NUI MAYNOOTH

Ollscoil na hÉireann Má Nuad

OLLSCOIL NA hÉIREANN MÁ NUAD
THE NATIONAL UNIVERSITY OF IRELAND MAYNOOTH

MATHEMATICAL PHYSICS

Year 3

Semester 1

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Quantum Mechanics I
MP363

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Time allowed: $1\frac{1}{2}$ hours

Answer **two** questions

All questions carry equal marks

1. (a) Bohr's model of the hydrogen atom assumes that the electron of mass m_e travels around the proton of mass m_p (which is 1836 times larger than m_e) at a nonrelativistic speed ($v \ll c$). Explain his model and show that the energies of the electron are given by

$$E_n = -\frac{\mu e^4}{8\epsilon_0^2 h^2 n^2}$$

where $n = 1, 2, \ldots$ and $\mu = m_e m_p/(m_e + m_p)$ is the reduced mass of the electron-proton system.

(b) Muonium and positronium are hydrogen-like "atoms" where an electron orbits, respectively, a positive mu lepton and a positron. Given that the mu lepton mass is 207 times larger than m_e and that the positron has the same mass as the electron, describe how the energy levels of muonium and positronium compare with those of hydrogen.

2. (a) Let $\mathcal{H}$ be a Hilbert space with inner product $\langle \,|\, \rangle$. If A is an operator acting on $\mathcal{H}$, give the definition of the *adjoint* operator $A^\dagger$ and define what is meant by a *Hermitian* operator.

(b) Show that the Hamiltonian for a one-dimesional quantum-mechanical system is a Hermitian operator, i.e. if we have a potential $V(x,t)$, then the differential operator

$$H = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x,t)$$

is a Hermitian operator with respect to the inner product

$$\langle \psi | \phi \rangle = \int_{-\infty}^{\infty} \overline{\psi(x,t)}\, \phi(x,t)\, \mathrm{d}x.$$

State *explicitly* any assumptions you make about the behaviour of the functions ψ and ϕ in this proof, and justify them.

3. Consider the one-dimensional quantum-mechanical simple harmonic oscillator, with $\{|e_n\rangle \,|n = 0, 1, \ldots\}$ being the orthonormalised energy eigenstates of energies $E_n = (n + 1/2)\hbar\omega)$. The creation and annihilation operators are defined as

$$a_\pm = P \pm im\omega X,$$

where P and X are the momentum and position operators, and they act on the eigenstates via

$$a_+ |e_n\rangle = \sqrt{2m\hbar\omega(n+1)}\, |e_{n+1}\rangle \,, \qquad a_- |e_n\rangle = \sqrt{2m\hbar\omega n}\, |e_{n-1}\rangle \,.$$

Suppose this oscillator is in the state $|\psi\rangle$ given by

$$|\psi\rangle = \frac{1}{\sqrt{6}} |e_0\rangle - \frac{1}{\sqrt{2}} |e_1\rangle + \frac{1}{\sqrt{3}} |e_2\rangle \,.$$

(a) Compute the average energy of this state, i.e. the expectation value of the Hamiltonian H.

(b) Compute the expectation values of the position operator X and the momentum operator P. (**Hint:** write these operators in terms of a_+ and a_-.)