

SIMULATION OF DECOHERENCE IN ONE-QUBIT SYSTEMS

Piotr Marecki

*Wyższa Szkoła Informatyki i Zarządzania
Ul. Legionów 81, 43-300 Bielsko-Biala, Poland*

Abstract

The paper addresses the problem of decay of quantum information. On the basis of a realistic model of the decay, which has been developed by the author in [4], numerical simulations are performed. These investigations give a detailed picture of the dynamics of the process in which the quantum information decays and a photon is emitted.

1. Introduction

One of the main difficulties in implementing quantum algorithms in real, physical systems is that the quantum information can reliably be stored only for a limited period of time. In other words the quantum information is subjected to a process, called decoherence, which disturbs it. The effects of this process accumulate in time and eventually render the quantum computing unreliable. The cause of decoherence is known; it always results from an interaction of the qubit with an environment.

This paper deals with decoherence caused by an interaction with quantum radiation field. Although in many cases qubits have environments which interact with them stronger than the quantum radiation field¹, and consequently lead to much shorter decoherence times, the interaction with quantum radiation field is fundamental in two ways:

- It is impossible² to eliminate the presence of this environment.
- It is very difficult to control this environment.

With regard to the second issue we note, in advance, that the vacuum fluctuations of the quantum radiation field are the cause of decoherence. Only recently, it became possible to

¹ By quantum radiation field we mean the quantum, electromagnetic field of photons.

control these fluctuations in experiments [1], although the ways of such a control are limited [2].

In the literature, the process of decoherence is usually presented as something spontaneous, which influences the quantum systems randomly with some assigned probability. This view is incorrect and reflects a confusion, which is also present in the physics literature. Our paper intends to clarify this issue. Our aim is to argue that the process of decoherence can be characterized completely, and the state of the full system (the quantum field + one qubit) is precisely known at each instant of time.

2. Model of decoherence

There are many experimental possibilities for implementing quantum computations [3]. The qubit is encoded either: in the energy levels of spatial motion³ or in some other degree of freedom⁴. In all cases the qubit is associated with some physical particle, whose properties (eg. the spin) are utilized in computations. This particle (eg. the nucleus of an atom in a molecule in NMR), has the possibility to move in space, and the motion too is governed by quantum mechanics. The state of the spatial motion of the particle is prescribed by a “wave-function” of the particle, $\mathbf{y}(\mathbf{x})$. According to quantum mechanics, the squared absolute value of $\mathbf{y}(\mathbf{x})$ describes the probability density of finding the

particle at a given point \mathbf{x} .

In what follows we shall consider a situation, where the quantum information is encoded in the energy levels of spatial motion. The basis vector $|0\rangle$ will correspond to the ground state of spatial motion⁵ with the wave-function $\mathbf{y}_0(\mathbf{x})$, while the basis vector $|1\rangle$ will correspond to an excited state with the wave-function $\mathbf{y}_1(\mathbf{x})$.

² This is a consequence of a theorem, not just an experimental fact.

³ This is the case in the implementation, which utilizes trapped ions.

⁴ For instance, the NMR implementation uses two positions of nuclear spin as the basis states for qubits.

⁵ In order to be specific we take the 1S and 2P states of the hydrogen atom. Their wave-functions are known. The energy difference of both states is 10.3eV.

The system which we will consider consists of a qubit coupled to the quantum radiation field. We shall assume that there will be only single excitations of the radiation field (one photon emitted). The state of the full system can be described by a vector:

$$S(t) = c(t)\mathbf{y}_1(\mathbf{x}) \otimes \Omega + \mathbf{y}_0(\mathbf{x}) \otimes |f(t)\rangle,$$

here Ω denotes the electromagnetic vacuum, $c(t)$ is a complex-valued function of time, and

$$|f(t)\rangle = \int d\mathbf{p} f^a(t, \mathbf{p}) a_a^*(\mathbf{p}) \Omega,$$

that is, $|f(t)\rangle$ describes a time-dependent state of one photon, with both polarizations (index \mathbf{a}) and with all possible momenta (index \mathbf{p}). The vector $S(t)$ corresponds therefore an entangled state. The first part of this state describes the state $|1\rangle$ of the qubit and to the vacuum of the radiation, while, the second part corresponds to qubit's $|0\rangle$ accompanied by the emitted photon. The functions $c(t), f^a(t, \mathbf{p})$ must be determined. Their initial values, at $t = 0$, are:

$$\begin{aligned} c(0) &= 1, \\ f^a(0, \mathbf{p}) &= 0. \end{aligned}$$

It is remarkable, that a closed integro-differential equation for the function $c(t)$ can be derived [4]:

$$\frac{dc}{dt}(t) = -e^{i\mathbf{w}t} \int_0^t ds e^{-i\mathbf{w}s} c(s) Q(t-s),$$

where \mathbf{w} denotes the energy difference of both states of the qubit, and

$$\begin{aligned} Q(t-s) &= \int \frac{d^3 p}{2|\mathbf{p}|} e^{-i\mathbf{p}(t-s)} \left(\mathbf{d}^{ij} - \frac{\mathbf{p}^i \mathbf{p}^j}{|\mathbf{p}|^2} \right) \bar{c}_i(\mathbf{p}) c_j(\mathbf{p}), \\ c_i(\mathbf{p}) &= \int d^3 x e^{-i\mathbf{p}\mathbf{x}} \bar{\mathbf{y}}_1(\mathbf{x}) \frac{\partial \mathbf{y}_0(\mathbf{x})}{\partial x^i}. \end{aligned}$$

In the above expressions \mathbf{d}^{ij} denotes the elements of the identity matrix, and the summation over the indices $i, j = 1, 2, 3$ was assumed wherever they appear twice in the formulas. The particular form of the function $Q(t-s)$ can be shown to come from the auto-correlation function⁶ of the quantum radiation field. As we see, the spatial wave-functions of both states of the qubit influence the function $Q(t-s)$. In the case of hydrogen atom we find

⁶ This function describes the fluctuations of the quantum radiation field.

$$Q(d) = \frac{24\mathbf{p}}{3} \int_0^{\infty} p dp e^{-ipd} \left[\frac{3/4 - p^2}{[(3/2)^2 + p^2]^3} \right]^2.$$

This function is continuous and differentiable up to the fifth order for all values of its argument. Moreover, it decays quickly for large d .

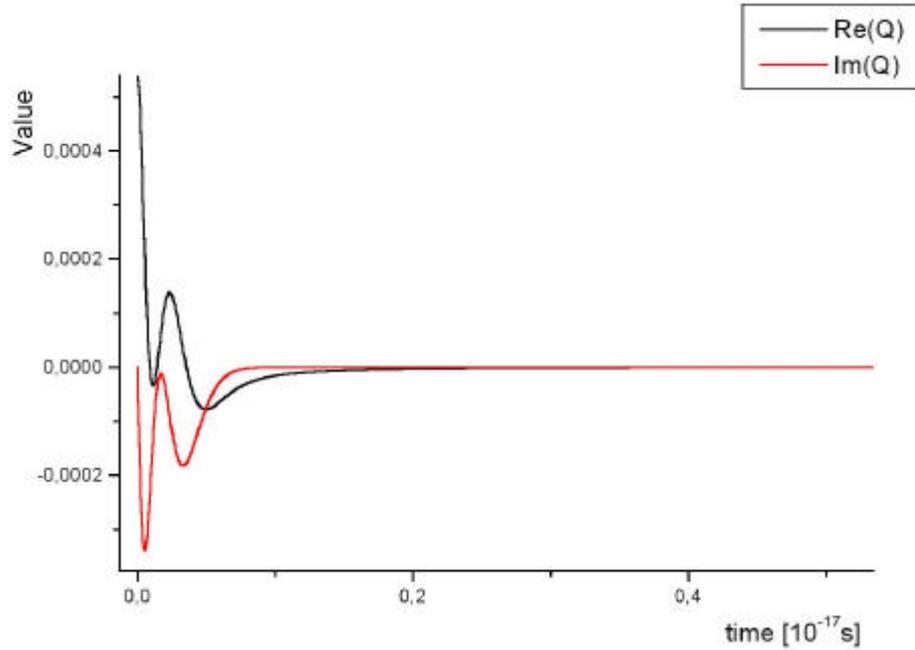


Figure 1. Behavior of the integral kernel, $Q(d)$, which enforces decoherence, in the case of 2P-1S transition of the hydrogen atom. The value of the function for larger arguments is very small.

3. Numerical simulation

The problem of decoherence has been reduced to one integro-differential equation with a smooth kernel $Q(t-s)$. The author knows no analytical solutions of this equation. A natural approach to deal with such an equation is to use numerical methods of integration. We have employed the simplest possible method⁷, namely the Euler method for differential equations accompanied by the trapezoidal rule for integration. In the course of simulation, in order to establish the change of the function $c(t)$ from t to $t+dt$ it is necessary to integrate the function $c(s)S(t-s)$ from 0 up to t . The unpleasant

⁷ This method is simultaneously the least accurate one.

consequence of this fact is, that the numerical algorithm is not efficient. Indeed, the greater the time t up to which the simulation has already been performed, the slower the evaluation of dc/dt , and consequently the slower the calculation of further data points. The results of the simulation have been presented below. As the numerical method is crude, we consider them preliminary.

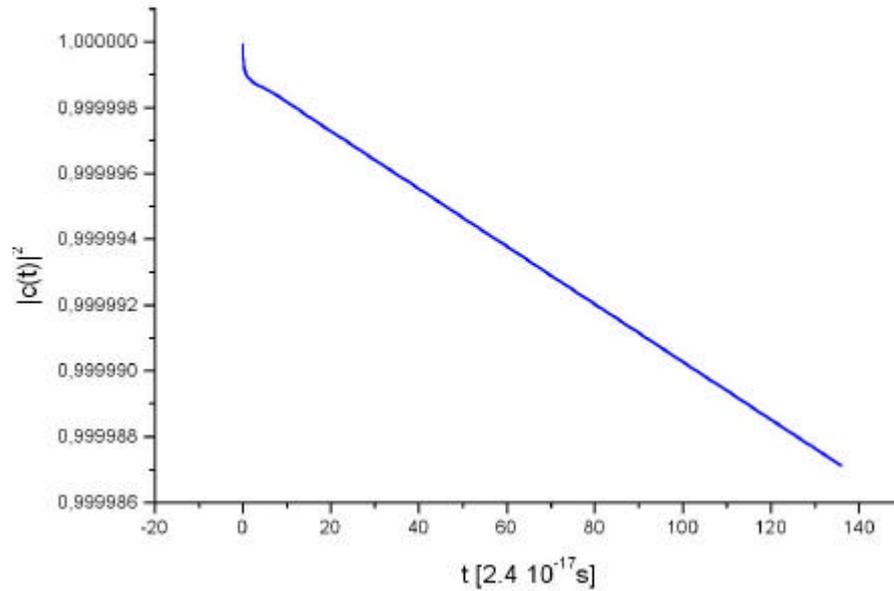


Figure 2. Numerical solution of the integro-differential equation for the case of 2P-1S transition of hydrogen. The dependence of $|c(t)|^2$ represents the decay of quantum information (the problem of decoherence).

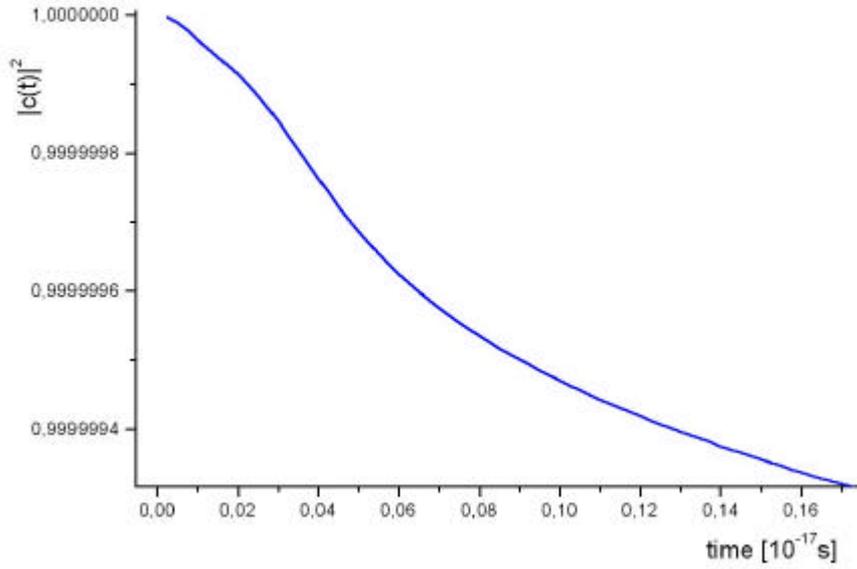


Figure 3. Numerical solution of the integro-differential equation for the case of 2P-1S transition of hydrogen. The beginning of the decay process has been zoomed (calculated with a greater accuracy).

The behavior presented above, captures the qualitative features of the process of decoherence, known from experiments. These features, in general, exhibit two phases:

- The early phase of decoherence, characterized by nontrivial behavior (such as oscillations or some quick decay, see fig. 3) of the probability, $|c(t)|^2$, for the qubit remaining in the excited state $|1\rangle$.
- The “tail” phase of decoherence, during which the probability $|c(t)|^2$ decays steadily. The characteristic shape of this decay, believed to be an exponential function, remains to be numerically determined (see the notes in the conclusions).

Appearance of these features is the source of our confidence in this model.

4. Concluding remarks and outlook

In this paper we have presented the model and the simulation of the dynamics system consisting of a qubit interacting with radiation. Our results show in what way the qubit, which is initially in the state $|1\rangle$, gradually decays into towards the state $|0\rangle$. Instead of looking at an open system consisting of a single qubit, we investigate the dynamics of a

closed system consisting of a qubit *and* a quantum electromagnetic field. Contrary to frequent assumptions the decay of quantum information is not spontaneous or random. Our results prove that the state of the full system can be reliably determined at each instant of time. However, as the state of the photon is typically unobservable, the state of the qubit appears to decay (decohere).

Simple numerical algorithms have been employed in order to solve the integro-differential equation, derived in an earlier work [3], which describes decoherence. In the simulation it was necessary to choose the time step, dt , even as small as 0,0001 in order to get a reliable precision. However, the experimentally measured decoherence times are of order of nanoseconds (which correspond to $t = 10^8$ in the numerical units). The problem of conducting the simulation for larger times, while maintaining an appropriate control of the quality of the results, remains open.

The following directions deserve in our opinion further attention as possible research areas:

- Higher order methods for the integro-differential equations could be pursued in order to expand the maximal time of the simulation. Indeed, the equation of interest can be trivially cast into the form of ordinary Volterra equation of second kind, and many higher order methods for such equations are available [5,6]. The long-time asymptotic behavior of the probability $|c(t)|^2$ is important, because it is this behavior which limits the practical quantum computations.
- In the model we have assumed the initial state of the qubit is $|1\rangle$, and that it is not influenced on purpose by some external control. However, in praxis decoherence limits quantum processors which execute some algorithms, not merely reside in the state $|1\rangle \otimes \dots \otimes |1\rangle$. At this stage we can only tell, that the decoherence process will surely be altered by a computation performed on the quantum processor. It would be interesting to see how the computation alters the decoherence time. Such a question can rather straightforwardly be addressed with the help of our methods.
- The decay process, caused by fluctuations of the quantum electromagnetic field, can be altered by means of modifying the properties of these fluctuations – as indicated in the introduction. Focussing the attention on the squeezed states [1], the fluctuations are lower (for such states) only for very short times, typically 10^{-14} s. After these times, the squeezed states exhibit vastly enhanced fluctuations.

However, the presence of a squeezed state (instead of the vacuum) in the background of a qubit seems to have greatly prolonged the coherence times, which are typically longer than 10^{-9} s. No explanation of this dilemma is known to the author. The presence of alternative background states of the quantum electromagnetic field appears to be a good method of prolonging the accessibility of quantum information.

5. Bibliography

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