Quantum mechanics: Problem Set 2

October 16^{th}, 2012

1. Define the inner product

$$\langle u|v\rangle$$

between two vectors u and v. Define the also the norm

$$\|v\|$$

of a vector v.

Show that it coincides with the ordinary dot product for vectors with real components in three dimensional space.

2. Prove, from the definition of $\langle u|v\rangle$, *any three* of the statements below

$$\begin{aligned}
\langle u+v|w\rangle &= \langle u|w\rangle + \langle v|w\rangle \\
\langle u|v+w\rangle &= \langle u|v\rangle + \langle u|w\rangle \\
\langle u|\lambda v\rangle &= \lambda \langle u|v\rangle \\
\langle \lambda u|v\rangle &= \bar{\lambda} \langle u|v\rangle \\
\langle v|u\rangle &= \overline{\langle u|v\rangle}
\end{aligned}$$

3. Let $S(\phi)$ denote the space of solutions to the Schrödinger equation. Prove that if

$$\phi,\, \psi \in S(\phi)$$

then $\alpha\phi + \beta\psi$ also belongs to $S(\phi)$.

4. Define the inner product on $S(\phi)$ and show that it satisfies any three of the properties quoted above in question **2.**

5.

Taking space, for simplicity, to be one dimensional, prove that the momentum operator

$$p = -i\hbar \frac{d}{dx}$$

is self-adjoint with respect to the inner product

$$\langle \psi|\phi\rangle = \int_{\infty}^{\infty} \overline{\psi}(x)\phi(x)\, dx$$

You should make make appropriate assumptions about the fall off of ψ at ∞.

6. Hence show that

$$\frac{p^2}{2m}$$

is also self-adjoint.

7. Given that $p^2/2m$ is self-adjoint prove that the Hamiltonian

$$H = \frac{p^2}{2m} + V$$

is self-adjoint (V is of course taken to be a real valued function).

8. Given that any two states ϕ and ψ satisfy The (Cauchy-Schwarz) inequality

$$|\langle\phi|\psi\rangle|^2 \leq \langle\phi|\phi\rangle \langle\psi|\psi\rangle$$

Explain how, for correctly normalised ϕ and ψ, this ensures that all probabilities lie between 0 and 1.

9. Prove that, if ψ is a wave function normalised so that

$$\langle\psi|\psi\rangle = 1$$

then this statement remains true for all time—i.e. prove that

$$\frac{\partial}{\partial t} \langle\psi|\psi\rangle = 0$$

10. Show that the position and momentum operators x and p do not commute but rather satisfy

$$[x, p] = i\hbar I$$