

Robustness of Multiqubit Entanglement

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ABSTRACT

We survey our results on the decay of multiqubit entanglement due to internal interactions between qubits. Dipole-dipole interaction induces decoherence of strongly entangled nuclear spins. The dynamics of spin clusters can be described as quantum decoherence due to an effective composite bath consisting of fully correlated and uncorrelated parts. The rate of decoherence scales up as a square root of the number of entangled spins, resulting in linear scaling of a measure of quantum noise. Our theory is consistent with a recent experiment.

Keywords: Spin, qubit, dipole, entanglement, NMR, decoherence, cluster

1. INTRODUCTION

Quantum computation is a new promising approach to solving some practical problems which are intractable (exponentially hard) for classical computers.¹ The quantum computational performance is achieved by exploiting complex quantum dynamics of many-particle systems in exponentially large Hilbert spaces, necessarily including evolution through entangled states. Experimental implementation of Shor’s quantum factoring algorithm² in a molecule by using seven nuclear spins of its atoms as qubits has been demonstrated.³

In order to achieve scalable implementations of quantum computation one has to protect the fragile entangled states from environmental decoherence. The first question to be asked when considering decoherence of a multiqubit state, is what is the relation between actions of noise on different qubits. There are different approaches to the problem. Some authors investigated the effect of a single bosonic bath representing noisy environment interacting with the whole cluster.^{4–7} An alternative approach in which the noise sources acting on each qubit in a cluster are independent was also explored.^{6,8} A realistic model of the environment should be somewhere between these two extremes. Still, quantitatively accounting for a partially correlated environment would significantly complicate the analysis,⁹ even for a two-particle system.¹⁰ The dynamics of decoherence of entangled multiqubit clusters has attracted much attention recently, motivated by an experimental breakthrough: decoherence of groups of up to 650 entangled nuclear spins was observed and quantified for the first time.¹¹ In this work we survey results¹² on decoherence of multispin clusters caused by both correlated and uncorrelated environmental quantum noise. We consider the experimentally studied¹¹ system consisting of nuclear spins $I = 1/2$.

2. THE MULTISPIN SYSTEM

Initially a system is in thermal equilibrium, with the density matrix

$$\rho_{eq} \simeq \frac{1}{2^N} \left(\mathbf{1} + \frac{\gamma \hbar H_0}{kT} \sum_j I_z^j \right), \quad (1)$$

where N is the number of spins, γ is the gyromagnetic ratio, H_0 is the constant applied magnetic field, k is the Boltzmann constant, T is the temperature, and I_z^j is the z component of the j th spin operator.

By applying a special sequence of radio-frequency pulses,¹¹ high-order correlations between the spins can be built, thereby creating an ensemble of spin clusters that are only weakly coupled with each other. To describe

the evolution of the spins in a sample, it suffices to consider only the dynamics of one such cluster with a well defined number of spins n . We assume^{11, 13} that after this pulse sequence, spins are in a state described by the density operator $\rho(0)$ with all the even coherences excited with equal probability.

Starting from a highly correlated multiqubit state, the system of spins then decays t due to the dipole-dipole interaction given by the Hamiltonian

$$H_{dd} = \sum_{j < k} d_{jk} (2I_z^j I_z^k - I_x^j I_x^k - I_y^j I_y^k), \quad (2)$$

where

$$d_{jk} = \frac{\hbar^2 \gamma^2 (1 - 3 \cos^2 \theta_{jk})}{2r_{jk}^3}, \quad (3)$$

and r_{jk} , θ_{jk} are, respectively, the absolute value and the angle with the z direction, of the vector connecting the j th and k th spins.

The system evolution is given by

$$\rho(t) = \exp\left(-\frac{i}{\hbar} H_{dd} t\right) \rho(0) \exp\left(\frac{i}{\hbar} H_{dd} t\right). \quad (4)$$

This evolution does not produce an experimentally observable signal. To observe the effect of the dipole-dipole interaction, a conversion step is carried out by another sequence of radio-frequency pulses.¹³ During this step multiple quantum coherences are converted back to observable single-spin longitudinal magnetization. The resulting longitudinal magnetization can be detected then by measuring the free induction decay amplitude

$$S(t) \propto \text{Tr}[\rho(t)\rho(0)]. \quad (5)$$

In this review, we focus on the dephasing effect of the dipole-dipole interaction, neglecting any energy exchange between spins. The latter is described by the flip-flop term

$$I_x^j I_x^k + I_y^j I_y^k \quad (6)$$

in the Hamiltonian (2). The truncated Hamiltonian has the form

$$H_{dd}^* = 2 \sum_{j < k} d_{jk} I_z^j I_z^k, \quad (7)$$

corresponding to the limit of “unlike spins.”¹⁴

We use the Zeeman basis $|a\rangle = |a_1 \dots a_n\rangle$, where $a_i = \pm 1$ and

$$I_z^i |a_i\rangle = (a_i/2) |a_i\rangle. \quad (8)$$

In this representation the off-diagonal density matrix elements evolve as

$$\begin{aligned} \rho_{ab}(t) &= \rho_{ab}(0) \exp\left[-\frac{i}{2} \sum_{j < k} d_{jk} (a_j a_k - b_j b_k) t\right] \\ &= \rho_{ab}(0) f_{ab}(t). \end{aligned} \quad (9)$$

The dynamics of the normalized NMR signal (5) can be expressed as

$$S(t) = \prod_{j < k} \cos^2\left(\frac{d_{jk} t}{2}\right) \propto \sum_{ab} |\rho_{ab}(0)|^2 f_{ab}(t). \quad (10)$$

One can experimentally extract from the overall signal, (10), the contributions $S_M(t)$ corresponding to different

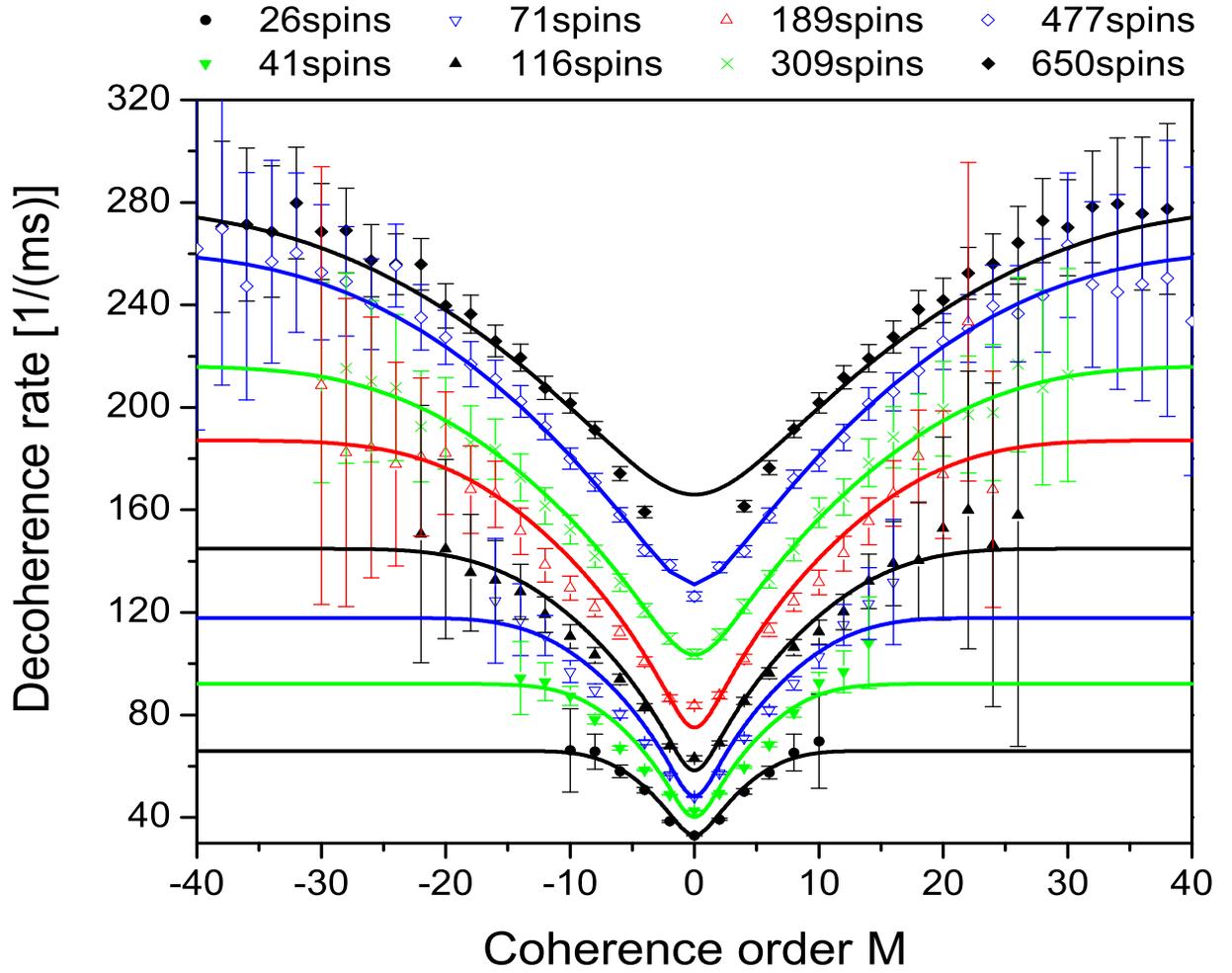


Figure 1. Decay of the signal from high correlated spin clusters for different spin cluster sizes. The points represent experimental values.¹¹ The solid lines are obtained in accordance with the theoretical formula (14).

coherence orders M ,¹¹

$$S(t) = \sum_M S_M(t), \quad (11)$$

where the signal contributions $S_M(t)$ can be evaluated using the formula

$$S_M(t) \propto \sum_{ab \subset M} |\rho_{ab}(0)|^2 f_{ab}(t). \quad (12)$$

Here the summation $\sum_{ab \subset M}$ is over all the possible configurations, with the additional condition

$$\sum_j (a_j - b_j) = 2M. \quad (13)$$

Table 1. The second moment and degree of correlation for adamantane samples, obtained from decoherence rates for different cluster sizes.

n	26	41	71	116	189	309	477	650
$M_2, 10^9\text{s}^{-2}$	1.50	1.65	1.60	1.60	1.65	1.60	1.60	1.55
p	0.27	0.28	0.33	0.33	0.32	0.32	0.32	0.32

For a large cluster size we obtain the expression for the normalized signal,

$$S_M(t) = p \exp(-M^2 \alpha t^2) + (1 - p) \exp\left(-\frac{n}{2} \alpha t^2\right), \quad (14)$$

exact up to second order in time, where the degree of correlation p is defined as

$$p = \frac{1}{n} \left(\sum_j d_{jk} \right)^2 / \sum_j d_{jk}^2, \quad (15)$$

so that $0 \leq p \leq 1$, $\alpha = M_2/9$.

Here

$$M_2 = \frac{9}{4\hbar^2} \sum_j d_{jk}^2 \quad (16)$$

is Van Vleck's expression for the second moment.¹⁴

The two terms in (14) can be regarded as, respectively, the contributions from the correlated and uncorrelated perturbations to the spin dynamics. The interaction described by the Hamiltonian (7) can be semiclassically interpreted as a perturbing magnetic field at the site of each spin (parallel or antiparallel to the strong external magnetic field) produced by all the other spins in the cluster. The resulting spread in Larmor frequencies for different spins in the cluster causes destructive interference, or dephasing, observable by the decay of the NMR signal. The limit of a totally correlated perturbation, $p = 1$, corresponds to the case $d_{jk} \equiv \text{const}$ leading to the same perturbing field for each spin in the cluster. In contrast, the case of absolutely random coefficients, $\langle d_{jk} \rangle_j = 0$, gives $p = 0$ and corresponds to fully uncorrelated dynamics.

3. COMPARISON WITH EXPERIMENT

We used recent experimental results¹¹ for the evaluation of the degree of correlation parameter, p , for spin clusters in adamantane samples. In Figure 1, we plot curves of decay rates of various coherence orders for different cluster sizes fitted to the experimental data. The decoherence rate was defined as the inverse of the $1/e$ decay time. The degree of correlation, p , extracted from the experimental data for different coherence orders, is presented in Table 1. As follows from formula (15), in the limit of large cluster size this parameter is determined by geometrical configurations and does not depend on the size, n . The remaining fluctuations around the average value, $\bar{p} = 0.33$, can be attributed to experimental noise and to corrections for small n .

The total magnetic resonance signal from the cluster, $S(t)$, can be obtained by summation over all the contributions from the different coherence orders $S_M(t)$, according to (14),

$$S(t) = \frac{p}{\sqrt{n \alpha t^2 + 1}} + (1 - p) \exp\left(-\frac{n}{2} \alpha t^2\right). \quad (17)$$

Taking the average values of $\bar{p} = 0.33$ and $\overline{M_2} = 1.8 \cdot 10^9 \text{s}^{-2}$, obtained previously, it is possible to predict the temporal dependence of the NMR signal resulting from highly correlated spin clusters. The results shown in Figure 2 are in good agreement with experiment. As can be seen from Figure 2, the formula (17) describes the

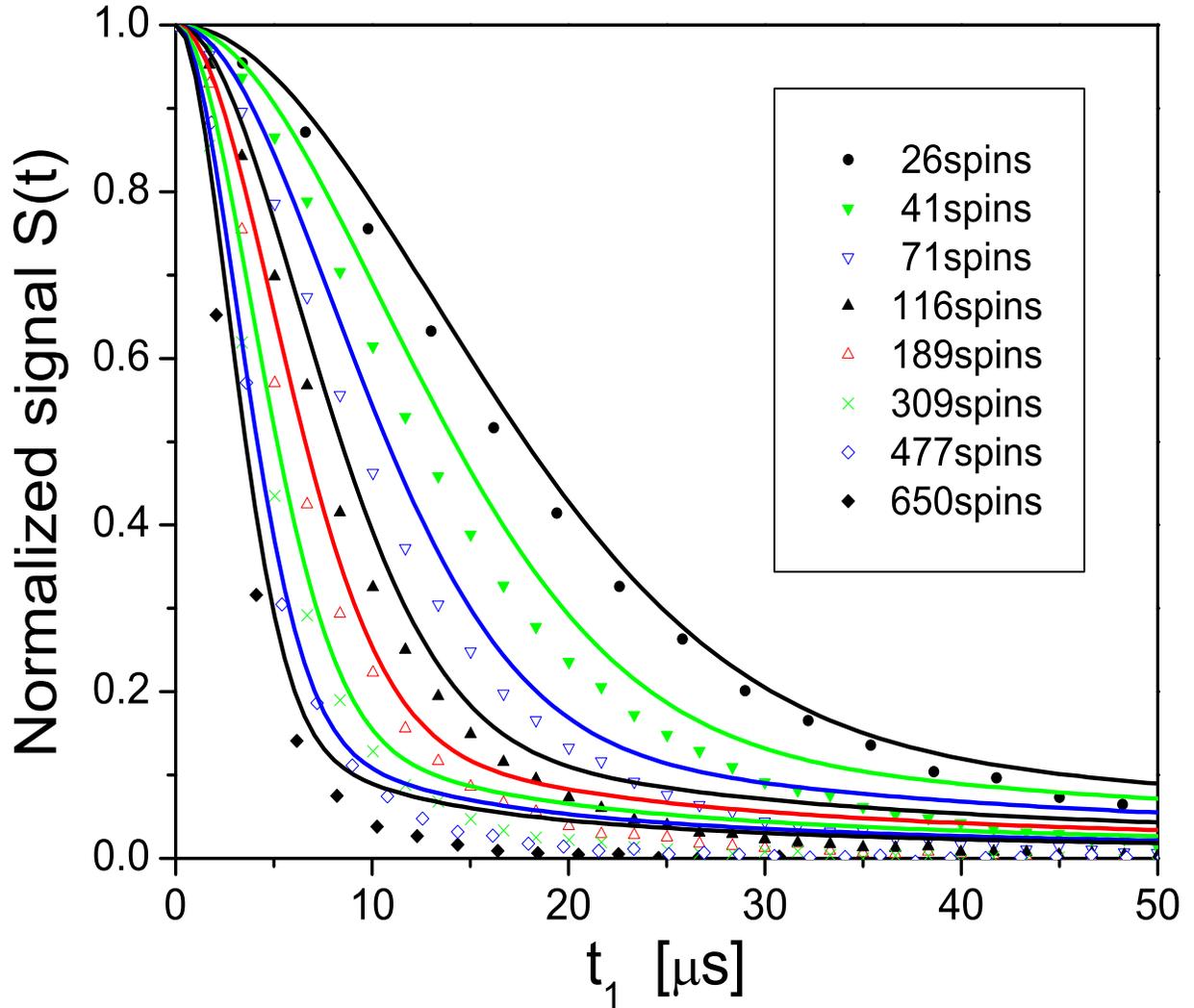


Figure 2. Decoherence rate as a function of coherence order for different spin cluster sizes. The points represent experimental values.¹¹ The solid lines are theoretical curves given by the formula (17).

initial fast drop of coherence with reasonable accuracy. A small divergence at large times between the formula (17), which is exact up to the second order in time, and the experimental results, can be attributed to the higher order terms.

To analyze the influence of the degree of correlation on spin dynamics we used formula (17). Figure 3 shows the decay of the NMR signal for the spin cluster size of intermediate value, $n = 116$, and three representative examples of the degree of correlation, p . One can see that initially all three curves decay equally. However, at later times the signal from the spin cluster subject to correlated perturbation exhibits slower decay compared

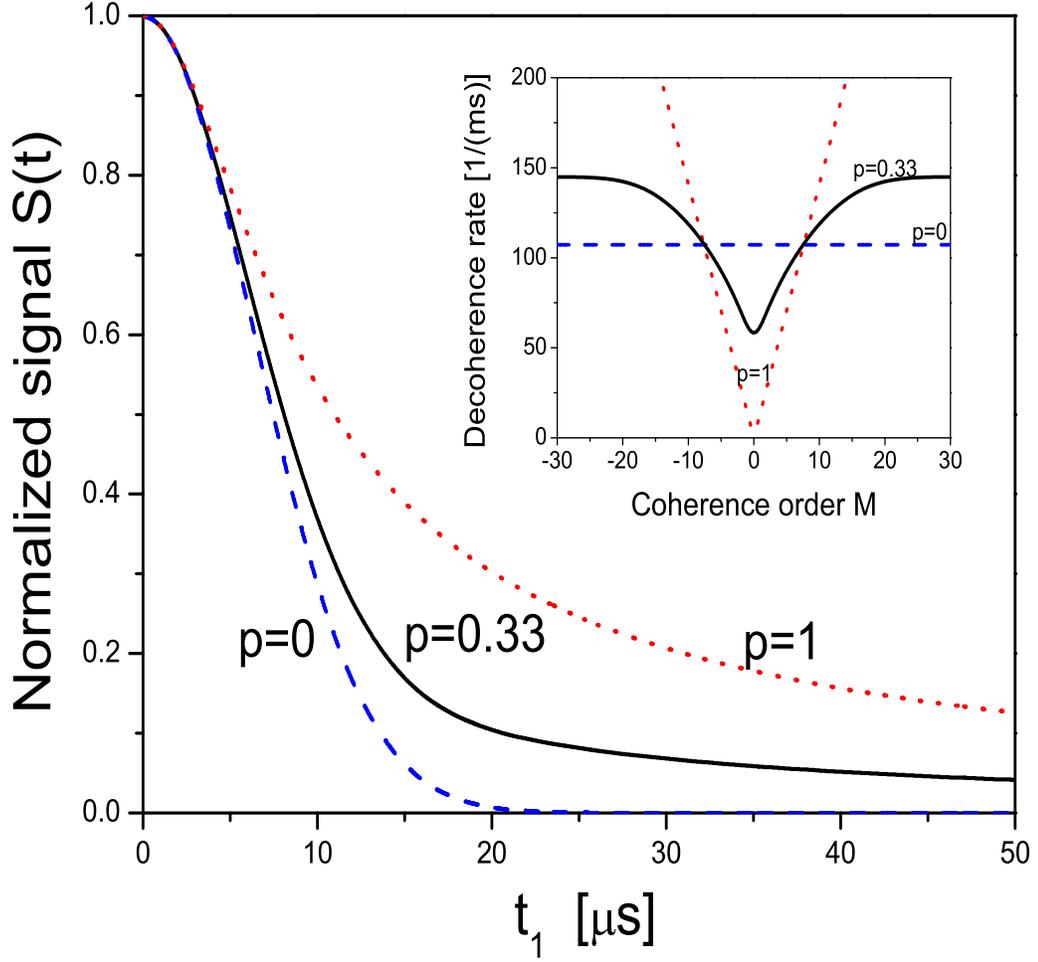


Figure 3. Dependence of the signal from a highly correlated spin cluster with size $n = 116$ and three values of degree of correlation p , i.e., $p = 0$ (dashed line, uncorrelated perturbation), $p = 0.33$ (solid line, partial correlation corresponding to the experimental situation), and $p = 1$ (dotted line, correlated perturbation). The inset shows the decoherence rate as a function of the coherence order, M .

to the uncorrelated perturbation. This result follows from the behavior of the decoherence rate as function of the coherence order, M . As can be seen from the inset in Figure 3, for uncorrelated perturbation all coherence orders decay with the same, comparatively high, rate $\sqrt{n\alpha}/2$. In contrast, the decay rate for the correlated spin dynamics increases linearly with the absolute value of M , as $|M|\sqrt{\alpha}$. For the most probable configurations, which, according to (1), are those with $M \approx 0$, the decay rate for the correlated perturbation is actually less than

that for the uncorrelated perturbation. The fact that correlated environment is less disruptive for the $M \approx 0$ states is not surprising. In particular, quantum computing error avoiding schemes based on decoherence free subspaces^{5,15} are based on a related property.

4. SCALING WITH THE NUMBER OF QUBITS

For implementation of large-scale quantum computation, scaling of the decoherence rate with the number of qubits is important. From the expression (17) it transpires that decoherence rate of a spin cluster, defined as inverse $1/e$ decay time, *always* increases as $\propto \sqrt{n}$ with the number of spins, n , although the corresponding prefactor depends on the degree of correlation, p . The square root of n scaling was indeed experimentally established recently by Krojanski and Suter.¹¹

For quantum information processing applications, it is also important to evaluate the error of a quantum computer, represented by a cluster of highly correlated spins, induced by the dipole-dipole interactions between the spins. The error is defined as the deviation, $\delta_n = 1 - S(t_g)$, of the NMR signal from its initial value, due to decoherence processes, during the time required for elementary gate operations, t_g . In order to enable successful implementation of quantum error correction schemes, one needs to maintain this error below a small threshold in order to allow fault-tolerant quantum error correction.¹ Taking the smallness of the parameter δ_n into account, one can utilize (17) to obtain

$$\delta_n \propto nt^2. \quad (18)$$

This shows that if the error is small, it scales linearly with the number of spins independently of the degree of correlation. This linear scaling of the error for short times, agrees with theoretical results for bosonic models of environment^{7,8} and suggests that the worst case scenario of “superdecoherence”⁴ is not realized for such systems.

5. CONCLUSIONS

In summary, we have surveyed a theory of coherence decay for entangled spin clusters due to internal dipole-dipole interactions. The dynamics is phenomenologically modeled by decoherence due to interaction with a composite bath consisting of fully correlated and uncorrelated parts. Perturbation due to the correlated terms leads to slower decay of coherence at larger times than that due to uncorrelated terms. The decoherence rate scales up as a square root of the number of spins. The results obtained can be useful in analysis of decoherence effects in spin-based quantum computers.

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