

Aims and Objectives Quantum Physics I, Session 3

WAVEPACKETS AND THE UNCERTAINTY PRINCIPLE

Aims (What I intend to do)

- 1) To discuss how particles may be represented as wavepackets.
- 2) To look at the relationship between wavepackets and the uncertainty principle.

Objectives (What you should be able to do after completing the lecture and worksheet)

- 1) To be able to explain how a sum of waves leads to the idea of a wavepacket.
 - 2) To be able to solve problems involving the uncertainty relations and identify the connection to wavepackets.
-

Quantum Physics 1 PHY2002 Worksheet 3

Task 1. Go over your lecture notes and read relevant section in a text book if needed. You can find some material in Young and Freedman, at the end of section 39.5 (12th ed).

Task 2. Show that the free particle wave function $\Psi_f = A \exp\left(\frac{i}{\hbar}(px - Et)\right)$ gives a probability of finding the particle that is independent of x. (Remember, the wavefunction is a complex quantity).

Task 3. Show that the function,

$$\Psi(x) = \int_{-\infty}^{+\infty} \exp\left(-\frac{\alpha^2 k^2}{2}\right) \exp(ikx) dk \quad (1)$$

results in a Gaussian form for the wavefunction. A working through of this problem now follows.

We can combine the exponentials in the above equation to give,

$$\Psi(x) = \int_{-\infty}^{+\infty} \exp\left(-\frac{\alpha^2 k^2}{2} + ikx\right) dk$$

Now we play a mathematical trick: we know that if we can make the argument of the exponential a perfect square, the integral becomes a standard one for which we know (or can calculate) the answer. A perfect square has the form:

$$(ak + bx)^2 = a^2 k^2 + 2abkx + b^2 x^2$$

where we can see that:

$$a^2 = -\frac{\alpha^2}{2} \text{ and } 2ab = i$$

we can solve these two equations and show that:

$$b^2 = \frac{(2ab)^2}{4a^2} = -\frac{1}{4a^2} = \frac{1}{2\alpha^2}$$

we therefore complete the square by adding the term $\frac{x^2}{2\alpha^2}$ to the argument of the exponential (and compensating for this by multiplying by another exponential):

$$\begin{aligned} \Psi(x) &= \int_{-\infty}^{+\infty} \exp\left(-\frac{\alpha^2 k^2}{2} + ikx + \frac{x^2}{2\alpha^2}\right) \exp\left(-\frac{x^2}{2\alpha^2}\right) dk \\ &= \exp\left(-\frac{x^2}{2\alpha^2}\right) \int_{-\infty}^{+\infty} \left(\frac{i\alpha k}{\sqrt{2}} + \frac{x}{\alpha\sqrt{2}}\right) dk \\ &= \exp\left(-\frac{x^2}{2\alpha^2}\right) \int_{-\infty}^{+\infty} \left(-\frac{\alpha^2}{2} \left(k - \frac{ix}{\alpha^2}\right)^2\right) dk \end{aligned}$$

making the substitution: $k' = k - \frac{ix}{\alpha^2}$ we then obtain,

$$\Psi(x) = \exp\left(-\frac{x^2}{2\alpha^2}\right) \int_{-\infty}^{+\infty} \exp\left(-\frac{\alpha^2 k'^2}{2}\right) dk'$$

The integral is now a standard integral which has a value $\sqrt{\frac{2\pi}{\alpha^2}}$, so that:

$$\Psi(x) = \sqrt{\frac{2\pi}{\alpha^2}} \exp\left(-\frac{x^2}{2\alpha^2}\right)$$

which is of the Gaussian form required, with $\alpha^2 = L$.

Notice that the probability distribution takes the form $\exp\left(-\frac{x^2}{L^2}\right)$, so the

Gaussian has a half-width L (half-width is the distance from the peak, to the point at which the function is 1/e of its value at the peak).

Task 4. If you wish to learn more about wavepackets and how they evolve consult one of the main QM texts – most of them discuss this. For example “An introduction to Quantum Physics” by French and Taylor.

Task 5. Background reading for next session (4), Stationary states and the Schrödinger equation, see section 39.5 of Young and Freedman (12th ed).

You can find out some interesting things about the Uncertainty Principle at,

<http://www.aip.org/history/heisenberg>

Solution to task 2 of Worksheet 2.

We have that the wavelength and pulse duration are 1.06×10^{-6} m and 3×10^{-11} s respectively. The energy-time form of the uncertainty relationship is given by, $\Delta E \Delta t \geq \hbar/2$. For minimum uncertainty the equality holds and we can re-arrange this expression to give,

$$\Delta E \Delta t = \hbar/2$$

with $\Delta E = h \Delta \nu$ we find,

$$\Delta E \Delta t = h \Delta \nu \Delta t = \hbar/2$$

so that,

$$\Delta \nu = \frac{1}{4\pi \Delta t}$$

substituting the numbers gives a frequency spread of 2.7×10^9 Hz.

To find the wavelength spread we note the usual relation, $c = \nu \lambda$, re-expressing this as, $\nu = c/\lambda$ and differentiating both sides we find ,

$$|d\nu| = \frac{c}{\lambda^2} d\lambda$$

from which the wavelength spread is found to be 1.01×10^{-11} m.

Solution to task 3 of Worksheet 2.

We first find calculate the energy of each photon, and find it to be 4.8×10^{-19} J (or equivalently 3 eV). The average photon flux is given by,

$$N = \frac{I}{h\nu}$$

substituting the values this gives 6.25×10^4 photons/m².s, or 6.25 photons per sq cm per second. So during a 1s time interval the average number of flashes on any 1.00 cm² area is 6.25. Since photons are quantized, here with an energy of 3.0 eV, it is impossible to observe a fraction of a photon. In any 1 s period, one always measures an integral number of flashes, perhaps five in one interval, eight in another etc..

Solution to task 4 of Worksheet 2.

Classical particle. The particle hits one detector only, Newtonian mechanics predicts which one.

Classical wave. All detectors will respond, by an amount proportional to the intensity.

Quantum mechanically. Only one detector responds, but it is impossible in advance to predict which one. If the experiment is repeated many times, $|\Psi|^2$ gives the probability that a specific detector will respond in a particular experiment.