

## 2.6 Invariance of the Interval

**Note.** In this section, we define a quantity called the “interval” between two events which is invariant under a change of spacetime coordinates from one inertial frame to another (analogous to “distance” in geometry). We will also derive equations for time and length dilation.

**Note.** Consider the experiment described in Figure II-8. In inertial frame  $S'$  a beam of light is emitted from the origin, travels a distance  $L$ , hits a mirror and returns to the origin. If  $\Delta t'$  is the amount of time it takes the light to return to the origin, then  $L = \Delta t'/2$  (recall that  $t'$  is multiplied by  $c$  in order to put it in geometric units). An observer in frame  $S$  sees the light follow the path of Figure II-8b in time  $\Delta t$ . Notice that the situation here is not symmetric since the laboratory observer requires two clocks (at two positions) to determine  $\Delta t$ , whereas the rocket observer only needs one clock (so the Principle of Relativity does not apply). In geometric units, we have:  $(\Delta t/2)^2 = (\Delta t'/2)^2 + (\Delta x/2)^2$  or  $(\Delta t')^2 = (\Delta t)^2 - (\Delta x)^2$  with  $\beta$  the velocity of  $S'$  relative to  $S$ , we have  $\beta = \Delta x/\Delta t$  and so  $\Delta x = \beta\Delta t$  and  $(\Delta t')^2 = (\Delta t)^2 - (\beta\Delta t)^2$  or

$$\Delta t' = \sqrt{1 - \beta^2}\Delta t. \quad (78)$$

Therefore we see that under the hypotheses of relativity, time is not absolute and the time between events depends on an observer’s motion relative to the events.

**Note.** You might be more familiar with equation (78) in the form:

$$\Delta t = \frac{1}{\sqrt{1 - \beta^2}} \Delta t'$$

where  $\Delta t'$  is an interval of time in the rocket frame and  $\Delta t$  is how the laboratory frame measures this time interval. Notice  $\Delta t \geq \Delta t'$  so that time is dilated (lengthened).

**Note.** Since  $\beta = v/c$ , for  $v \ll c$ ,  $\beta \approx 0$  and  $\Delta t' \approx \Delta t$ .

**Definition.** Suppose events  $A$  and  $B$  occur in inertial frame  $S$  at  $(t_1, x_1, y_1, z_1)$  and  $(t_2, x_2, y_2, z_2)$ , respectively, where  $y_1 = y_2$  and  $z_1 = z_2$ . Then define the *interval* (or *proper time*) between  $A$  and  $B$  as  $\Delta\tau = \sqrt{(\Delta t)^2 - (\Delta x)^2}$  where  $\Delta t = t_2 - t_1$  and  $\Delta x = x_2 - x_1$ .

**Note.** As shown above, in the  $S'$  frame

$$(\Delta t')^2 - (\Delta x')^2 = (\Delta t)^2 - (\Delta x)^2$$

(recall  $\Delta x' = 0$ ). So  $\Delta\tau$  is the same in  $S'$ . That is, the interval is invariant from  $S$  to  $S'$ . As the text says “The interval is to spacetime geometry what the distance is to Euclidean geometry.”

**Note.** We could extend the definition of interval to motion more complicated than motion along the  $x$ -axis as follows:

$$\Delta\tau = \{(\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2\}^{1/2}$$

or

$$(\text{interval})^2 = (\text{time separation})^2 - (\text{space separation})^2.$$

**Note.** Let's explore this "time dilation" in more detail. In our example, we have events  $A$  and  $B$  occurring in the  $S'$  frame at the same position ( $\Delta x' = 0$ ), but at different times. Suppose for example that events  $A$  and  $B$  are separated by one time unit in the  $S'$  frame ( $\Delta t' = 1$ ). We could then represent the ticking of a second hand on a watch which is stationary in the  $S'$  frame by these two events. An observer in the  $S$  frame then measures this  $\Delta t' = 1$  as

$$\Delta t = \frac{1}{\sqrt{1 - \beta^2}} \Delta t'.$$

That is, an observer in the  $S$  frame sees the one time unit *stretched* (dilated) to a length of  $\frac{1}{\sqrt{1 - \beta^2}} \geq 1$  time unit. So the factor  $\frac{1}{\sqrt{1 - \beta^2}}$  shows how much slower a moving clock ticks in comparison to a stationary clock. The Principle of Relativity implies that an observer in frame  $S'$  will see a clock stationary in the  $S$  frame tick slowly as well.

**However**, the Principle of Relativity does not apply in our example above (see p. 123) and both an observer in  $S$  and an observer in  $S'$  agree that  $\Delta t$  and  $\Delta t'$  are related by

$$\Delta t = \frac{1}{\sqrt{1 - \beta^2}} \Delta t'.$$

So **both** agree that  $\Delta t \geq \Delta t'$  *in this case*. This seems strange initially, but will make more sense when we explore the *interval* below. (Remember,  $\Delta x \neq 0$ .)

**Definition.** An interval in which time separation dominates and  $(\Delta\tau)^2 > 0$  is *timelike*. An interval in which space separation dominates and  $(\Delta\tau)^2 < 0$  is *spacelike*. An interval for which  $\Delta\tau = 0$  is *lightlike*.

**Note.** If it is possible for a material particle to be present at two events, then the events are separated by a timelike interval. No material object can be present at two events which are separated by a spacelike interval (the particle would have to go faster than light). If a ray of light can travel between two events then the events are separated by an interval which is lightlike. We see this in more detail when we look at spacetime diagrams (Section 2.8).

**Note.** If an observer in frame  $S'$  passes a “platform” (all the train talk is due to Einstein’s original work) of length  $L$  in frame  $S$  at a speed of  $\beta$ , then a laboratory observer on the platform sees the rocket observer pass the platform in a time  $\Delta t = L/\beta$ . As argued above, the rocket observer measures this time period as  $\Delta t' = \Delta t\sqrt{1 - \beta^2}$ . Therefore, the rocket observer sees the platform go by in time  $\Delta t'$  and so measures the length of the platform as

$$L' = \beta\Delta t' = \beta\Delta t\sqrt{1 - \beta^2} = L\sqrt{1 - \beta^2}.$$

Therefore we see that the time dilation also implies a length contraction:

$$L' = L\sqrt{1 - \beta^2}. \quad (83)$$

**Note.** Equation (83) implies that lengths are contracted when an object is moving fast relative to the observer. Notice that with  $\beta \approx 0$ ,  $L' \approx L$ .

**Example (Exercise 2.6.2).** Pions are subatomic particles which decay radioactively. At rest, they have a half-life of  $1.8 \times 10^{-8}$  sec. A pion beam is accelerated to  $\beta = 0.99$ . According to classical physics, this beam should drop to one-half its original intensity after traveling for  $(0.99)(3 \times 10^8)(1.8 \times 10^{-8}) \approx 5.3\text{m}$ . However, it is found that it drops to about one-half intensity after traveling 38m. Explain, using either time dilation or length contraction.

**Solution.** Time is not absolute and a given amount of time  $\Delta t'$  in one inertial frame (the pion's frame, say) is observed to be dilated in another inertial frame (the particle accelerator's) to  $\Delta t = \Delta t' / \sqrt{1 - \beta^2}$ . So with  $\Delta t' = 1.8 \times 10^{-8}\text{sec}$  and  $\beta = 0.99$ ,

$$\Delta t = \frac{1}{\sqrt{1 - .99^2}}(1.8 \times 10^{-8}\text{sec}) = 1.28 \times 10^{-7}\text{sec}.$$

Now with  $\beta = .99$ , the speed of the pion is  $(.99)(3 \times 10^8\text{m/sec}) = 2.97 \times 10^8\text{m/sec}$  and in the inertial frame of the accelerator the pion travels

$$(2.97 \times 10^8\text{m/sec})(1.28 \times 10^{-7}\text{sec}) = 38\text{m}.$$

In terms of length contraction, the accelerator's length of 38m is contracted to a length of

$$L' = L\sqrt{1 - \beta^2} = (38\text{m})\sqrt{1 - .99^2} = 5.3\text{m}$$

in the pion's frame. With  $v = .99c$ , the pion travels this distance in

$$\frac{5.3\text{m}}{(.99)(3 \times 10^8\text{m/sec})} = 1.8 \times 10^{-8}\text{sec.}$$

This is the half-life and therefore the pion drops to 1/2 its intensity after traveling 38m in the accelerator's frame.