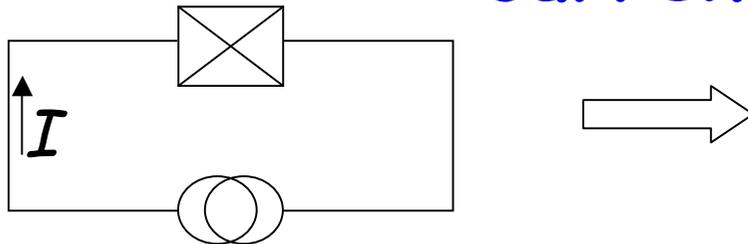
- Frequency tuning of the qubit
- Coupling of the qubit to microwaves in order to manipulate its state
- Coupling qubits with each other or with a microwave or radiofrequency resonator to realize quantum gates
- Detection of the qubit with a state selective device

We will now describe a device, the phase qubit, in which all these essential functions can be implemented.

V-B

The phase qubit

Josephson junction driven by constant current



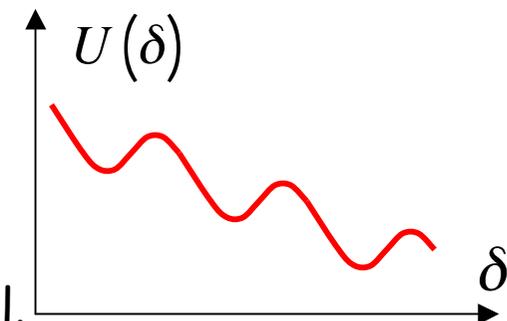
When the JJ is fed by a dc current I produced by an external source, the current conservation in the circuit can be written by expressing the dc and ac Josephson laws: :

$$I = I_0 \sin \delta + \frac{dQ}{dt} = I_0 \sin \delta + C \frac{dV}{dt} = I_0 \sin \delta + \frac{\hbar C}{2e} \frac{d^2 \delta}{dt^2}$$

This equation describes the acceleration of δ produced by the sum of two «forces»: The first, proportional to $I_0 \sin \delta$, is the non-linear restoring force of the Josephson resonator. The other, proportional to I , is an applied force imposed by the source of current. The sum of these forces derives from a potential proportional to $-I_0 \cos \delta - I \delta$. Comparing with the Hamiltonian of the open circuit JJ, we immediately get the current-driven Hamiltonian:

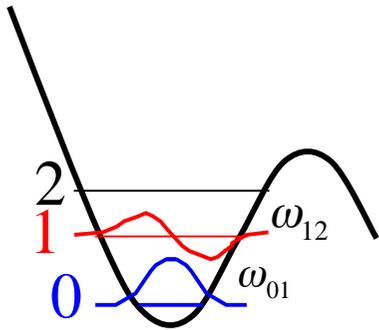
$$H(I) = \frac{2e^2}{C} p^2 - \frac{\hbar}{2e} (I\delta + I_0 \cos \delta)$$

which rules the dynamics of a quantum effective particle with conjugate coordinates δ and p in a washboard potential.



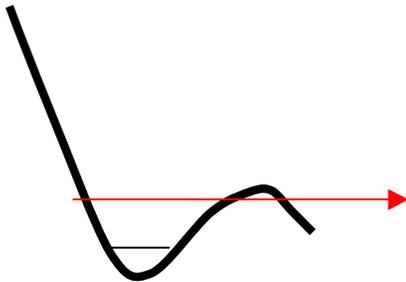
The Josephson phase qubit

For $I < I_0$, $U(\delta)$ has minima (around which it is quasi-harmonic) and maxima. Let us focus on the system's dynamics around a minimum. The ground state and the first excited state in the potential well associated to this minimum are separated by frequency ω_{01} (typically a few GHz). The second excited state is linked to the first by a transition with frequency $\omega_{12} \neq \omega_{01}$ due to the potential anharmonicity. We can thus selectively excite the $0 \rightarrow 1$ transition and realize an effective two-level system.



State selective detection by tunnel effect across barrier:

By increasing I , we lower the barrier between two wells until we reach a configuration where state 1 has an energy just below the potential maximum. If the qubit is in state 1, the effective particle escapes by tunneling through the barrier and δ undergoes an accelerated motion down the washboard. When $d\delta/dt$ exceeds a critical value, the junction transits to the normal phase and a voltage appears between its ports, which gives a detection signal selectively detecting the qubit in state 1. The state 0 remains stable in the well and undetected by this effect. In order to selectively detect 0, we can transfer the system from 0 to 1 by a resonant microwave pulse (see below) and then detect state 1.



Tuning the phase qubit by varying current I

The frequency ω_{01} of the qubit depends upon the current I which controls the operating point δ_0 of the JJ and the shape of the washboard potential. The positions of the minima of $U(\delta)$ correspond to the cancellations of the force acting on the effective particle and are given by solving $I - I_0 \sin \delta_0 = 0$ or $\delta_0 = \text{Arc sin } I/I_0$, with the additional condition that the second derivative of $U(\delta)$ is positive. At these points, the system is classically in equilibrium and the relation $I = I_0 \sin \delta$ means that the current goes across the « intrinsic » JJ without component across the capacity (classically we have at this point $dp/dt = 0$). In the vicinity of this equilibrium, we have:

$$U(\delta) = U(\delta_0) + \frac{(\delta - \delta_0)^2}{2} \frac{d^2 U(\delta_0)}{d\delta^2} = Cte + \frac{I_0 \hbar \cos \delta_0}{4e} (\delta - \delta_0)^2$$

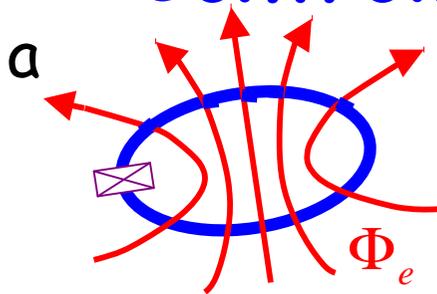
The classical frequency of the linearized Josephson oscillator is then:

$$\omega(I) = \sqrt{\frac{2eI_0 \cos \delta_0}{\hbar C}} = \sqrt{\frac{2eI_0}{\hbar C} \left[1 - \frac{I^2}{I_0^2}\right]^{1/4}}$$

It is I-dependent and cancels for $I = I_0$. The frequency of the qubit ω_{01} is $\sim 0.95\omega(I)$, the correction factor being due to the JJ anharmonicity. We see that by varying I one can tune the qubit.

By manipulating the qubit with a variable current I one can thus tune its frequency *and* detect its state (by lowering the height of the barrier). These operations are carried out sequentially. We will now describe a variant in which the control and the detection are performed by an inductive coupling of the qubit to external circuits (control and detection by the flux).

Controlling the phase qubit by the flux



Instead of controlling the qubit by a dc current, one can do it inductively by sending a magnetic flux Φ_e across the circuit (fig. a).

In practice, Φ_e is produced through a superconducting dc transformer (fig. b) coupling the qubit circuit to an external one. M is their mutual inductance and L is the "classical" self-inductance of the qubit circuit (note: do not confuse L with L_0 defined above which was the intrinsic linearized inductance of the JJ). The controlling current I_Φ produces the flux $\Phi_e = M I_\Phi$ across the circuit qubit. An induced current I appears in L which opposes the incident flux, producing a total flux:

$$\Phi = \Phi_e - LI$$

Applying the phase quantization relation, we get the condition satisfied by the JJ phase:

$$\delta = 2\pi \frac{\Phi_e - LI}{\Phi_0} \quad ; \quad \Phi_0 = \frac{h}{2e}$$

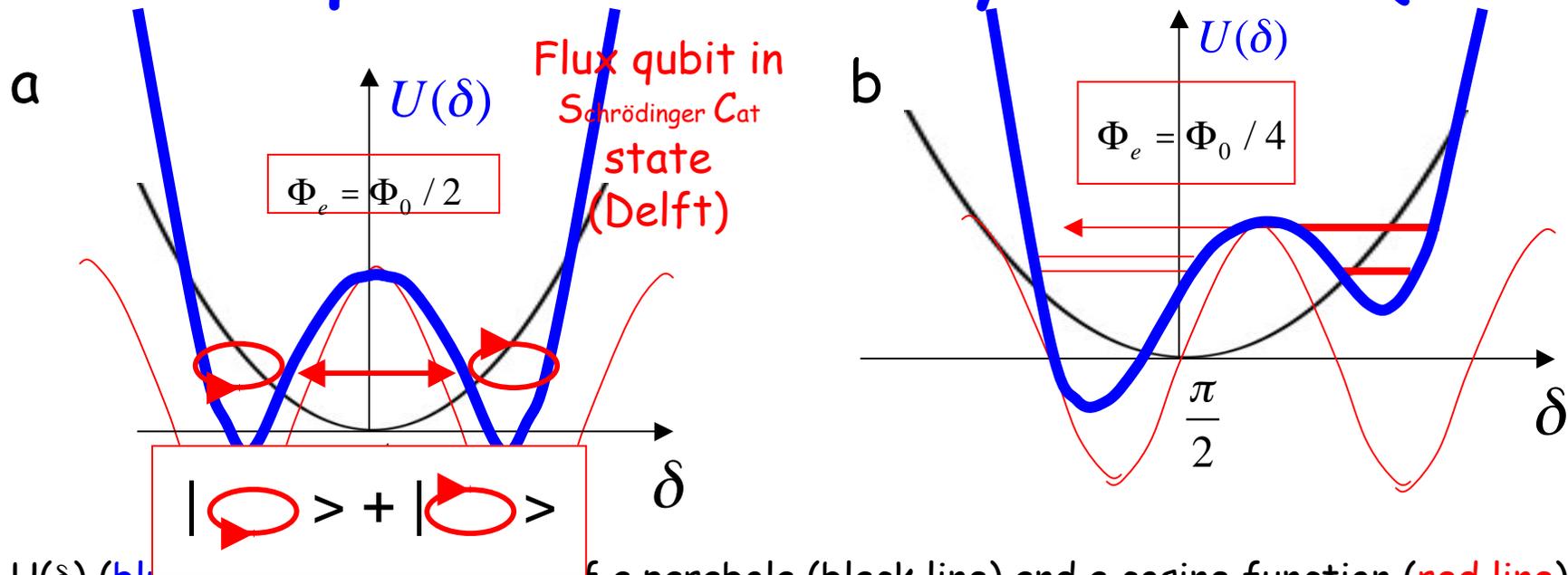
which yields the expression of the magnetic energy of the L inductance versus δ :

$$\frac{1}{2} LI^2 = \frac{\Phi_0^2}{2L} \left[\frac{\delta}{2\pi} - \frac{\Phi_e}{\Phi_0} \right]^2$$

and leads, after adding the capacitive and intrinsic inductive contributions, to the hamiltonian of the qubit controlled by the flux Φ_e :

$$H = \frac{2e^2}{C} p^2 + \frac{\Phi_0^2}{2L} \left[\frac{\delta}{2\pi} - \frac{\Phi_e}{\Phi_0} \right]^2 - \frac{\Phi_0}{2\pi} I_0 \cos \delta$$

Phase qubit controlled by the flux (cont'd)



$U(\delta)$ (blue line) is now the sum of a parabola (black line) and a cosine function (red line). By changing Φ_e , we vary the phase of the cosine at the minimum of the parabola, which changes the shape of $U(\delta)$. When $2\pi I_0 L / \Phi_0 \geq 1$, $U(\delta)$ has a double well shape. The potential is symmetrical for $\Phi_e / \Phi_0 = 1/2$ (Fig.a), dissymetrical otherwise (the case $\Phi_e / \Phi_0 = 1/4$ is shown in Fig.b). The symmetrical situation corresponds to the flux qubit. Classically, the system admits two stable operating points, corresponding to currents flowing in opposite directions. Quantum tunneling through the barrier admixes these classical solutions, leading to a non-classical ground state, which is a linear superposition of these two classical situations. We will not study this case here and focus on the dissymetrical case (b). For a convenient choice of L , the right well admits a small number of bound states, the two lowest ones forming the qubit. Detection is made by tunnel effect, as for the current controlled qubit, by adjusting the flux to vary the height of the barrier. The system leaks towards the deeper well at left, which has a big density of excited states in the vicinity of the barrier summit. By adjusting finely Φ_e we can also tune the qubit frequency (change of the potential curvature).

Order of magnitude of phase variance

By developing the qubit Hamiltonian up to second order in $\delta - \delta_0$, we obtain the harmonic oscillator expression:

$$H \approx E_C p^2 + \frac{E_J'}{2} (\delta - \delta_0)^2 \quad (E_J' \neq E_J = \hbar I_0 / 2e \text{ of open circuit JJ})$$

whose angular frequency, close to ω_{01} , is:

$$\omega_{oh} \approx \omega_{01} = \frac{1}{\hbar} \sqrt{2E_C E_J'}$$

The variance of δ in the ground state is given by:

$$\frac{E_J'}{2} \langle (\delta - \delta_0)^2 \rangle = \frac{\hbar \omega_{01}}{4}$$

We eliminate E_J' between the last two eqs. and we note that $E_C = 2e^2/C$. We then get a simple expression linking $\nu_{01} = \omega_{01}/2\pi$, the flux quantum, the junction capacity and the variance of δ :

$$\langle (\delta - \delta_0)^2 \rangle = \frac{e}{\Phi_0 C \nu_{01}}$$

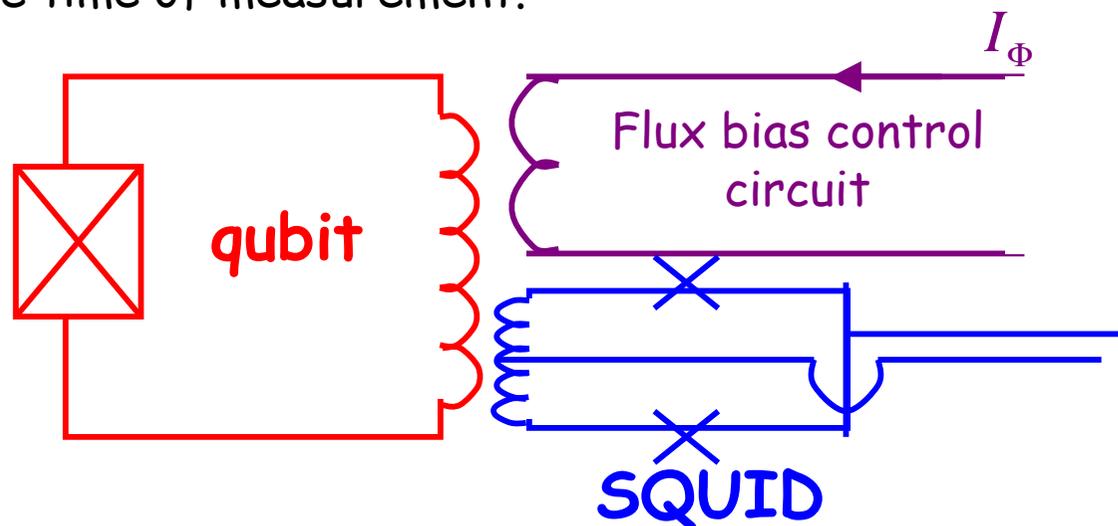
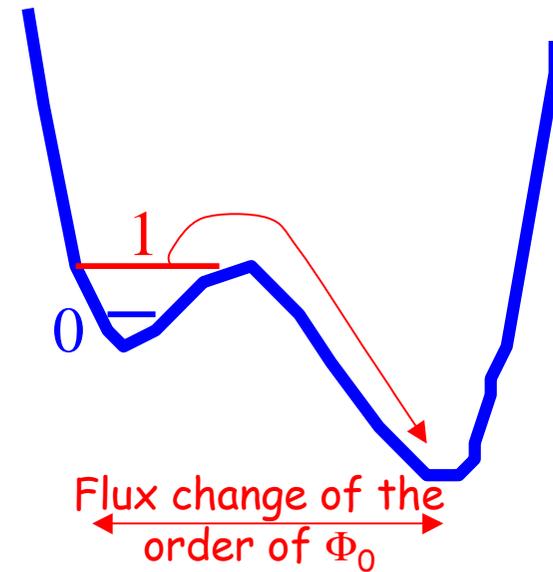
A typical JJ capacity being $C \sim 10^{-12} \text{F}$, we get, by injecting the values of e and Φ_0 :

$$\Phi_0 = 2.10^{-15} \text{T.m}^2 \quad ; \quad C \approx 10^{-12} \text{F} \quad \rightarrow \quad \langle (\delta - \delta_0)^2 \rangle \approx \frac{8.10^7}{\nu_{01}} \text{rad}^2$$

For a typical qubit frequency $\nu_{01} = 5 \text{GHz}$, we have $\Delta\delta = \sqrt{\langle (\delta - \delta_0)^2 \rangle} \sim 0.13$ radian. The phase is relatively well-defined (hence the name « phase qubit ») while the conjugate variable variance is large: $\Delta p \sim 1/(2\Delta\delta) \sim 5$. The charge of the phase qubit is "fuzzy".

Detecting the phase qubit with a SQUID

When the qubit transits from one well to the other, δ changes by about π , which corresponds to a change of about I_0 of the current in the qubit circuit and to a flux variation of about $LI_0 \sim \Phi_0$. This sudden flux jump of about one flux quantum is detected by a SQUID inductively coupled to the qubit (a voltage appears between the ports of the SQUID when the qubit transits from one well to the other). The flux controlling circuit (flux bias) is used to finely tune the qubit frequency and to bring it suddenly to the threshold of selective detection of its quantum states at the time of measurement.

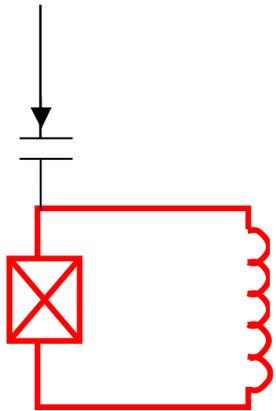


Sketch of the phase qubit with its flux bias control circuit and the detecting SQUID.

Manipulating the qubit state with rf pulses

An ac current $-I_{rf} \sin(\omega t - \varphi)$ with frequency close to ω_{01} can be used to excite the qubit across a coupling capacitor and to realize rotations of its Bloch vector. The qubit Hamiltonian becomes:

$I_{rf}(t)$



$$H(t) = H_{QB} + \frac{\Phi_0}{2\pi} \delta I_{rf} \sin(\omega t - \varphi)$$

The interaction term, proportional to the position δ of the effective particle representing the qubit, is reminiscent of the electric dipole coupling term in an atom. We develop the sine function keeping only the resonant term (RWA approximation) and we move into the frame rotating at angular frequency ω . We get:

$$H_{QB}^{(r)} = \frac{\hbar\Delta}{2} \sigma_z + i \frac{\Phi_0 \langle 0 | \delta | 1 \rangle I_{rf}}{8\pi} [\sigma_- e^{i\varphi} - \sigma_+ e^{-i\varphi}] = \frac{\hbar}{2} [\Delta \sigma_z - \Omega \sin \varphi \sigma_x + \Omega \cos \varphi \sigma_y]$$

where we have introduced the qubit Pauli matrices and their linear combinations:

$$\sigma_+ = |1\rangle\langle 0| = \frac{\sigma_x + i\sigma_y}{2} \quad ; \quad \sigma_- = |0\rangle\langle 1| = \frac{\sigma_x - i\sigma_y}{2}$$

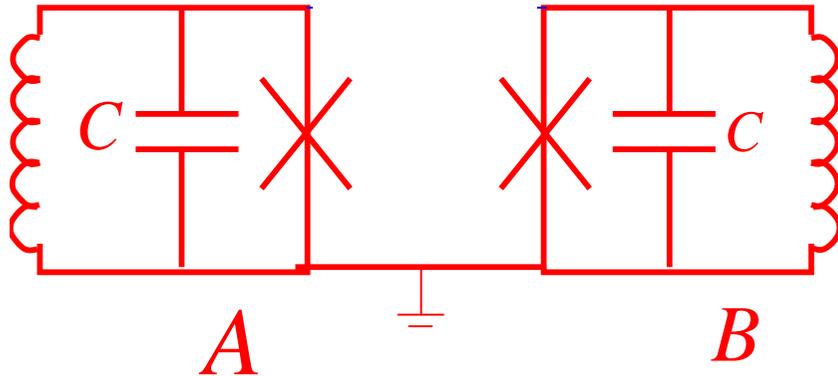
and defined the frequency mismatch Δ and the Rabi frequency Ω :

$$\Delta = \omega_{01} - \omega \quad ; \quad \Omega = \frac{\Phi_0 \langle 0 | \delta | 1 \rangle I_{rf}}{2h}$$

We retrieve the formulas describing the rotations of a qubit analyzed in lecture 1.

With $\langle 0 | \delta | 1 \rangle \sim 0,1$, we find $\Omega/2\pi \sim 3 \cdot 10^{16}$ Hz/A. A 1nA rf current yields a Rabi frequency of about 30 MHz, corresponding to 15 ns for a π pulse.

Capacitive coupling of two qubits



Consider two identical phase qubits A and B. We couple them by a capacitor whose capacity C_x is very small compared to the capacity C of the JJ's. The voltages V_A and V_B on the two sides of C_x are $2ep_a/C$ and $2ep_b/C$ respectively (p_a and p_b are the the canonic momenta of the two circuits). Hence, the coupling energy induced by C_x :

$$H_{\text{int}} = \frac{1}{2} C_x [V_A - V_B]^2 = \frac{2e^2 C_x}{C^2} [p_A - p_B]^2$$

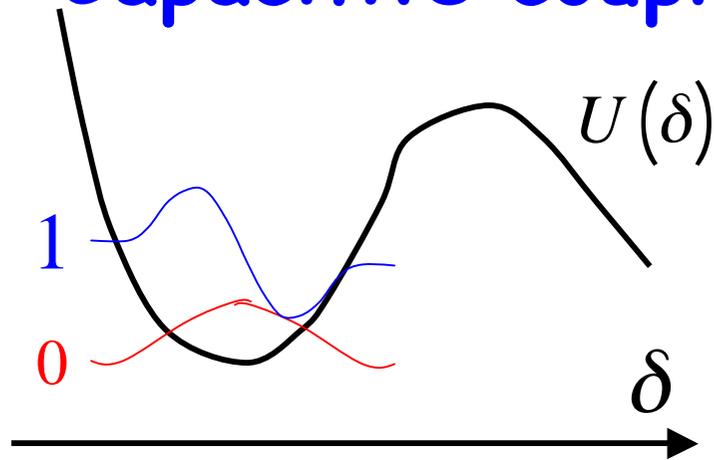
By grouping this term with the two qubit hamiltonians, we write the total Hamiltonian as:

$$H = H_A + H_B + H_{\text{int}} = H'_A + H'_B + H'_{\text{int}}$$

$$\text{with } H'_i = \frac{2e^2(C + C_x)}{C^2} p_i^2 + U(\delta_i) \quad (i = A, B) \quad \text{et} \quad H'_{\text{int}} = -\frac{4e^2 C_x}{C^2} p_A p_B$$

The qubits « mass » is slightly renormalized, which modifies their common frequency, and a coupling term appears, which is proportional to the product of the canonical momenta. This coupling lifts the qubit degeneracy and produces a frequency doublet in the spectrum of the coupled systems.

Capacitive coupling of two qubits (cont'd)



The p_i operators have a zero expectation value in states 0 and 1 and a non-zero matrix element between these states. The coupling between the tensor product states $|0_A, 1_B\rangle$ and $|1_A, 0_B\rangle$ (which are degenerate without coupling) is:

$$\langle 0, 1 | H'_{\text{int}} | 1, 0 \rangle = -\frac{4e^2 C_x}{C^2} |\langle 0 | p | 1 \rangle|^2$$

To evaluate this expression, let us express the zero-point kinetic energy of each qubit oscillator:

$$\frac{2e^2 \langle 0 | p^2 | 0 \rangle}{C} = \frac{\hbar \omega_{01}}{4} \approx \frac{2e^2 \langle 0 | p | 1 \rangle \langle 1 | p | 0 \rangle}{C}$$

We have indeed: $\langle 0 | p^2 | 0 \rangle = \sum_j \langle 0 | p | j \rangle \langle j | p | 0 \rangle \approx \langle 0 | p | 1 \rangle \langle 1 | p | 0 \rangle$

where the sum involves all states j of the qubit (bound and unbound), the first term giving alone the order of magnitude. We then get from the first and second eq.:

$$\langle 0, 1 | H'_{\text{int}} | 1, 0 \rangle \approx -\frac{C_x}{2C} \hbar \omega_{01} = -\hbar \frac{g}{2} \quad \text{with} \quad g = \frac{C_x}{C} \omega_{01}$$

The parameter $g/2\pi$ is the energy exchange frequency between the coupled qubits. It is equal to their frequency $\omega_{01}/2\pi$, divided by the ratio C/C_x (usually of the order of 1000). The coupling $g/2\pi$ is typically of the order of a few MHz.

Entanglement of capacitively coupled qubits

At resonance ($\omega_{01A} = \omega_{01B} = \omega_{01}$), the Hamiltonian of the two coupled qubits is:

$$H = \frac{\hbar\omega_{01}}{2} [\sigma_{Az} + \sigma_{Bz}] - \frac{\hbar g}{2} [\sigma_{A+}\sigma_{B-} + \sigma_{A-}\sigma_{B+}]$$

If the initial state is $|0\rangle_A |1\rangle_B$ and the qubits are coupled during time t , the system oscillates between $|0\rangle_A |1\rangle_B$ and $|0\rangle_B |1\rangle_A$ with frequency g :

$$|\Psi_{AB}(t)\rangle = \cos\frac{gt}{2} |0\rangle_A |1\rangle_B + i \sin\frac{gt}{2} |1\rangle_A |0\rangle_B$$

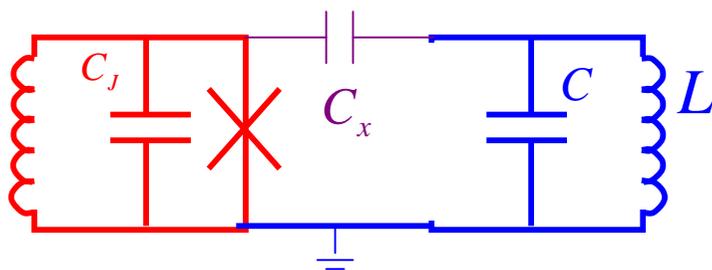
It is generally at time t in an entangled state. If $t = \pi/2g$, the entanglement is maximal:

$$|\Psi_{AB}(t = \pi/g)\rangle = \frac{1}{\sqrt{2}} [|0\rangle_A |1\rangle_B + i |1\rangle_A |0\rangle_B]$$

To prepare this state from two qubits initialized in the ground state $|0\rangle_A |0\rangle_B$ one applies first a π pulse to qubit B, then let interact A and B during time $t = \pi/2g$. In order to switch g on and off, one tunes swiftly the qubit frequency difference $\omega_{01A} - \omega_{01B}$ over a frequency domain large compared to g . This is done by using the flux bias controls of one of the qubits. When $\omega_{01A} - \omega_{01B} \gg g$ the probability of energy exchange between $|0\rangle_A |1\rangle_B$ and $|1\rangle_A |0\rangle_B$ is negligible and the qubits are effectively decoupled. This general method, also extended to the coupling of qubits with LC resonators (see below), has been used to entangle qubits, realize quantum gates and test Bell inequalities with mesoscopic superconducting circuits.

Coupling a qubit to an rf LC resonator: Circuit QED

The simplest rf resonator in an LC circuit. The capacitive coupling of this circuit to the phase qubit is similar to the coupling of two qubits. We write first the quantum hamiltonian of the LC circuit:



$$H_R = \frac{Q^2}{2C} + \frac{\Phi^2}{2L} \quad ; \quad [Q, \Phi] = i\hbar$$

$$Q = \sqrt{\frac{\hbar\omega_{LC}}{2}} (a + a^\dagger) \quad ; \quad \omega_{LC} = \frac{1}{\sqrt{LC}} \quad \left(\frac{\langle Q^2 \rangle_0}{2C} = \frac{\hbar\omega_{LC}}{4} \right)$$

$$\text{Voltage applied to } C_X : \frac{2ep}{C_j} - \frac{Q}{C}$$

$$\text{Coupling energy: } \frac{1}{2} C_X \left[\frac{2ep}{C_j} - \frac{Q}{C} \right]^2 \rightarrow H'_{\text{int}} = -2e \frac{C_X}{C_j C} pQ \approx -C_X \sqrt{\frac{\hbar\omega_{01}}{2C_j}} \sqrt{\frac{\hbar\omega_{LC}}{2C}} (\sigma_+ + \sigma_-) (a + a^\dagger)$$

Near resonance ($\omega_{01} \sim \omega_{LC}$) and neglecting antiresonant terms:

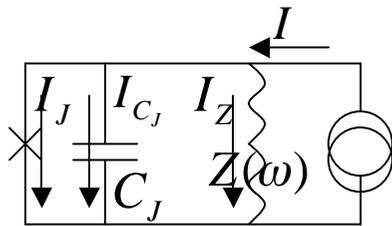
$$H'_{\text{int}} \approx -\frac{C_X \hbar\omega_{LC}}{2\sqrt{CC_j}} (\sigma_+ a + \sigma_- a^\dagger) = -\frac{\hbar\Omega}{2} (\sigma_+ a + \sigma_- a^\dagger) \quad ; \quad \Omega = \frac{C_X}{\sqrt{CC_j}} \omega_{LC} = \frac{C_X}{C} \frac{1}{\sqrt{LC_j}}$$

We recover the Jaynes-Cummings Hamiltonian of CQED. The coupling of a qubit with a coaxial line resonator (electrically equivalent to a network of LC circuits) is described by a similar Hamiltonian. « Circuit QED » experiments are realized with such systems (see lecture 6).

Qubit relaxation

Qubit relaxation corresponds to the damping of the effective particle, whose evolution is ruled by the qubit Hamiltonian. Phenomenologically, we describe it by adding in the circuit a resistive element with a complex impedance $Z(\omega)$, which describes all the losses (radiative losses, losses in the JJ and capacitance dielectrics...) at the qubit frequency ω . In the case of a phase qubit driven by a constant current, the current conservation yields:

$$I = I_J + I_Z + I_{C_J} = I_0 \sin \delta + \frac{\hbar}{2e} \frac{1}{Z} \frac{d\delta}{dt} + \frac{C_J \hbar}{2e} \frac{d^2 \delta}{dt^2}$$



Near $\delta = \delta_0$, we get a damped oscillator equation:

$$\frac{C_J \hbar}{2e} \frac{d^2 \delta}{dt^2} + \frac{\hbar}{2e} \frac{1}{Z} \frac{d\delta}{dt} + I_0 [\delta - \delta_0] \cos \delta_0 = 0$$

The solution is an exponential $\delta - \delta_0 = \alpha e^{-izt}$ whose complex frequency z is the solution of:

$$\frac{C_J \hbar}{2e} z^2 + i \frac{\hbar}{2e} \frac{1}{Z} z - I_0 \cos \delta_0 = 0$$

We find $z_{\pm} = \pm \omega - i\gamma/2$ with:

$$\omega \approx \sqrt{\frac{2I_0 e \cos \delta_0}{\hbar C_J}} \quad ; \quad \gamma = \frac{1}{RC_J} \quad (\text{with } R = \text{Re}[Z(\omega)] \gg \frac{1}{C_J \omega}, \text{Im}[Z(\omega)])$$

γ is the damping rate of the qubit energy, i.e. the reciprocal of its excited state lifetime T_1 :

$$T_1 = \frac{1}{\gamma} = RC_J$$

For a flux biased qubit, the impedance can be fairly large, $R \sim 10^5$ Ohm, resulting in $T_1 \sim 1 \mu\text{s}$ (for $C_J = 10^{-11}$ F). The T_2 relaxation time describes the damping of the qubit coherence. It is sensitive not only to energy damping, but also to all kinds of classical noise affecting the phase of the qubit. In the best case, T_2 is of the order of T_1 , also in the $1 \mu\text{s}$ range.

Conclusion of lecture 5

The charge and phase variables of a circuit involving a Josephson junction behave as conjugate quantum observables. Their evolution is ruled by a hamiltonian whose form depends upon the details of the circuit. The potential associated to this hamiltonian exhibits a local minimum around which the system behaves as a non-linear harmonic oscillator whose two lowest energy levels define a qubit. The frequency of the phase qubit is tuned by adjusting a flux bias and the qubit detection is achieved by having the excited qubit state tunneling through a potential barrier whose summit is properly fixed, again by playing with the flux bias. The tunneling is monitored by a SQUID circuit coupled to the qubit. Manipulation of the qubit state is achieved by injecting in the circuit a resonant rf pulse. Interaction of qubit circuits with each other or with a linear rf resonator of the LC type is achieved via capacitor coupling.

The decoherence of the superconducting qubit is mainly due to losses in the dielectrics (T_1) and to fluctuations of electric and magnetic fields inducing random variations of qubit frequency (T_2). In phase qubits, the T_1 and T_2 orders of magnitudes are in the microsecond range, while the qubit frequency is in the few GHz range and the couplings to rf, to each other and to rf resonators are in the few MHz range.

The phase qubit described here is the one realized in J.Martinis group at USBC. Other kinds of devices (flux qubits and charge qubits called qutonium and transmon) have also been developed in various labs (Delft, Yale, Saclay, ETH-Zurich...). For a discussion of superconducting qubits, see College de France lectures by M.Devoret and the following references:

- J.Martinis & M.Devoret, Superconducting qubits in « Quantum entanglement and information processing », Ecole des Houches sessions **79** (2004).
- R.Schoelkopf & S.Girvin, Wiring-up quantum systems, Nature, **451**, 664 (2008)
- J.Clarke & F.Wilhelm, Superconducting quantum bits, Nature, **453**, 1031 (2008)