

# Qubit stabilizer states are complex projective 3-designs

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A *complex spherical  $t$ -design* is a configuration of vectors which is “evenly distributed” on a sphere in the sense that it reproduces Haar measure up to  $t$ -th moments. We show that the set of all  $n$ -qubit stabilizer states forms a complex spherical 3-design in dimension  $2^n$ . Stabilizer states had previously only been known to constitute 2-designs. The problem is reduced to the task of counting the number of stabilizer states with pre-described overlap with respect to a reference state. This, in turn, reduces to a counting problem in discrete symplectic vector spaces for which we find a simple formula.

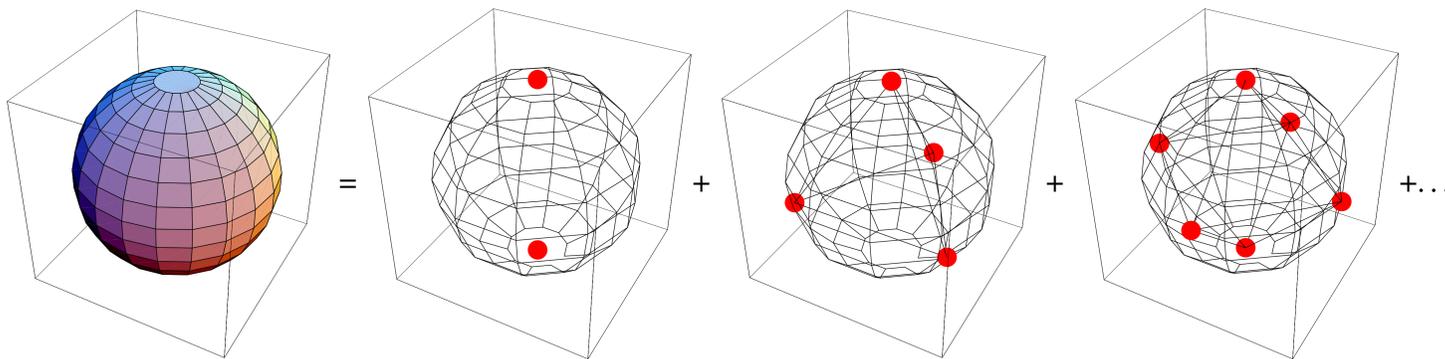


Figure 1: Caricature of spherical designs as a “series expansion” of Haar measure. The three configurations shown are, respectively, a 1-, 2-, and a 3-design on the 2-sphere (with the final set corresponding to  $n = 1$  in our theory).

## Complex spherical $t$ -designs

Consider a finite dimensional Hilbert space  $\mathcal{H}$  and a set of vectors  $X = \{x_i\}$  in  $\mathcal{H}$ . Define:

- $t$ -design:  $X$  obeying  $|X|^{-1} \sum_i (|x_i\rangle\langle x_i|)^{\otimes t} = \int dx (|x\rangle\langle x|)^{\otimes t}$ ,
- $t$ -th frame potential  $\mathcal{F}_t(X) = |X|^{-2} \sum_{i,j} |\langle x_i | x_j \rangle|^{2t}$ ,
- $t$ -th Welch bound:  $\mathcal{W}_t(\mathcal{H}) = \binom{\dim \mathcal{H} + t - 1}{t}^{-1}$ ,
- Theorem [1]:  $X$  is  $t$ -design  $\iff \mathcal{F}_t(X) = \mathcal{W}_t(\mathcal{H})$ .

Examples:

ONBs and tight frames are 1-designs,  
SIC-POVMs and MUBs are 2-designs.

## Qubit stabilizer states: Vanilla approach

Recall standard presentation of stabilizer states [2,3]:

- *Pauli matrices*:  $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ ,  $Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ ,  $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ ,
- *Pauli Group*:  $G_1 = \{\pm I, \pm iI, \pm X, \pm iX, \pm Y, \pm iY, \pm Z, \pm iZ\}$  and  $G_n$  consists of all possible tensor products of elements of  $G_1$ .
- *Stabilizer states*: unique states that are invariant under  $n$  elements of  $G_n$ .

Example: The EPR state  $|EPR\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$  is the unique state invariant under  $X \otimes X, Z \otimes Z \in G_2$ .

However, this description is not well adapted to our purpose.

## Stabilizer states: the phase space lens

We aim to compute  $\mathcal{F}_t(\text{stabilizer states})$ :

- *Clifford Invariance*  $\implies$  Fix one stabilizer state and count # of other ones with a given overlap,
- Unclear how to do this in vanilla approach...  
 $\implies$  must treat stabilizer states more *structurally*. Central tenet:

Combinatorics of stabilizer states is governed by *discrete symplectic geometry* [4].

- Pauli operators form a *projective representation* of the *discrete symplectic vector space*  $V := \mathbb{F}_d^n \oplus \mathbb{F}_d^n$ ,
- Commuting stabilizer groups are images of *isotropic subspaces* of  $V$   
 $\implies$  Stabilizer bases are labeled by *Lagrangian subspaces*  $M$  of  $V$ .

Example (continued): For  $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2$  we have  $V \simeq \mathbb{F}_2^2 \oplus \mathbb{F}_2^2$  and  $M = \text{span}\{(1, 1; 0, 0), (0, 0; 1, 1)\}$  is a Lagrangian subspace.  $M$  specifies the stabilizer state  $|M\rangle = |EPR\rangle$ .

## Discrete symplectics combinatorics

Calculating  $\mathcal{F}_t(\text{stabilizer states})$  reduces to fixing one Lagrangian  $M$  and computing # of Lagrangians  $N$  that obey  $\dim M \cap N = k$ .

Fix a Lagrangian  $M$  of  $V$  and define:

- $\mathcal{G}(M, k)$ : *Grassmannian* of  $k$ -dimensional subspaces of  $M$ ,
- $\mathcal{T}(M)$ : number of *polarizations* containing  $M$ ,
- $\tau(V) := |\mathcal{T}(M)|$  (does not depend on choice of  $M$ ).

Theorem 1 [RK, DG, 2013]: Fix  $M$  in  $V$ . The # of Lagrangians  $N$  that obey  $\dim M \cap N = k$  equals  $|\mathcal{G}(M, k)| \tau(\hat{V})$ , where  $\hat{V} = \mathbb{F}_d^{n-k} \oplus \mathbb{F}_d^{n-k}$ .

- Our proof uses *symplectic reduction*,
- “Theory developed for planetary motion helps to compute stabilizer frame potentials...”

## Our result

- Theorem [RK, DG, 2013]  $\implies \mathcal{F}_t(\text{stabilizer states}) = \mathcal{F}_t(\mathcal{H})$  (a purely combinatorial expression),
- Comparing  $\mathcal{F}_t(\mathcal{H})$  to the Welch bound yields:

Theorem 2 [RK, DG, 2013]:  $\mathcal{F}_3(\text{stabilizer states}) = \mathcal{W}_3(\mathcal{H})$  iff  $\dim \mathcal{H} = 2^n$ .

Corollary: *The set of all qubit stabilizer states is a 3-design.*

- Subcorollary: Single qubit MUBs (e.g.  $|0\rangle, |1\rangle, |+\rangle, |-\rangle, |\odot\rangle, |\oslash\rangle$ ) are 3-designs.
- Remark: A similar statement has been known in the field of algebraic combinatorics [5] using significantly more involved methods.

## References

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