NUI MAYNOOTH

Ollscoil na hÉireann Má Nuad

OLLSCOIL NA hÉIREANN MÁ NUAD

THE NATIONAL UNIVERSITY OF IRELAND MAYNOOTH

MATHEMATICAL PHYSICS

Year 3

Autumn Repeat Examination

2010–2011

Quantum Mechanics

MP363

Professor P. H. Damgaard, Professor D. M. Heffernan, Professor C. Nash.

Time allowed: $1\frac{1}{2}$ hours

Answer **two** questions

All questions carry equal marks

1.

Describe how the Rayleigh-Jeans formula

$$I(\nu, T) = \frac{2\nu^2}{c^2} kT$$

for a radiating black body was modified by Planck. Show that for small frequencies the Planck and Rayleigh-Jeans formulae agree.

Describe the model of the hydrogen atom proposed by Bohr. The muon μ has the same charge as an electron but a mass 207 times greater than the electron mass m. A muon is captured by a proton and forms an atom of muonic hydrogen. Find the radius of the first Bohr orbit of this atom.
Note: $e = 1.602 \times 10^{-19}\, coul.$, $h = 6.63 \times 10^{-34} J\, s$, $m = 9.109 \times 10^{-31}\, kg$, $\epsilon_0 = 8.85 \times 10^{-12}\, volt - metre/coul.$, $c = 3 \times 10^8\, m\, s^{-1}$.

2. Explain what is meant by an observable in quantum mechanics. Define the expectation $\langle A \rangle_\psi$ of an observable A in a state ψ and explain *briefly* its physical significance. Show also that $\langle A \rangle_\psi$ is always real.

A particle has Hamiltonian H and a wave function $\psi(x, t)$ whose spatial dependence is given by

$$\psi(x, 0) = \frac{1}{\sqrt{2}} \left(\phi_1(x) + \phi_2(x)\right)$$

where $\phi_1(x)$ and $\phi_2(x)$ are normalised eigenstates of H with eigenvalues E_1 and E_2 respectively.

Find $\psi(x, t)$ the wave function at time t and then show that the expectation of an observable A in the state $\psi(x, t)$ is given by

$$\langle A \rangle_\psi = \frac{1}{2} \int_{-\infty}^{\infty} \left(\overline{\phi_1(x)} A \phi_1(x) + \overline{\phi_2(x)} A \phi_2(x) \right) dx + Re \left(e^{i(E_1 - E_2)t/\hbar} \int_{-\infty}^{\infty} \overline{\phi_1(x)} A \phi_2(x) dx \right)$$

where Re denotes real part.

3. What is meant by the statement than an operator A self-adjoint? Show that self-adjoint operators have *real* eigenvalues.

A particle of mass m moves in one dimension subject to a real potential $V(x)$. Show that the momentum operator

$$p = -i\hbar \frac{d}{dx}$$

is self-adjoint. with respect to the inner product

$$\langle \psi | \phi \rangle = \int_{\infty}^{\infty} \overline{\psi}(x) \phi(x)\, dx$$

You should make appropriate assumptions about the fall off of ψ at ∞.

Show further that the Hamiltonian

$$H = \frac{p^2}{2m} + V$$

is self-adjoint.

Define the expectation $\langle A \rangle_\psi$ of an observable A in a state ψ. If $H = p^2/(2m)$ is the Hamiltonian for a free particle assume that

$$\frac{dA}{dt} = \frac{i}{\hbar}[H, A]$$

for any observable and show that

$$\frac{d}{dt}\langle mx \rangle_\psi = \langle p \rangle_\psi$$

4. Verify that the position and momentum operators x and p respectively satisfy the commutation relation

$$[x, p] = i\hbar\, I$$

calculate

$$[x^n, p]$$

where n is a positive integer.

The motion of a particle in one dimension is described by the Hamiltonian H whose normalized eigenstates $\psi_n(x),\ n = 0, 1, 2, \ldots$, satisfy the Schrodinger equation

$$H\psi_n(x) = E_n\psi_n(x)$$

with

$$\int_{-\infty\infty} \overline{\psi_n(x)}\psi_m(x)\, dx = \delta_{m,n}$$

The particle is in a state represented by the wavefunction $\Psi(x)$ where

$$\Psi(x) = \sum_{n=0}^{\infty} \left(\frac{1}{\sqrt{2}}\right)^{n+1} \psi_n(x)$$

show that $\Psi(x)$ is normalised to unity and calculate the probability that the particle has energy E_n for some given n.