

## **Aims and Objectives Quantum Physics I Session 16**

### **MEASUREMENTS OF TWO DYNAMICAL VARIABLES, COMMUTATORS AND THE UNCERTAINTY PRINCIPLE**

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#### **Aims (What I intend to do)**

- 1) To explore what happens when we try to measure two different quantities of a quantum system.
- 2) To investigate the role of commutators, to see what it means to say two operators commute and look at the significance of such commutation on the measurement of two different quantities.
- 3) To examine the relationship between commutators and the uncertainty principle.

#### **Objectives (What you should be able to do after completing the lecture and worksheet)**

- 1) To be able to describe why the order in which measurements on a quantum system are undertaken matter.
- 2) To be able to evaluate commutators and thus determine which combinations of measurements can be made simultaneously, and which combinations cannot be made simultaneously.

## Quantum Physics 1 PHY2002 Worksheet 16

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**Task 1.** Go over your lecture notes and consult sections 4.4 of Rae if you wish.

**Task 2.** In section 4.4 of Rae you will find the commutator of  $\hat{p}_x$  and  $\hat{x}$  is evaluated to be  $[\hat{p}_x, \hat{x}] = -i\hbar$ . Try following a similar procedure for  $\hat{p}_y$  and  $\hat{x}$  and thus show that  $[\hat{p}_y, \hat{x}] = 0$ .

**Task 3.** Rae section 4.5 shows how these ideas lead to,

$$\Delta A \Delta B \geq \frac{1}{2} \left| \langle [\hat{A}, \hat{B}] \rangle \right|$$

(In fact Rae uses R and Q rather than A and B.) The proof of this is beyond the scope of the module, but I have included it here for the enthusiastic. We saw in the lecture that the commutator of  $\hat{p}_x$  and  $\hat{x}$   $[\hat{p}_x, \hat{x}] = -i\hbar$  gives the familiar Heisenberg uncertainty principle when substituted into this expression.

Now, to go further we must first define what we mean by “uncertainty”: The expectation value of the variable A is

$$\langle A \rangle = \int \Psi^* \hat{A} \Psi dx$$

So the operator describing the uncertainty in A is  $\hat{A}' = (\hat{A} - \langle A \rangle)$ . Note that if  $\hat{A}$  is Hermitian (as it must be) then so is  $\hat{A}'$ . If we want an average uncertainty, then we have to take the root-mean-square, so

$$\begin{aligned} \Delta A^2 &= \int \Psi^* (\hat{A} - \langle \hat{A} \rangle)^2 \Psi dx \\ &= \int \Psi^* \hat{A}'^2 \Psi dx \end{aligned}$$

Using the Hermitian property of  $\hat{A}'$ , we have

$$\begin{aligned} \Delta A^2 &= \int \Psi^* \hat{A}'^2 \Psi dx \\ &= \int (\hat{A}' \Psi) (\hat{A}'^* \Psi^*) dx \\ &= \int |\hat{A}' \Psi|^2 dx \end{aligned}$$

We can write down a similar expression for the quantity B:

$$\Delta B^2 = \int |\hat{B}'\Psi|^2 dx$$

Note that (by inspection)  $[\hat{A}', \hat{B}'] = [\hat{A}, \hat{B}]$

We now construct the complex function

$$\phi = \hat{A}'\Psi + i\lambda\hat{B}'\Psi$$

and consider

$$I(\lambda) = \int \phi^* \phi dx \geq 0$$

(The inequality is clearly satisfied because the integrand is never negative.)

The idea will be to minimise this integral with respect to  $\lambda$ , and this will give us the required uncertainty relation.

$$\begin{aligned} I(\lambda) &= \int (\hat{A}'\Psi + i\lambda\hat{B}'\Psi)^* (\hat{A}'\Psi + i\lambda\hat{B}'\Psi) dx \\ &= \int (\hat{A}'\Psi)^* (\hat{A}'\Psi) dx + \lambda^2 \int (\hat{B}'\Psi)^* (\hat{B}'\Psi) dx \\ &\quad + i\lambda \int [(\hat{A}'\Psi)^* (\hat{B}'\Psi) - (\hat{B}'\Psi)^* (\hat{A}'\Psi)] dx \\ &= \int \Psi^* (\hat{A}'^2 + \lambda^2 \hat{B}'^2 + i\lambda [\hat{A}', \hat{B}']) \Psi dx \end{aligned}$$

where in the last line we have used the Hermitian property of both operators. Hence

$$I(\lambda) = \Delta A^2 + \lambda^2 \Delta B^2 + i\lambda \langle [\hat{A}, \hat{B}] \rangle \geq 0 \quad (*)$$

Differentiating with respect to  $\lambda$ , the minimum of this will occur when  $\lambda$  satisfies:

$$2\lambda\Delta B^2 + i \langle [\hat{A}, \hat{B}] \rangle = 0$$

So

$$\lambda = -i \frac{\langle [\hat{A}, \hat{B}] \rangle}{2\Delta B^2}$$

Substituting this in to inequality (\*),

$$\begin{aligned} \Delta A^2 - \frac{\langle [\hat{A}, \hat{B}] \rangle^2}{4\Delta B^2} + \frac{\langle [\hat{A}, \hat{B}] \rangle^2}{2\Delta B^2} &\geq 0 \\ \text{i.e. } \Delta A^2 \Delta B^2 &\geq \frac{1}{4} \langle i[\hat{A}, \hat{B}] \rangle^2 \\ \text{or } \Delta A \Delta B &\geq \frac{1}{2} \langle [\hat{A}, \hat{B}] \rangle \end{aligned}$$

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