

MP463

Problem Set 3

Particle in a central potential. Hydrogen atom.

1. Consider a particle in a spherically symmetric potential.
 - (i) Write down a complete set of commuting observables for the system,
 - (ii) the relevant eigenvalue equations, and
 - (iii) the corresponding eigenstates in a general form in the coordinate representation.
2. Consider an isotropic three-dimensional harmonic oscillator of mass m and angular frequency ω . The Hamiltonian operator in the Cartesian coordinate representation is given as

$$H = -\frac{\hbar^2}{2m}\nabla^2 + \frac{m\omega^2}{2}\vec{R}^2 \quad (1)$$

Using the polar coordinates, show that the Hamiltonian describes a particle moving in a central potential.

3. The radial eigenvalue equation for the isotropic three-dimensional harmonic oscillator in polar coordinates can be written as

$$\left[-\frac{\hbar^2}{2\mu} \frac{1}{r} \frac{d^2}{dr^2} r + \frac{1}{2}\mu\omega^2 r^2 + \frac{l(l+1)\hbar^2}{2\mu r^2}\right]R_{k,l}(r) = E_{k,l}R_{k,l}(r) \quad (2)$$

- (i) Explain what physical quantities the individual terms in the equation correspond to;
- (ii) assume that at the limit $r \rightarrow 0$, $R_{k,l}(r)$ is approximately Cr^s , examine the behaviour of the wavefunction at this limit, i.e. derive the expression for s in terms of l and explain what are the physically acceptable conditions it has to satisfy;
- (iii) using $R_{k,l}(r) = \frac{1}{r}u_{k,l}(r)$, $\epsilon_{k,l} = \frac{2\mu E_{k,l}}{\hbar^2}$, $\beta = \sqrt{\frac{\mu\omega}{\hbar}}$, simplify the equation above, and, based on the task (ii), explain what condition

$u_{k,l}(r)$ has to satisfy in addition to the simplified eigenvalue equation for in order to be physically meaningful.

4. The radial eigenvalue equation for the isotropic three-dimensional harmonic oscillator in polar coordinates can be simplified to the following form

$$\left[\frac{d^2}{dr^2} - \beta^4 r^2 - \frac{l(l+1)}{r^2} + \epsilon_{k,l}\right]u_{k,l}(r) = 0 \quad (3)$$

with the additional condition $u_{k,l}(0) = 0$ to ensure that the solution is square-integrable at the origin. Here $u_{k,l}(r) = rR_{k,l}(r)$, $\epsilon_{k,l} = \frac{2\mu E_{k,l}}{\hbar^2}$ and $\beta = \sqrt{\frac{\mu\omega}{\hbar}}$.

- (i) Find all solutions of this equation for the limit $r \rightarrow \infty$, and
(ii) determine which of the solution is physically acceptable and explain why.

5. In the course of solving the problem of the hydrogen atom, we study the solution of the eigenvalue equation in the form $u_{k,l}(\rho) = e^{-\rho\lambda_{k,l}}y_{k,l}(\rho)$ where $y_{k,l}(\rho)$ must satisfy

$$\left\{\frac{d^2}{d\rho^2} - 2\lambda_{k,l}\frac{d}{d\rho} + \left[\frac{2}{\rho} - \frac{l(l+1)}{\rho^2}\right]\right\}y_{k,l}(\rho) = 0 \quad (4)$$

and the condition $y_{k,l}(0) = 0$. The solution of this eigenvalue problem is sought in the form of a power series

$$y_{k,l}(\rho) = \rho^s \sum_{q=0}^{\infty} c_q \rho^q \quad (5)$$

which must be finite to be physically acceptable. It leads to the recursion relation

$$q(q+2l+1)c_q = 2[(q+l)\lambda_{k,l} - 1]c_{q-1}. \quad (6)$$

- (i) Explain how this recursion is derived,
(ii) derive the relation between s and l considering the lowest order term ρ^{s-2} in the eigenvalue problem above, and
(iii) derive the expression for $E_{k,l}$ assuming that the recursion above terminates for a finite value of q and using $\lambda_{k,l} = \sqrt{-E_{k,l}/E_I}$.

Carry out the same derivations for the isotropic three-dimensional harmonic oscillator. Find the formulas analogous to those above in your lecture notes.

6. Provide the formula for the energy levels of the hydrogen atom, sketch the low-energy levels graphically, explain what are their degeneracies and describe the relevant quantum numbers.
7. Provide the formula for the energy levels of the isotropic three-dimensional harmonic oscillator, sketch the low-energy levels graphically, explain what are their degeneracies and describe the relevant quantum numbers.
8. Consider the hydrogen atom. The ionization potential of the hydrogen atom is given as $E_I = \mu e^4 / 2\hbar^2 = 13.6 \text{ eV}$. Calculate the numerical values of the energies of the energy levels which are characterized by the principal quantum numbers $n = 1, 2$ and 3 .
9. Calculate the wavelength of the light which can be spontaneously emitted by the hydrogen atom excited to $2p$ state.
 $E_I = \mu e^4 / 2\hbar^2 = 13.6 \text{ eV}$, $1 \text{ eV} = 1.6 \times 10^{-19} \text{ J}$, the speed of light is $c = 3 \times 10^8 \text{ ms}^{-1}$, and $\hbar = h/2\pi = 1 \times 10^{-34}$.
10. Calculate how much the energy of the ground state changes if, instead of the hydrogen atom, you consider the muonium, i.e. the system consisting of a muon μ^+ and an electron e^- . The mass of the electron is $m_e = 9.1 \times 10^{-31} \text{ kg}$ and the mass of the proton is $m_p = 1.67 \times 10^{-27} \text{ kg}$. The muon is 207 times heavier than electron. Calculate the same for the positronium, i.e. the system consisting of a positron e^+ (antiparticle of electron) and an electron e^- .