MP 472 Quantum Information and Computation

http://www.thphys.may.ie/staff/jvala/MP472.htm

Outline

Open quantum systems

The density operator

Quantum noise (decoherence)
- Quantum operations
- Operator sum representation

Quantum error correction

Fault-tolerant quantum computation

Operator sum representation (review)

is a representation of quantum operations in terms of the operators on the (principal) system only (important!):

Let $|e_k\rangle$ be the orthonormal basis for the (finite dimensional) Hilbert space of the environment, and let $\rho = |e_0\rangle\langle e_0|$ be the initial (pure) state of the environment. Then we can express quantum operations as

$$\mathcal{E}(\rho) = \sum_k \langle e_k|U(\rho \otimes |e_0\rangle\langle e_0|)U^+|e_k\rangle = \sum_k E_k \rho E_k^+$$

where $E_k = \langle e_k|U|e_0\rangle$ is an operator on the Hilbert space of the (principal) system; It is called an operation element of quantum operation.

The operation elements satisfy an important constraint known as the completeness relation

$$\sum_k E_k^+ E_k = I$$

The completeness relation is satisfied by quantum operations $\mathcal{E}(\rho)$ which are trace-preserving.

In general, there are non-trace-preserving operations for which $\sum_k E_k^+ E_k \leqslant I$ but they describe processes in which extra information about what occurred in the process is obtained by measurement.

Physical interpretation of OSR (review)

Imagine that a measurement of the environment is performed in the basis $|e_k\rangle$ after the unitary operation U has been applied. By the principle of implicite measurement (Notes 7), such a measurement affects only the state of the environment.
Let ρ_k be the state of the principal system given that outcome k occurs, so

$$\rho_k \propto \mathrm{tr}_E(|e_k\rangle\langle e_k|U(\rho\otimes|e_0\rangle\langle e_0|)U^+|e_k\rangle\langle e_k| = \langle e_k|U(\rho\otimes|e_0\rangle\langle e_0|)U^+|e_k\rangle = E_k\rho E_k^+$$

Normalizing ρ_k

$$\rho_k = E_k\rho E_k^+/\mathrm{tr}(E_k\rho E_k^+)$$

The probability of outcome k is given by

$$p(k) = \mathrm{tr}(|e_k\rangle\langle e_k|U(\rho\otimes|e_0\rangle\langle e_0|)U^+|e_k\rangle\langle e_k|) = \mathrm{tr}(E_k\rho E_k^+)$$

Thus

$$\mathcal{E}(\rho) = \Sigma_k\, p(k)\, \rho_k = \Sigma_k E_k\rho E_k^+$$

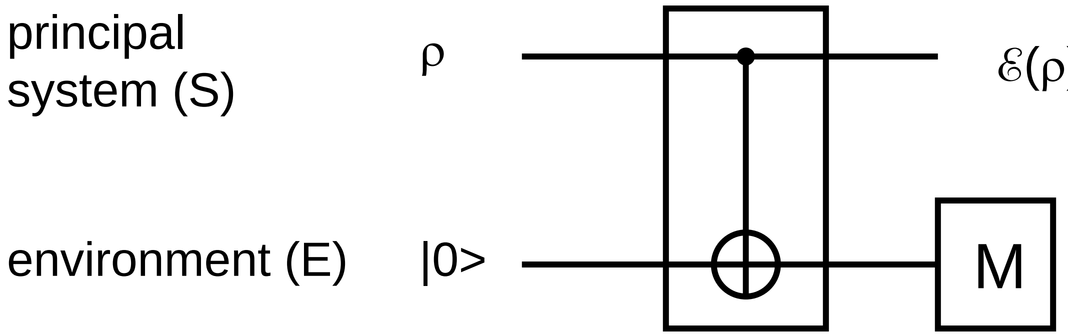
The action of the quantum operation is equivalent to taking the state ρ and randomly replacing it by

$$\rho_k = E_k\rho E_k^+/\mathrm{tr}(E_k\rho E_k^+)$$

with probability

$$p_k = \mathrm{tr}(E_k\rho E_k^+)$$

Physical interpretation of OSR: example (review)



$\mathcal{E}(\rho) = \mathrm{tr}_{env}\,[U_{CNOT}\,(\rho \otimes |0\rangle\langle 0|)U_{CNOT}^{+}]$

Suppose the states $|e_k\rangle$ are chosen as $|0_E\rangle$ and $|1_E\rangle$.
Measurement in the computational basis of the environment qubit does not change the state of the principal system:

$$U_{CNOT} = |0_S0_E\rangle\langle 0_S0_E| + |0_S1_E\rangle\langle 0_S1_E| + |1_S1_E\rangle\langle 1_S0_E| + |1_S0_E\rangle\langle 1_S1_E|$$

Thus

$$E_0 = \langle 0_E|U_{CNOT}|0_E\rangle = |0_S\rangle\langle 0_S|$$

$$E_1 = \langle 1_E|U_{CNOT}|1_E\rangle = |1_S\rangle\langle 1_S|$$

and therefore

$$\mathcal{E}(\rho) = P_0\rho P_0 + P_1\rho P_1$$

This result is in agreement with the result of our previous example (see previous notes - slide 5).

Measurements and OSR

How do we determine OSR for a given open quantum system?

A) Unitary dynamics

$$E_k = \langle e_k|U|e_0\rangle$$

B) Measurement on the combined system-environment (S-E) (here $\Sigma_k E_k^+ E_k \leqslant I$)

Let an initial (product) state $\rho = \rho_s \otimes \rho_{env}$ to evolve under unitary dynamics U and then allow (projective) measurement on S-E; the final quantum state of S-E is

$$P_m U (\rho_s \otimes \rho_{env}) U^+ P_m \;/\; tr(P_m U (\rho_s \otimes \rho_{env}) U^+ P_m)$$

The final state of S only is obtained by tracing out E:

$$tr_E (P_m U (\rho_s \otimes \rho_{env}) U^+ P_m) \;/\; tr(P_m U (\rho_s \otimes \rho_{env}) U^+ P_m)$$

Define a map

$$\mathcal{E}_m(\rho) = tr_E (P_m U (\rho_s \otimes \rho_{env}) U^+ P_m) = \qquad \rho_{env} = \Sigma_j q_j |j\rangle\langle j|$$

and using orthonormal basis $|e_k\rangle$ for the environment we obtain

$$= \Sigma_{jk}\, q_j \langle e_k|P_m U (\rho_s \otimes |j\rangle\langle j| U^+ P_m|e_k\rangle = \Sigma_{jk} E_{jk}\rho_s E_{jk}^+$$

$$E_{jk} = (q_j)^{1/2} \langle e_k|P_m U|j\rangle$$

System-environment models for any OSR

Given $\{E_k\}$, is there a reasonable model environment and dynamics to produce a quantum operation with the operational elements $\{E_k\}$?

For any quantum operation (trace-preserving or non-trace-preserving) $\mathcal{E}$, with operational elements $\{E_k\}$, there exist a model environment E, starting in a pure state $|e_0\rangle$, and model dynamics specified by a unitary operator U and projector P onto E s.t.

$$\mathcal{E}(\rho) = \mathrm{tr}_E \left(PU\left(\rho_s \otimes |e_0\rangle\langle e_0|\right)U^+P\right)$$

To show this, let's assume that $\mathcal{E}$ is a trace-preserving quantum operation, with operator sum representation generated by operation elements $\{E_k\}$, satisfying the completeness relation. In this case, we thus need to find only an appropriate unitary operator U to model the dynamics. Let $|e_k\rangle$ be an orthonormal basis set for environment (with one-to-one correspondence with the index k for E_k). Define an operator U s.t.

$$U|\psi\rangle|e_0\rangle = \Sigma_k E_k|\psi\rangle|e_0\rangle$$

where $|e_0\rangle$ is some standard state of the environment. For arbitrary states of the principal system $|\psi\rangle$ and $|\phi\rangle$, $\langle\psi|\langle e_0|U^+U|\phi\rangle|e_0\rangle = \langle\psi|\phi\rangle$ (by the completeness relation) so the operator U acts unitarily on the S-E state space, and tracing the state over E

$$\mathrm{tr}_E \left(U\left(\rho_s \otimes |e_0\rangle\langle e_0|\right)U^+\right) = \Sigma_k E_k \rho E_k^+$$

shows that this model provides a realization of the quantum operation $\mathcal{E}$ with $\{E_k\}$.

Axiomatic approach to quantum operations

Abstract but powerful!

Quantum operation $\mathcal{E}$ is defined as a map from the set of density operators of the input space S_1 to the output space S_2, with the following axioms:

(A 1)
$\mathrm{tr}[\mathcal{E}(\rho)]$ is the probability that the process represented by $\mathcal{E}$ occurs, when ρ is the initial state; thus, $0 \leqslant \mathrm{tr}[\mathcal{E}(\rho)] \leqslant 1$ for any state ρ.

(A 2)
$\mathcal{E}$ is a convex-linear map on the set of density matrices, i.e. for probabilities $\{p_i\}$,

$$\mathcal{E}\left(\Sigma_i p_i \rho_i\right) = \Sigma_i p_i \, \mathcal{E}\left(\rho_i\right).$$

(A 3)
$\mathcal{E}$ is a completely positive map. That is if $\mathcal{E}$ maps density operators of system S_1 to density operators of system S_2, then $\mathcal{E}$ (A) must be positive for any positive operator A. Furthermore, if we introduce an extra system R of arbitrary dimensionality, it must be true that $(\mathcal{I} \otimes \mathcal{E})(A)$ is positive for any positive operator A on the combined system RS_1, where $\mathcal{I}$ denotes the identity map on system R.

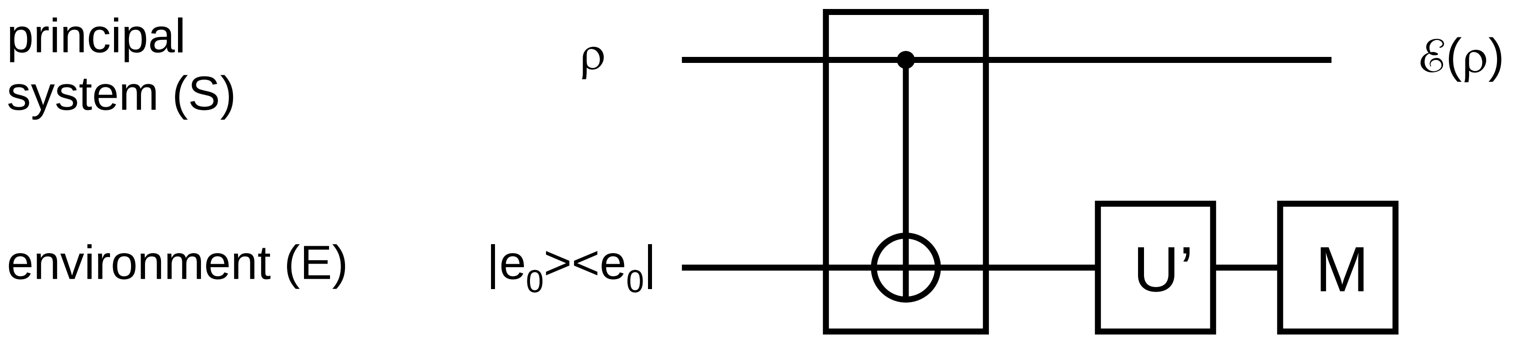
<u>Theorem</u>: The map $\mathcal{E}$ satisfies axioms A1, A2 and A3 iff

$$\mathcal{E}(\rho) = \Sigma_k E_k \rho E_k^+$$

for some set of operators $\{E_k\}$ which map the input Hilbert space to the output Hilbert space, and $\Sigma_k E_k^+ E_k \leqslant I$.

Unitary freedom in OSR

The operational elements in an operator sum representation for a quantum operation are not unique.



Theorem:

Suppose $\{E_i, .., E_m\}$ and $\{F_i, \ldots, F_n\}$ are operation elements giving rise to quantum operations $\mathcal{E}$ and $\mathcal{F}$, respectively. By appending zero operators to the shorter list of operational elements we may ensure that m=n. Then $\mathcal{E}=\mathcal{F}$ iff there exist complex numbers u_{ij} s.t. $E_i = \sum_j u_{ji} F_j$, and u is an m-by-m matrix.