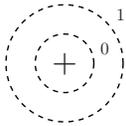


## 1 Physical Qubits

We now give three examples of physical realizations of qubits, but there are many more.

### Energy levels of hydrogen atom

Consider the electron in a hydrogen atom. It can be in its ground state (i.e. an  $s$  orbital) or in an excited state. If this were a classical system, we could store a bit of information in the state of the electron: ground = 0, excited = 1. So we can also store a qubit of information in the quantum state of the electron, i.e., in the superposition  $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ . Note that the electron actually has an infinite number of energy levels (indexed by quantum number  $n$ , with  $E_n \propto -1/n^2$ ), but that as long as we can isolate two of them, we can use these two as a qubit.



### Photon Polarization

There is a qubit associated with photon - its polarization. Recall that a photon moving along the  $z$ -axis has an associated electric field in the  $x$ - $y$  plane. The frequency of the field is determined by the frequency of the photon. However, this still leaves the  $x$ - $y$  components of the electric field unspecified. The 2-dimensional quantity specifying this field is the polarization of the photon.

See notes on polarization on the web page Science Trek at <http://www.colorado.edu/physics/2000>.

### Spin

Qubit systems can always be mapped onto an effective spin  $1/2$  system so it is important to understand what this is and where it comes from.

Elementary particles and composite particles carry an intrinsic angular momentum called spin. For our purposes, the most important particles are electrons and protons. To each of these is associated an angular momentum vector that can point up  $|\uparrow\rangle$  or down  $|\downarrow\rangle$ . The quantum mechanical spin state of an electron or proton is thus  $|\psi\rangle = \alpha|\uparrow\rangle + \beta|\downarrow\rangle$ . Therefore, spins can be used as qubits with  $|0\rangle = |\uparrow\rangle$ ,  $|1\rangle = |\downarrow\rangle$ .

The spin angular momentum is intrinsic and signals the presence of an intrinsic magnetic moment. Uhlenbeck and Goudsmit introduced the concept of 'spin' in 1925 to explain the behavior of hydrogen atoms in a magnetic field:

The extra transitions can be explained if an electron has an *intrinsic* magnetic moment  $\vec{\mu}$ , since a magnetic moment in a magnetic field  $\vec{B}$  has an energy  $E = -\vec{\mu} \cdot \vec{B}$ . In the context of QM, new energy levels can derive from  $\vec{\mu}$  being oriented parallel or anti-parallel to  $\vec{B}$ .

Where does  $\vec{\mu}$  come from?

The simplest explanation is "classical": classically, a magnetic moment  $\vec{\mu}$  comes from a loop of current.

The energy  $E = -\vec{\mu} \cdot \vec{B}$  comes from  $\vec{I} \times \vec{B}$  force of current in a B-field (Lorentz force). The lowest energy,

and thereby the place where "the system wants to go", is obtained when the magnetic moment and B-field line up.

If an isolated electron has "intrinsic"  $\vec{\mu}$  then the simplest explanation for this is that electron spins about some axis. This is independent of its orbital motion in an atom, just like the Earth's "spin" about the north pole is independent of its orbit around the sun.

Since  $\vec{\mu}$  is associated with a "spinning" charge, then we can write  $\vec{\mu}$  in terms of angular momentum. Anything that spins has angular momentum!

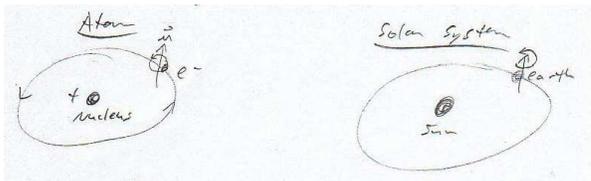
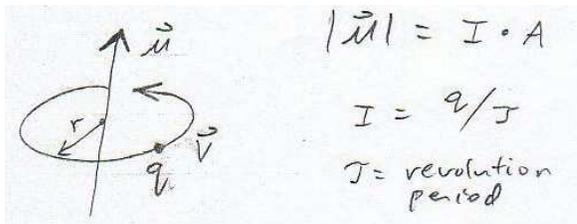
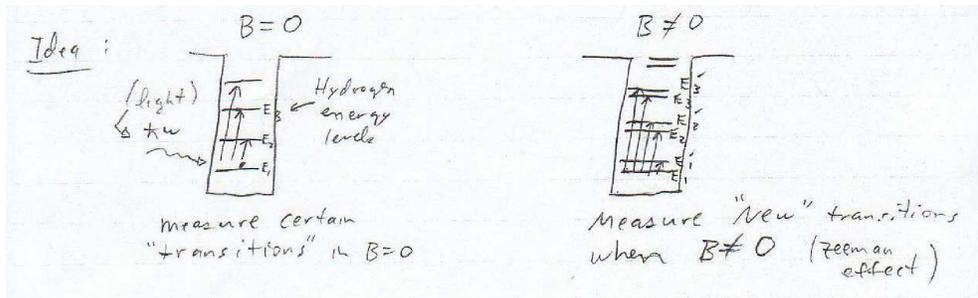
The simplest way to see this is classically for a spinning charge. For an electron the charge  $q$  is equal to  $-e$ . Angular momentum is given by  $\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v}$ .  $L = mvr$  for a charge of mass  $= m$  moving in a circle with velocity  $= v$ . The magnetic moment can be obtained as follows:

$$\mu = (\text{current})(\text{Area}) = \frac{q}{\tau} \cdot \pi r^2$$

But the revolution period  $\tau = \frac{2\pi r}{v}$ . Substituting for  $\tau$  and  $v$  in terms of  $L$ , we obtain

$$\vec{\mu} = \frac{q}{2m} \vec{L}$$

Now comes the tricky part. *The electron is not actually spinning about some axis!* It only acts as though it is. Electrons are point particles which, as far as we know, have no "size" in the traditional sense. Therefore the  $r$  in the previous discussion of spinning charge is not meaningful. The *intrinsic* angular momentum of



an electron has nothing to do with "orbital" motion, but it does lead to an intrinsic  $\vec{\mu}$ . This is a relativistic effect that can be derived from the Dirac Equation (Relativistic Schrodinger equation for spin- $\frac{1}{2}$  particles), but it holds for electrons that are not moving fast.

This intrinsic angular momentum is called "spin" =  $\vec{S}$ .

For an electron, classically:  $\vec{\mu} = -\frac{e}{2m}\vec{L}$ , while quantum mechanically:  $\vec{\mu} = -\frac{ge}{2m}\vec{S}$ .

What is  $g$ ?  $g$  is called the g-factor and it is a unitless correction factor due to QM. For electrons,  $g \approx 2$ . For protons,  $g \approx 5.6$ . You should also note that  $\frac{m_{proton}}{m_{electron}} \approx 2000$ , so we conclude that  $\mu_{proton} \ll \mu_{electron}$ .

So, to understand behavior of the electron's intrinsic magnetic moment  $\vec{\mu}$  (which is an observable we can measure) then we must understand the behavior of its intrinsic angular momentum =  $\vec{S}$ . This is why spin is important. Since the electron is small,  $\vec{S}$  must be described by QM.

See also notes "electrons in atoms" (look for "spin" section) on the web page Science Trek at <http://www.colorado.edu/phy>

## 2 One-qubit Unitaries/Gates

Rotations over a complex vector space are called unitary transformations. For example, rotation by  $\theta$  is unitary. Reflection about the line  $\theta/2$  is also unitary. Unitary operations  $U$  satisfy

$$UU^\dagger = U^\dagger U = 1$$

i.e.,  $U^\dagger = U^{-1}$ , the adjoint of the operator is equal to its inverse. (Recall that in the matrix representation we have  $[U^\dagger]_{ij} = [U^*]_{ji} = [U^T]_{ij}^*$ )

One very important unitary is the time evolution operator

$$U = \exp(-iHt)$$

where  $H$  is the Hamiltonian operator of the quantum system. In computer science we usually analyze quantum operations in terms of unitaries, or "gates". To physically realize these gates we need to implement the corresponding Hamiltonian operators  $H$ .

In order to manipulate a qubit, we must manipulate its state:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

This is done by acting on  $|\psi\rangle$  with unitary operators (i.e. gates) such that

$$\hat{U}|\psi\rangle = \alpha'|0\rangle + \beta'|1\rangle$$

where  $\hat{U}$  is a  $2 \times 2$  unitary matrix.

### Hadamard gate:

The Hadamard gate is a reflection about the line  $\theta = \pi/8$ . This reflection maps the  $x$ -axis to the  $45^\circ$  line, and the  $y$ -axis to the  $-45^\circ$  line. That is

$$|0\rangle \xrightarrow{H} \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle \equiv |+\rangle \tag{1}$$

$$|1\rangle \xrightarrow{H} \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle \equiv |-\rangle \tag{2}$$

In matrix form, we write

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} .$$

Notice that, starting in  $|\psi\rangle$  either  $|0\rangle$  or  $|1\rangle$ ,  $H|\psi\rangle$  when measured is equally likely to give 0 and 1. There is no longer any distinguishing information in the bit. This information has moved to the phase (in the computational basis).

In a quantum circuit diagram, we imagine the qubit travelling from left to right along the wire. The following diagram shows the application of a Hadamard gate.



### 3 Two-qubit gate: CNOT

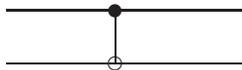
The controlled-not (CNOT) gate exors the first qubit into the second qubit ( $|a,b\rangle \rightarrow |a, a \oplus b\rangle = |a, a + b \bmod 2\rangle$ ). Thus it permutes the four basis states as follows:

$$\begin{array}{ll} 00 \rightarrow 00 & 01 \rightarrow 01 \\ 10 \rightarrow 11 & 11 \rightarrow 10 \end{array} .$$

As a unitary  $4 \times 4$  matrix, the CNOT gate is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

In a quantum circuit diagram, the CNOT gate has the following representation. The upper wire is called the control bit, and the lower wire the target bit.



It turns out that this is the only two qubit gate we need to think about ...

### 4 Bell states (EPR pairs)

There are four Bell states:

$$\begin{aligned} |\Phi^\pm\rangle &= \frac{1}{\sqrt{2}} (|00\rangle \pm |11\rangle) \\ |\Psi^\pm\rangle &= \frac{1}{\sqrt{2}} (|01\rangle \pm |10\rangle) . \end{aligned}$$

These are maximally entangled states on two qubits. They cannot be product states because there are no cross terms.

Consider one of the Bell states (also known as a EPR pair):

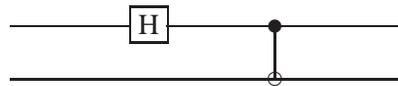
$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

Measuring the first qubit of  $|\Psi^-\rangle$  in the standard basis yields a 0 with probability 1/2, and 1 with probability 1/2. Likewise, measuring the second qubit of  $|\Psi^-\rangle$  yields the same outcomes with the same probabilities. Thus measuring one, and only one, qubit of this state yields a perfectly random outcome.

However, determining either qubit *exactly* determines the other. For example, if qubit 1 is measured and gives a 0, this projects the Bell state onto the state  $|01\rangle$  and the second qubit is then definitely a 1.

Furthermore, measurement of  $|\Psi^-\rangle$  in any basis will yield opposite outcomes for the two qubits. To see this, check that  $|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|v v^\perp\rangle - |v^\perp v\rangle)$ , for any  $|v\rangle = \alpha|0\rangle + \beta|1\rangle$ ,  $|v^\perp\rangle = \bar{\alpha}|1\rangle - \bar{\beta}|0\rangle$ .

We can generate the Bell states with a Hadamard gate and a CNOT gate. Consider the following diagram:



The first qubit is passed through a Hadamard gate and then both qubits are entangled by a CNOT gate.

If the input to the system is  $|0\rangle \otimes |0\rangle$ , then the Hadamard gate changes the state to

$$\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |0\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|10\rangle ,$$

and after the CNOT gate the state becomes  $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ , the Bell state  $|\Phi^+\rangle$ . In fact, one can verify that the four possible inputs produce the four Bell states:

$$\begin{aligned} |00\rangle &\mapsto \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = |\Phi^+\rangle; & |01\rangle &\mapsto \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) = |\Psi^+\rangle; \\ |10\rangle &\mapsto \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) = |\Phi^-\rangle; & |11\rangle &\mapsto \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) = |\Psi^-\rangle. \end{aligned}$$

## 5 EPR Paradox

In 1935, Einstein, Podolsky and Rosen (EPR) wrote a paper "Can quantum mechanics be complete?" [Phys. Rev. 47, 777, Available online via PRLA: [http://prola.aps.org/abstract/PR/v47/i10/p777\\_1](http://prola.aps.org/abstract/PR/v47/i10/p777_1)]

For example, consider coin-flipping. We can model coin-flipping as a random process giving heads 50% of the time, and tails 50% of the time. This model is perfectly predictive, but incomplete. With a more accurate experimental setup it would in principle be possible to follow the dynamics of the coin and to determine precisely the range of initial parameters for which the coin ends up heads, and the range for which it ends up tails.

We saw above that for a Bell state, when you measure first qubit, the second qubit is completely determined. However, if two qubits are far apart, then the second qubit must have had a determined state in some time interval *before* measurement, since the speed of light is finite. Moreover this holds in any basis. This appears analogous to the coin flipping example, i.e., there might be a more complete description which allows the

qubit states to be predicted. EPR therefore suggested that there is a more complete theory where “God does not throw dice”.

EPR made two assumptions:

- i) reality principle - the values of physical quantities have physical reality independent of whether a measurement of them is made or not.
- ii) locality principle - the result of a measurement on one system cannot instantaneously influence the result of a measurement on the second system

These two assumptions give rise to a contradiction, nicely illustrated by the analysis of the two qubits in a Bell state, due to Bohm. See Benenti et al., Sec. 2.5. A source emits the Bell state

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

and sends one qubit to Alice, and one to Bob. If Alice measures her qubit in the standard basis and e.g., gets a 1, then Bob will get a 0 upon measuring in the standard basis. On the other hand, if Alice measures her qubit in the Hadamard basis  $\{|+\rangle, |-\rangle\}$ , and gets a +, then Bob will get a - in the Hadamard basis. However, the states  $|0\rangle$  and  $|-\rangle$  are not the same, they differ by a Hadamard rotation. Which state Bob ends up with depends on the measurement made by Alice. This contradicts the assumption of locality. EPR concluded that such situations imply that quantum mechanics is not a complete theory of the physical world.

What would a more complete theory look like? Here is the most extravagant framework. . . When the entangled state is created, the two particles each make up a (very long!) list of all possible experiments that they might be subjected to, and decide how they will behave under each such experiment. When the two particles separate and can no longer communicate, they consult their respective lists to coordinate their actions. To describe such behavior one would have to invoke the existence of ‘local hidden variables’ that are not evident in the quantum description.

It was not until 1964, almost three decades later, that a verifiable and quantitative measure of the local realism assumption was provided. This was given by Bell, who constructed correlation functions of the measurements of Alice and Bob that satisfy a strict inequality under the assumptions of local realism. However, the quantum analog of the correlation functions can violate the inequality for certain choices of measurement basis. The Bell inequality was subsequently tested experimentally in 1981 by Aspect and co-workers, using Bell (EPR) pairs constructed from photon polarization states. Aspect et al. found that indeed nature does not obey the Bell inequalities and so violates local realism. This is consistent with the predictions of quantum mechanics for EPR pairs summarized above and supports the view that nothing *can* be known about the quantum state until a measurement is made. It also tells us that the quantum correlations in an EPR pair are ‘stronger’ than classical correlations. A detailed analysis of the Bell inequality for  $|\Psi^-\rangle$  can be found at <http://minty.caltech.edu/Ph195/downloads.htm> (lecture 10/24, pp. 11-15).

For further reading on the EPR paradox, Bell’s inequality, and the experimental verification of violation of this by quantum systems, see the recommended texts and D. Mermin, *Physics Today* vol. 38(4), April 1985, pp. 38-47.