

Special Topic: Black Holes

Primary Source: *A Short Course in General Relativity*, 2nd Ed., J. Foster and J.D. Nightingale, Springer-Verlag, N.Y., 1995.

Note. Suppose we have an isolated spherically symmetric mass M with radius r_B which is at rest at the origin of our coordinate system. Then we have seen that the solution to the field equations in this situation is the *Schwarzschild solution*:

$$d\tau^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2 d\phi^2 - r^2 \sin^2 \phi d\theta.$$

Notice that at $r = 2M$, the metric coefficient g_{11} is undefined. Therefore this solution is only valid for $r > 2M$. Also, the solution was derived for points outside of the mass, and so r must be greater than the radius of the mass r_B . Therefore the Schwarzschild solution is only valid for $r > \max\{2M, r_B\}$.

Definition. For a spherically symmetric mass M as above, the value $r_S = 2M$ is the *Schwarzschild radius* of the mass. If the radius of the mass is less than the Schwarzschild radius (i.e. $r_B < r_S$) then the object is called a *black hole*.

Note. In terms of “traditional” units, $r_S = 2GM/c^2$. For the Sun, $r_S = 2.95$ km and for the Earth, $r_S = 8.86$ mm.

Note. Since the coordinates (t, r, ϕ, θ) are inadequate for $r \leq r_S$, we introduce a new coordinate which will give metric coefficients which are valid for all r . In this way, we can explore what happens inside of a black hole!

Note. We keep r , ϕ , and θ but replace t with

$$v = t + r + 2M \ln \left| \frac{r}{2M} - 1 \right|. \quad (*)$$

Theorem. In terms of (v, r, ϕ, θ) , the Schwarzschild solution is

$$d\tau^2 = (1 - 2M/r)dv^2 - 2dv dr - r^2 d\phi^2 - r^2 \sin^2 \phi d\theta. \quad (**)$$

These new coordinates are the *Eddington-Finkelstein coordinates*.

Proof. Homework! (Calculate dt^2 in terms of dv and dr , then substitute into the Schwarzschild solution.)

Note. Each of the coefficients of $d\tau^2$ in Eddington-Finkelstein coordinates is defined for all nonzero $r > r_B$. Therefore we can explore what happens for $r < r_S$ in a black hole. We are particularly interested in light cones.

Note. Let's consider what happens to photons emitted at a given distance from the center of a black hole. We will ignore ϕ and θ and take $d\phi = d\theta = 0$. We want to study the radial path that photons follow (i.e. radial lightlike geodesics). Therefore we consider $d\tau = 0$.

Then (**) implies

$$\begin{aligned}(1 - 2M/r)dv^2 - 2dv dr &= 0, \\ (1 - 2M/r)\frac{dv^2}{dr^2} - 2\frac{dv}{dr} &= 0, \\ \left(\frac{dv}{dr}\right) \left((1 - 2M/r)\frac{dv}{dr} - 2\right) &= 0.\end{aligned}$$

Therefore we have a lightlike geodesic if $\frac{dv}{dr} = 0$ or if $\frac{dv}{dr} = \frac{2}{1 - 2M/r}$.

Note. First, let's consider radial lightlike geodesics for $r > r_S$. Differentiating (*) gives

$$\begin{aligned}\frac{dv}{dr} &= \frac{dt}{dr} + 1 + \frac{1}{r/2M - 1} \\ &= \frac{dt}{dr} + \frac{r/2M}{r/2M - 1} = \frac{dt}{dr} + \frac{1}{1 - 2M/r}.\end{aligned}$$

With the solution $dv/dr = 0$, we find

$$\frac{dt}{dr} = \frac{-1}{1 - 2M/r}.$$

Notice that this implies that $dt/dr < 0$ for $r > 2M$. Therefore for $dv/dr = 0$ we see that as time (t) increases, distance from the origin (r) decreases. Therefore $dv/dr = 0$ gives the *ingoing lightlike geodesics*.

With the solution $dv/dr = 2/(1 - 2M/r)$, we find

$$\frac{dt}{dr} = \frac{2}{1 - 2M/r} - \frac{1}{1 - 2M/r} = \frac{1}{1 - 2M/r}.$$

Notice that this implies that $dt/dr > 0$ for $r > 2M$. Therefore for $dv/dr = 2/(1 - 2M/r)$, we see that as time (t) increases, distance from the origin (r) increases. Therefore $dv/dr = 2/(1 - 2M/r)$ gives the *outgoing lightlike geodesics*. Therefore for $r > 2M$, a flash of light at

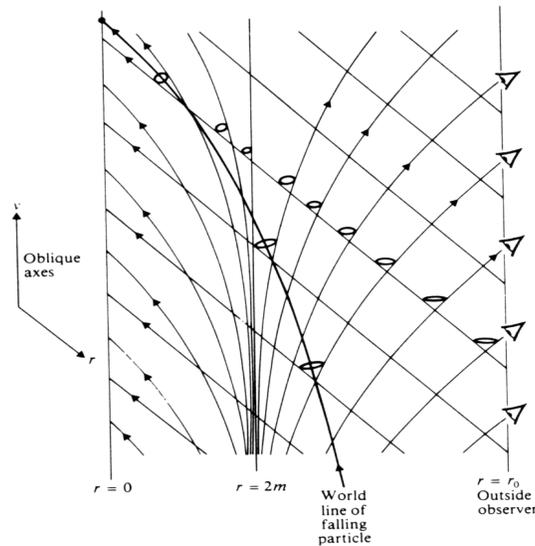
position r will result in photons that go towards the black hole and photons that go away from the black hole (remember, we are only considering radial motion).

Note. Second, let's consider radial lightlike geodesics for $r < 2M$. Integrating the solution $dv/dr = 0$ gives $v = A$ (A constant). Integrating the solution $dv/dr = 2/(1 - 2M/r)$ gives

$$v = \int \frac{dv}{dr} dr = \int \frac{2 dr}{1 - 2M/r} = \int \frac{2r dr}{r - 2M}$$

$$= 2 \int \left(1 + \frac{2M}{r - 2M} \right) dr = 2r + 4M \ln |r - 2M| + B,$$

B constant. In the following figure, we use oblique axes and choose A such that $v = A$ gives a line 45° to the horizontal (as in flat spacetime). The choice of B just corresponds to a vertical shift in the graph of $v = 2r + 4M \ln |r - 2M|$ and does not change the shape of the graph (so we can take $B = 0$).

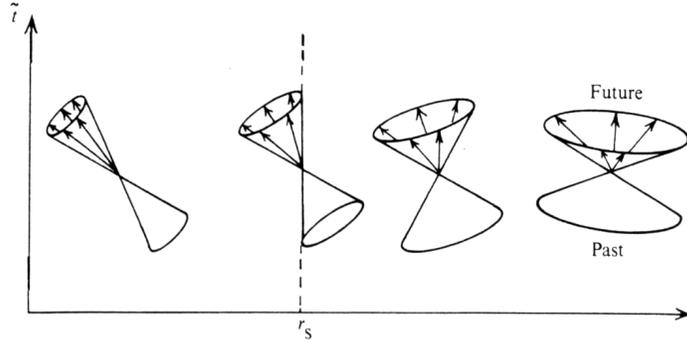


The little circles represent small local lightcones. Notice that a photon

emitted towards the center of the black hole will travel to the center of the black hole (or at least to r_B). A photon emitted away from the center of the black hole will escape the black hole if it is emitted at $r > r_S = 2M$. However, such photons are “pulled” towards $r = 0$ if they are emitted at $r < r_S$. Therefore, any light emitted at $r < r_S$ will not escape the black hole and therefore cannot be seen by an observer located at $r > r_S$. Thus the name *black* hole. Similarly, an observer outside of the black hole cannot see any events that occur in $r \leq r_S$ and the sphere $r = r_S$ is called the *event horizon* of the black hole.

Note. Notice the worldline of a particle which falls into the black hole. If it periodically releases a flash of light, then the outside observer will see the time between the flashes taking a longer and longer amount of time. There will therefore be a *gravitational redshift* of photons emitted near $r = r_S$ ($r > r_S$). Also notice that the outside observer will see the falling particle take longer and longer to reach $r = r_S$. Therefore the outside observer sees this particle fall towards $r = r_S$, but the particle appears to move slower and slower.

Note. Notice that the radial lightlike geodesics determined by $dv/dr = 2(1 - 2M/r)$ have an asymptote at $r = r_S$. This will result in lightcones tilting over towards the black hole as we approach $r = r_S$:



(Figure 46, page 93 of *Principles of Cosmology and Gravitation*, M. Berry, Cambridge University Press, 1976.) Again, far from the black hole, light cones are as they appear in flat spacetime. For $r \approx r_S$ and $r > r_S$, light cones tilt over towards the black hole, but photons can still escape the black hole. At $r = r_S$, photons are either trapped at $r = r_S$ (those emitted radially to the black hole) or are drawn into the black hole. For $r < r_S$, all worldlines are directed towards $r = 0$. Therefore *anything* inside r_S will be drawn to $r = 0$. All matter in a black hole is therefore concentrated at $r = 0$ in a *singularity* of infinite density.

Note. The Schwarzschild solution is an *exact* solution to the field equations. Such solutions are rare, and sometimes are not appreciated in their fullness when introduced. Here is a brief history from *Black Holes and Time Warps: Einstein's Outrageous Legacy* by Kip Thorne (W.W. Norton and Company, 1994):

1915 Einstein (and David Hilbert) formulate the field equations (which Einstein published in 1916).

1916 Karl Schwarzschild presents his solution which later will describe nonspinning, uncharged black holes.

1916 & 1918 Hans Reissner and Gunnar Nordström give their solutions, which later will describe nonspinning, charged black holes. (The ideas of black holes, white dwarfs, and neutron stars did not become part of astrophysics until the 1930's, so these early solutions to the field equations were not intended to address any questions involving black holes.)

1958 David Finkelstein introduces a new reference frame for the Schwarzschild solution, resolving the 1939 Oppenheimer-Snyder paradox in which an imploding star freezes at the critical (Schwarzschild) radius as seen from outside, but implodes through the critical radius as seen from outside.

1963 Roy Kerr gives his solution to the field equations.

1965 Boyer and Lindquist, Carter, and Penrose discover that Kerr's solution describes a spinning black hole.

Some other highlights include:

1967 Werner Israel proves rigorously the first piece of the black hole “no hair” conjecture: a nonspinning black hole must be precisely spherical.

1968 Brandon Carter uses the Kerr solution to show frame dragging around a spinning black hole.

1969 Roger Penrose describes how the rotational energy of a black hole can be extracted.

1972 Carter, Hawking, and Israel prove the “no hair” conjecture for spinning black holes. The implication of the no hair theorem is that a black hole is described by three parameters: mass, rotational rate, and charge.

1974 Stephan Hawking shows that it is possible to associate a temperature and entropy with a black hole. He uses quantum theory to show that black holes can radiate (the so-called *Hawking radiation*).

1993 Hulse and Taylor are awarded the Nobel Prize for an indirect detection of gravitational waves from a binary pulsar.

Early 2000s Two Laser Interferometer Gravity Wave Observatories (LIGO) are activated, one in Washington the other in Louisiana.

2011 NASA’s Gravity Probe B confirms frame dragging around the Earth.