MP 472 Quantum Information and Computation

http://www.thphys.may.ie/staff/jvala/MP472.htm

Outline

Open quantum systems

The density operator

Quantum noise (decoherence)
• Quantum operations
• Operator sum representation

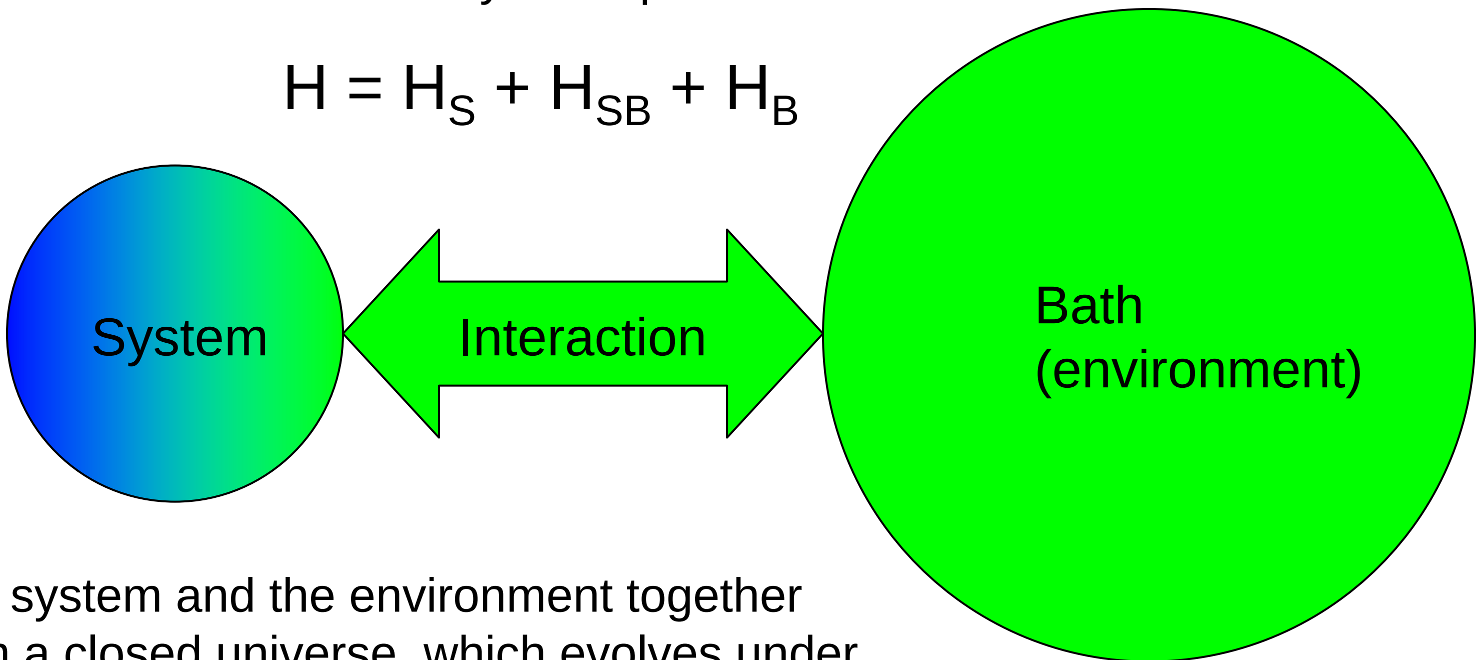
Quantum error correction

Fault-tolerant quantum computation

Open quantum systems (review)

No physical systems are closed (isolated). They are open as they interact with environment formed by other particles and fields:

$$H = H_S + H_{SB} + H_B$$



The system and the environment together form a closed universe, which evolves under unitary dynamics generated by the total Hamiltonian H

$$i\hbar \, d|\psi(t)\rangle/dt = H(t)|\psi(t)\rangle$$

The system interacting with environment however evolves as open quantum system under reduced dynamics which is NOT unitary. The effect of environment appears as noise onto the system intrinsic dynamics (generated by H_s). Quantum states of the system and of the environment interact and become entangled, they are loosing their purity and become MIXED.

Quantum operations

constitute theoretical framework for description of the evolution of quantum mechanical systems in most general circumstances:

$$\rho \longrightarrow \rho' = \mathcal{E}(\rho)$$

Examples:

A) Unitary evolution

$$\mathcal{E}(\rho) = U\rho U^{+}$$

Homework:
pure states evolve under unitary transformation as $|\psi\rangle \rightarrow U|\psi\rangle$, show that equivalently $\rho \rightarrow \mathcal{E}(\rho) = U\rho U^{+}$, for $\rho = |\psi\rangle\langle\psi|$.

B) Measurement

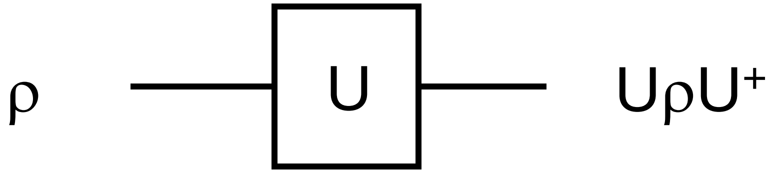
$$\mathcal{E}_m(\rho) = M_m \rho M_m^{+}$$

Homework:
A quantum measurement with outcomes labelled by m is described by the set of measurement operators M_m s.t. $\Sigma_m M_m^{+} M_m = I$. Let the state before the measurement be ρ,

- show that for $\mathcal{E}_m(\rho) = M_m \rho M_m^{+}$, the state after the measurement is $\mathcal{E}_m(\rho)/\mathrm{tr}(\mathcal{E}_m(\rho))$;
- show that the probability of obtaining the result m is $p(m) = \mathrm{tr}(\mathcal{E}_m(\rho))$.

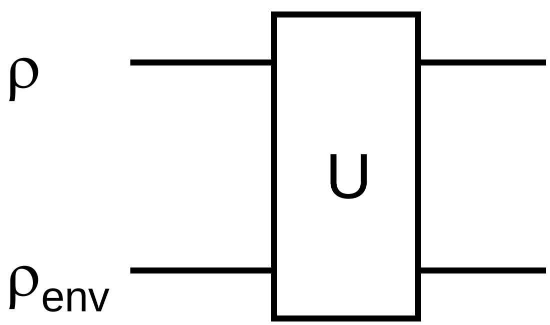
Environment and quantum operations

The dynamics of closed quantum systems is unitary



The dynamics of open quantum systems is not unitary in general:

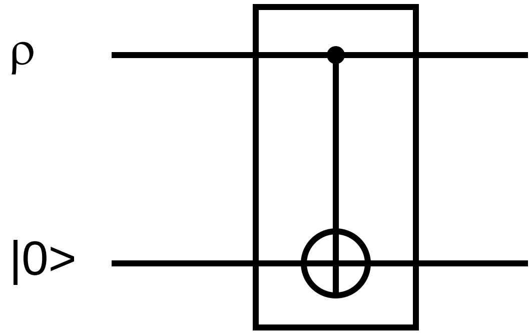
Assume the system-environment initial state is a product state $\rho \otimes \rho_{env}$. This composite system (universe) then evolves under the unitary operation U applied for certain duration of time. After then the system no longer interacts with the environment, so we perform partial trace over the environment to obtain the final state, the reduced density matrix, of the system alone:



$$\mathcal{E}(\rho) = \mathrm{tr}_{env}\left[U(\rho \otimes \rho_{env})U^{+}\right]$$

Environment and quantum operations: example

Assume the system is one qubit, the environment is one qubit in the initial state |0>, and the unitary operation is controlled-NOT operation with the system qubit as the control qubit:



$\mathcal{E}(\rho) = \mathrm{tr}_{env}\,[U_{CNOT}\,(\rho \otimes |0\rangle\langle 0|)U_{CNOT}^{+}]$

$$\mathcal{E}(\rho) = \mathrm{tr}_{env}\,[U_{CNOT}\,(\rho \otimes |0\rangle\langle 0|)U_{CNOT}^{+}] =$$

$$= \mathrm{tr}_{env}\,[(P_0 \otimes I + P_1 \otimes X)(\rho \otimes |0\rangle\langle 0|)(P_0 \otimes I + P_1 \otimes X)] =$$

$$= \mathrm{tr}_{env}\,[(P_0 \otimes I)\,(\rho \otimes |0\rangle\langle 0|)(P_0 \otimes I)\; + (P_0 \otimes I)\,(\rho \otimes |0\rangle\langle 0|)(P_1 \otimes X)$$

$$+ (P_1 \otimes X)\,(\rho \otimes |0\rangle\langle 0|)(P_0 \otimes I) + (P_1 \otimes X)\,(\rho \otimes |0\rangle\langle 0|)(P_1 \otimes X)] =$$

$$= \mathrm{tr}_{env}\,[P_0 \rho P_0 \otimes |0\rangle\langle 0| + P_0 \rho P_1 \otimes |0\rangle\langle 0|X + P_1 \rho P_0 \otimes X|0\rangle\langle 0| + P_1 \rho P_1 \otimes X|0\rangle\langle 0|X] =$$

$$= \mathrm{tr}_{env}\,[P_0 \rho P_0 \otimes |0\rangle\langle 0| + P_0 \rho P_1 \otimes |0\rangle\langle 1| + P_1 \rho P_0 \otimes |1\rangle\langle 0| + P_1 \rho P_1 \otimes |1\rangle\langle 1|] =$$

$$= P_0 \rho P_0 \otimes \langle 0|0\rangle + P_0 \rho P_1 \otimes \langle 1|0\rangle + P_1 \rho P_0 \otimes \langle 0|1\rangle + P_1 \rho P_1 \otimes \langle 1|1\rangle =$$

$$= P_0 \rho P_0 + P_1 \rho P_1$$

Operator sum representation (OSR)

is a representation of quantum operations in terms of the operators on the (principal) system only (important!):

Let $|e_k\rangle$ be the orthonormal basis for the (finite dimensional) Hilbert space of the environment, and let $\rho = |e_0\rangle\langle e_0|$ be the initial (pure) state of the environment. Then we can express quantum operations as

$$\mathcal{E}(\rho) = \Sigma_k \langle e_k|U(\rho \otimes |e_0\rangle\langle e_0|)U^+|e_k\rangle = \Sigma_k E_k \rho E_k^+$$

where $E_k = \langle e_k|U|e_0\rangle$ is an operator on the Hilbert space of the (principal) system; It is called an operation element of quantum operation.

The operation elements satisfy an important constraint known as the completeness relation

$$\Sigma_k E_k^+ E_k = I$$

The completeness relation is satisfied by quantum operations $\mathcal{E}(\rho)$ which are trace-preserving.

In general, there are non-trace-preserving operations for which $\Sigma_k E_k^+ E_k \leqslant I$ but they describe processes in which extra information about what occurred in the process is obtained by measurement.

Physical interpretation of OSR

Imagine that a measurement of the environment is performed in the basis $|e_k>$ after the unitary operation U has been applied. By the principle of implicit measurement (Lecture 7), such a measurement affects only the state of the environment.
Let ρ_k be the state of the principal system given that outcome k occurs, so

$$\rho_k \propto tr_E(|e_k><e_k|U(\rho \otimes |e_0><e_0|)U^+|e_k><e_k| = <e_k|U(\rho \otimes |e_0><e_0|)U^+|e_k> = E_k\rho E_k^+$$

Normalizing ρ_k

$$\rho_k = E_k\rho E_k^+/tr(E_k\rho E_k^+)$$

The probability of outcome k is given by

$$p(k) = tr(|e_k><e_k|U(\rho \otimes |e_0><e_0|)U^+|e_k><e_k|) = tr(E_k\rho E_k^+)$$

Thus

$$\mathcal{E}(\rho) = \Sigma_k\ p(k)\ \rho_k = \Sigma_k E_k\rho E_k^+$$

The action of the quantum operation is equivalent to taking the state ρ and randomly replacing it by

$$\rho_k = E_k\rho E_k^+/tr(E_k\rho E_k^+)$$

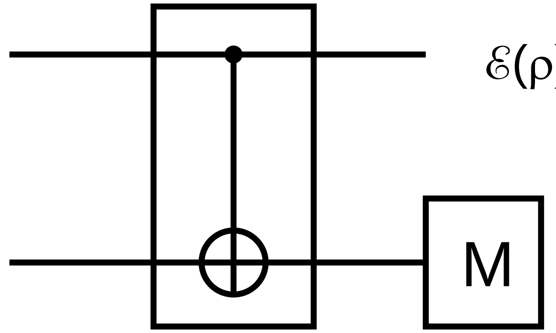
with probability

$$p_k = tr(E_k\rho E_k^+)$$

Physical interpretation of OSR: example

principal system (S) ρ

environment (E) $|0>$



$$\mathcal{E}(\rho) = tr_{env}[U_{CNOT}(\rho \otimes |0><0|)U_{CNOT}^{+}]$$

Suppose the states $|e_k>$ are chosen as $|0_E>$ and $|1_E>$.
Measurement in the computational basis of the environment qubit does not change the state of the principal system:

$$U_{CNOT} = |0_S 0_E><0_S 0_E| + |0_S 1_E><0_S 1_E| + |1_S 1_E><1_S 0_E| + |1_S 0_E><1_S 1_E|$$

Thus

$$E_0 = <0_E|U_{CNOT}|0_E> = |0_S><0_S|$$
$$E_1 = <1_E|U_{CNOT}|1_E> = |1_S><1_S|$$

and therefore

$$\mathcal{E}(\rho) = P_0 \rho P_0 + P_1 \rho P_1$$

This result is in agreement with the result of our previous example (see slide 5).