

## Dirac notation of photon polarization states

A horizontally polarized photon is denoted by  $|h\rangle$ , a vertically polarized photon by  $|v\rangle$ . A

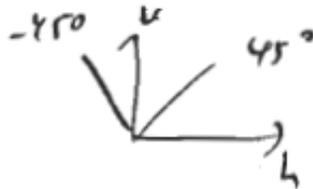


Figure I: Definition of the photon polarization bases  $|h\rangle$ ,  $|v\rangle$  and  $|+45^\circ\rangle$ ,  $|-45^\circ\rangle$ .

photon polarized along  $\pm 45^\circ$  is in the state

$$|\pm 45^\circ\rangle = \frac{1}{\sqrt{2}}(|h\rangle \pm |v\rangle). \quad (24-1)$$

Right and left circularly polarized photons are described by the states

$$|R\rangle = \frac{1}{\sqrt{2}}(|h\rangle + i|v\rangle) \quad (24-2)$$

$$|L\rangle = \frac{1}{\sqrt{2}}(|h\rangle - i|v\rangle) \quad (24-3)$$

A photon polarized at an angle  $\theta$  is denoted by

$$|\theta\rangle = \cos \theta |h\rangle + \sin \theta |v\rangle. \quad (24-4)$$

The probability amplitude of a vertically polarized photon to pass through a polarizer at angle  $\theta$  is given by the projection

$$\langle \theta | v \rangle = (\cos \theta \langle h | + \sin \theta \langle v |) \cdot |v\rangle = \sin \theta \langle v | v \rangle = \sin \theta. \quad (24-5)$$

The probability is given by  $\sin^2 \theta$  etc.

Different polarization states can be converted into each other by using material where the index of refraction depends on the polarization direction.

- $\frac{\lambda}{4}$  plate: object for which accumulated phase along one axis (the “slow” axis) is  $\frac{\pi}{2}$  less than for other (fast) axis. If the slow and fast axis are oriented at  $\pm 45^\circ$ , then  $|h\rangle \rightarrow |R\rangle$ ,  $|v\rangle \rightarrow |L\rangle$

$$|h\rangle = \frac{1}{\sqrt{2}}(|45^\circ\rangle + |-45^\circ\rangle) \rightarrow \frac{1}{\sqrt{2}}(|45^\circ\rangle + e^{i\frac{\pi}{2}}|-45^\circ\rangle) = |R\rangle \quad (24-6)$$

$$|v\rangle = \frac{1}{\sqrt{2}}(|45^\circ\rangle - |-45^\circ\rangle) \rightarrow \frac{1}{\sqrt{2}}(|45^\circ\rangle + e^{-i\frac{\pi}{2}}|-45^\circ\rangle) = |L\rangle \quad (24-7)$$

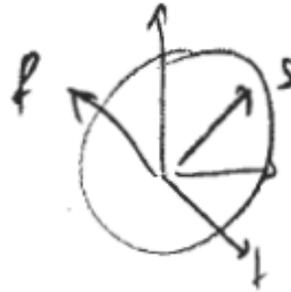


Figure II: Fast and slow axis on a  $\lambda/4$ -plate.

- $\frac{\lambda}{2}$  plate: object for which accumulated phase along slow and fast axes differs by  $\pi$ . A  $\frac{\lambda}{2}$  plate at angle  $\theta$  converts  $h$  into linearly polarized light at angle  $2\theta$ .

## The concept of uncertainty in hidden-variable theories and in the standard interpretation of QM

In the standard interpretation of **QM**, certain combinations of measurable quantities (those that do not commute with each other) are “not simultaneously predictable”, e.g., we cannot predict with certainty both the outcome of a position and of a momentum measurement. This is an intrinsic feature of the standard interpretation, and not due to some limitation in the resolution of the measurement apparatus etc. It is a feature of the state of the system, and the structure of the vectorspace of states.

In contrast, a “hidden-variables” theory would postulate that randomness arises because we do not know the complete state of the system. (There is some information about the system that is not contained in the wavefunction, but that in principle exists.) This corresponds to a randomness somewhat similar to that encountered in classical statistical mechanics. According to hidden-variables theories, the quantum mechanical state description is thus an incomplete representation of the physical reality. Randomness then arises not as an inherent feature of the physical reality, but of the **QM** description of the reality.

This is similar to classical statistical mechanics, where, not having access to or not being able to track the variables of  $10^{23}$  particles, we resort to average values. Then fluctuations around these average values naturally result from our incomplete knowledge of the microscopic dynamics.

For many decades, it was believed that hidden-variables theories cannot be distinguished in their predictions, let alone experimentally, from the standard interpretation of **QM**. The John Bell came up with a simple situation where the predictions differ, and where experiments can decide between the two views.

**Example: angular momentum and uncertainty**

Let us consider a system with  $l = 1$  in a standard state ( $m = 1$ ) The standard interpretation

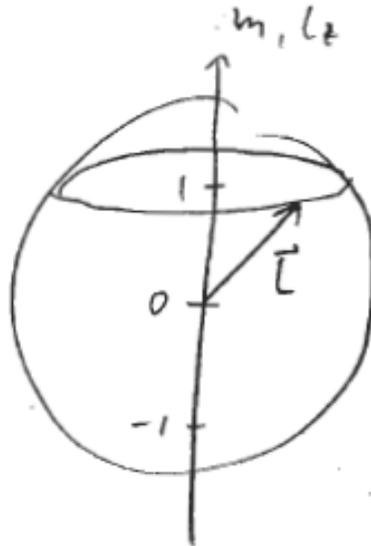


Figure III: ???

of **QM** asserts that, in this state, the components  $L_x$  and  $L_y$  (and, hence, the direction of  $\mathbf{L}$ ) are uncertain in the sense that they are “unknowable” or “unpredictable”, and that the **QM** specification of the quantum numbers of the operators that commute ( $\mathbf{L}, L_z$ ) tells us all that can be possibly known about the system:

Two systems with the same quantum numbers  $l, m$  are identical, even if a measurement of, say  $L_x$  on the two systems yields different outcomes. The uncertainty is inherent in that it is not dependent on the “exactness” of preparation of the system.

An uncertainty of a different kind appears in statistical mechanics or classically chaotic systems. A small uncertainty  $\epsilon$  in the initial preparation leads to an exponentially growing uncertainty in time, so that after a very short time the state of the system is unpredictable:

Two systems, that yield different measurement outcomes were **not** prepared identically, had we known the exact preparation parameters, we could have predicted the measurement outcome, and two sufficiently identically prepared systems would yield the same results when subjected to some measurement.

Some physicists, among them Einstein, believed that the uncertainty in **QM** should be of the same kind, mainly based on philosophical grounds:

It should be possible to assign “physical reality” to the component  $L_x$  of angular momentum that will be measured **before** the measurement is made. According to this view of “local realism”, if two systems prepared in the same state  $|l, m\rangle$  yield different outcomes where  $L_x$  is measured, it is because they were actually **not** prepared identically.

Local realistic hidden theories assume that there is a “hidden variable” not contained within **QM** that specifies the  $L_x$  component of angular momentum. If we knew the value of this hidden variable, we could predict  $L_x$ . According to this view, **QM** is incomplete in that it does not specify the state of the system completely, it is more some sort of “average theory” just like statistical mechanics. The problem is sharpened further if we consider:

### Entangled states of two particles

For two particles, **QM** allows us to prepare entangled states where the measurement on each particle yields an uncertain outcome, but the outcomes of the two measurements are correlated. As a simple example, consider the following polarization state of two photons:

$$|\text{Bell}\rangle = \frac{1}{\sqrt{2}} \left( |h\rangle_A |v\rangle_B - |v\rangle_A |h\rangle_B \right) \quad (24-8)$$

We call this state a Bell state in honor of the Irish physicist John Bell who first showed the non-equivalence of hidden-variable theories and the standard interpretation of **QM**. Here, the subscripts  $A, B$  label the two photons. A state of two particles is defined to be **entangled** (the expression was invented by Schrödinger, German: ‘Verschränkung’) if no basis exists where the state can be written as product state  $|\dots\rangle_A |\dots\rangle_B$ . The states of the two photons are correlated if person  $A$  (Alice) measures her photon to be horizontally polarized, then person  $B$  (Bob) will measure his photon to be vertically polarized, and vice versa. Based on her measurement, the outcome of which is completely uncertain, Alice can predict the outcome of Bob’s measurement. However, the state is more correlated than in a classical system where, say, a blue and a red ball are distributed between Alice and Bob, but we do not know who got which. To see this, let us write the Bell state in the  $|\pm 45^\circ\rangle$  basis:

Since,

$$|h\rangle = \frac{1}{\sqrt{2}}(|+45^\circ\rangle + |-45^\circ\rangle) = \frac{1}{\sqrt{2}}(|\nearrow\rangle + |\nwarrow\rangle) \quad (24-9)$$

$$|v\rangle = \frac{1}{\sqrt{2}}(|+45^\circ\rangle - |-45^\circ\rangle) = \frac{1}{\sqrt{2}}(|\nearrow\rangle - |\nwarrow\rangle) \quad (24-10)$$

$$|\text{Bell}\rangle = \{|h\rangle_A |v\rangle_B - |v\rangle_A |h\rangle_B\} \quad (24-11)$$

$$= \frac{1}{2\sqrt{2}} \left\{ (|\nearrow\rangle_A + |\nwarrow\rangle_A) (|\nearrow\rangle_B + |\nwarrow\rangle_B) - (|\nearrow\rangle_A - |\nwarrow\rangle_A) (|\nearrow\rangle_B + |\nwarrow\rangle_B) \right\} \quad (24-12)$$

$$= \frac{1}{2\sqrt{2}} \left\{ 2|\nwarrow\rangle_A |\nearrow\rangle_B - 2|\nearrow\rangle_A |\nwarrow\rangle_B \right\} \quad (24-13)$$

$$= \frac{1}{\sqrt{2}} \left\{ |-45^\circ\rangle_A |+45^\circ\rangle_B - |+45^\circ\rangle_A |-45^\circ\rangle_B \right\} \quad (24-14)$$

The states of the two particles are also orthogonally polarized in the  $\pm 45^\circ$  basis! In fact, one can show that they are orthogonal in any basis. This means that if Alice chooses to measure her photon in the  $\pm 45^\circ$  basis, she can predict the outcome of Bob's measurement in that basis.

*Note.* Right after Alice's measurement of a  $|h\rangle$ -polarized photon, Bob's photon's state is  $|v\rangle$ , while immediately after Alice's measurement in the  $|45^\circ\rangle$  basis, say with outcome  $|+45^\circ\rangle$ , Bob's photon's state will be  $|-45^\circ\rangle$ , even if Bob is light years away.

Although Alice cannot use this fact to transmit information faster than the speed of light (if Bob does not communicate with Alice, his probabilities are  $\frac{1}{2}$  for measurements in any basis, and he gains no information), we are reaching treacherous ground: Does Alice's measurement constitute immediate action at a distance, i.e., is **QM** non-local? After all, Bob can measure immediately after Alice (they could have synchronized their clocks initially), both can measure randomly in either the  $|h\rangle, |v\rangle$  or the  $|\pm 45^\circ\rangle$ , basis and when they compare notes later, they will find that whenever they happened to measure in the same basis, Bob had the opposite polarization from that of Alice. Local theories are very dear to physicists, and we do not like to give up the notion of locality easily.

## Example: Local vs. global conservation laws

### Global charge conservation law

"The total charge in the universe is conserved." Such conservation laws are useless for all practical purposes (cannot be falsified).

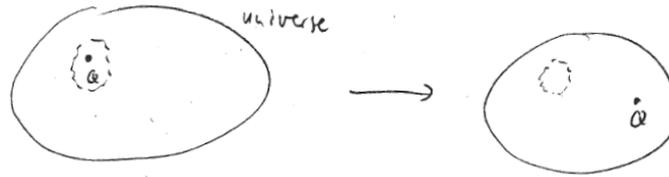


Figure IV: If a local conservation law is valid, a charge leaving a volume must pass through its bounding surface. Global conservation laws, where a charge disappears from a volume without passing through the surface, and appears elsewhere in the universe, are useless.

### Local charge conservation law

If charge disappears from a volume, it has to flow through the surface into the neighboring volume

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0 \quad \rightarrow \quad \text{differential form} \quad (24-15)$$

$$\frac{\partial}{\partial t} \int \rho dV + \int \mathbf{j} \cdot d\mathbf{A} = 0 \quad \rightarrow \quad \text{integral form} \quad (24-16)$$

of continuity equation describing local conservation of charge.



Figure V: Charge  $Q$  in a volume bounded by a surface  $A$ .

### Bell's argument and inequality

If a local hidden parameter exists that completely determines the results of the measurements of Alice and Bob, then there must exist a function  $A(\lambda, \hat{e})$  with values  $A(\lambda, \hat{e}_A) = \pm 1$  for Alice's measurements and  $B(\lambda, \hat{e}_B) = \pm 1$  for Bob's measurements. Here  $\hat{e}_A$  ( $\hat{e}_B$ ) defines the direction along which Alice (Bob) chooses to measure (sets up her (his) polarizer), and we call the value of the function  $+1$  if the photon passes through the polarizer, and  $-1$  if it does not.

The main point is that the local hidden-variables theory assumes that Bob's outcome function  $B(\lambda, \hat{e}_B)$  does not depend on the direction in which Alice set up her polarizer, or what outcome she measured. We define a correlation function  $E(\hat{e}_A, \hat{e}_B)$  as the product of

Alice's and Bob's outcome functions. For a hidden-variable theory with some unknown distribution function  $P(\lambda)$  for the variable  $\lambda$ , with  $P(\lambda) \geq 0$ ,  $\int d\lambda P(\lambda) = 1$ , we can write

$$E(\hat{e}_A, \hat{e}_B) = \int P(\lambda) A(\lambda, \hat{e}_A) B(\lambda, \hat{e}_B) \quad (24-17)$$

*Note.*  $P(\lambda)$  does not depend on the analyzer angles  $\hat{e}_A, \hat{e}_B$  chosen by Alice and Bob: the photon pair can be prepared before Alice and Bob decide how to set their polarizers.

### Bell's theorem

For a local hidden-variable theory, the quantity

$$S = E(\hat{e}_A, \hat{e}_B) + E(\hat{e}'_A, \hat{e}'_B) + E(\hat{e}_A, \hat{e}'_B) - E(\hat{e}'_A, \hat{e}_B) \quad (24-18)$$

for any choice of measurement angles  $\hat{e}_A, \hat{e}'_A, \hat{e}_B, \hat{e}'_B$  satisfies

$$|S| \leq 2. \quad (24-19)$$

This inequality can be violated by the predictions of **QM**.

*Proof.* For hidden-variables theories we define the quantity

$$S(\lambda) = A(\lambda, \hat{e}_A)B(\lambda, \hat{e}_B) + A(\lambda, \hat{e}'_A)B(\lambda, \hat{e}_B) + A(\lambda, \hat{e}_A)B(\lambda, \hat{e}'_B) - A(\lambda, \hat{e}'_A)B(\lambda, \hat{e}_B) \quad (24-20)$$

with  $S = \int d\lambda P(\lambda) S(\lambda)$ .  $S(\lambda)$  can also be written as

$$S(\lambda) = A(\lambda, \hat{e}_A)[B(\lambda, \hat{e}_B) + B(\lambda, \hat{e}'_B)] + A(\lambda, \hat{e}'_A)[B(\lambda, \hat{e}'_B) - B(\lambda, \hat{e}_B)] \quad (24-21)$$

Since Bob's outcome function is always  $B = +1$  or  $B = -1$ , either the first or the second term always vanishes. Consequently,

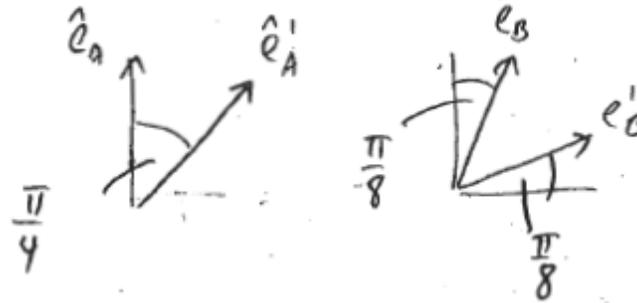
$$S = \int d\lambda P(\lambda) S(\lambda) = \int d\lambda P(\lambda) 2A(\lambda, \hat{e}'_A). \quad (24-22)$$

Since  $A$  takes on only the values  $\pm 1$ ,  $-2 \leq S \leq 2$ . Thus for hidden-variables theories,  $|S| \leq 2$ .  $\square$

For **QM**, consider the Bell state

$$|\text{Bell}\rangle = \frac{1}{\sqrt{2}} \left\{ |h\rangle_A |v\rangle_B - |v\rangle_A |h\rangle_B \right\} \quad (24-23)$$

and the polarizer angles. What is the **QM** prediction for the correlation function  $E(\hat{e}_A, \hat{e}_B)$ ? Consider Alice's measurement along  $\hat{e}_a$ :



(a) Polarizer angle for Alice. (b) Polarizer angle for Bob.

Figure VI: Polarizer angles.

If she measures her polarization along  $\hat{e}_A$  (with probability  $\frac{1}{2}$ ), corresponding to her photon being  $\nu$  polarized, and her outcome function value being  $A = +1$ , then she projects the Bell state into

$$|\text{Bell}\rangle \rightarrow -\frac{1}{\sqrt{2}}|\nu\rangle_A|h\rangle_B \quad (24-24)$$

Consequently, Bob has a horizontally polarized photon and if he chooses the measurement axis  $\hat{e}_B$ , he will find a photon along that axis ( $B = 1$ ) with a probability  $\cos^2 \frac{3\pi}{8} = \sin^2 \frac{\pi}{8}$ . If he choose  $\hat{e}_B$  instead, he will find  $B = 1$  with probability  $\cos^2 \frac{\pi}{8}$  etc. Generally, we can write for an angle  $\alpha$  of Alice's polarizer relative to the same axis, with the notation  $\alpha_\perp = \alpha + \frac{\pi}{2}$ ,  $\beta_\perp = \beta + \frac{\pi}{2}$ :

$$|h\rangle_A = \cos \alpha |\hat{e}_\alpha\rangle_A + \sin \alpha |\hat{e}_{\alpha_\perp}\rangle_A \quad (24-25)$$

$$|\nu\rangle_A = -\sin \alpha |\hat{e}_\alpha\rangle_A + \cos \alpha |\hat{e}_{\alpha_\perp}\rangle_A \quad (24-26)$$

$$|h\rangle_B = \cos \beta |\hat{e}_\beta\rangle_B + \sin \beta |\hat{e}_{\beta_\perp}\rangle_B \quad (24-27)$$

$$|\nu\rangle_B = -\sin \beta |\hat{e}_\beta\rangle_B + \cos \beta |\hat{e}_{\beta_\perp}\rangle_B \quad (24-28)$$

In terms of these polarization states, the Bell state can be rewritten as (substitute this into

definition of Bell state)  $|\text{Bell}\rangle = \frac{1}{\sqrt{2}}(|h\rangle_A|v\rangle_B - |v\rangle_A|h\rangle_B)$ :

$$\begin{aligned} |\text{Bell}\rangle &= \frac{1}{\sqrt{2}} \left\{ (-\cos\alpha \sin\beta + \sin\alpha \cos\beta) |\hat{e}_\alpha\rangle_A |\hat{e}_\beta\rangle_B \right. \\ &\quad + (-\sin\alpha \sin\beta - \cos\alpha \cos\beta) |\hat{e}_{\alpha_\perp}\rangle_A |\hat{e}_\beta\rangle_B \\ &\quad + (\cos\alpha \cos\beta + \sin\alpha \sin\beta) |\hat{e}_\alpha\rangle_A |\hat{e}_{\beta_\perp}\rangle_B \\ &\quad \left. + (\sin\alpha \cos\beta - \cos\alpha \sin\beta) |\hat{e}_{\alpha_\perp}\rangle_A |\hat{e}_{\beta_\perp}\rangle_B \right\} \end{aligned} \quad (24-29)$$

$$\begin{aligned} &= \frac{1}{\sqrt{2}} \left\{ \sin(\alpha - \beta) |\hat{e}_\alpha\rangle_A |\hat{e}_\beta\rangle_B - \cos(\alpha - \beta) |\hat{e}_{\alpha_\perp}\rangle_A |\hat{e}_\beta\rangle_B \right. \\ &\quad \left. + \cos(\alpha - \beta) |\hat{e}_\alpha\rangle_A |\hat{e}_{\beta_\perp}\rangle_B - \sin(\alpha - \beta) |\hat{e}_{\alpha_\perp}\rangle_A |\hat{e}_{\beta_\perp}\rangle_B \right\} \end{aligned} \quad (24-30)$$

Since  $\hat{e}_\alpha$  ( $\hat{e}_\beta$ ) is associated with outcome function  $A = 1$  ( $B = 1$ ) and  $\hat{e}_{\alpha_\perp}$  ( $\hat{e}_{\beta_\perp}$ ) is associated with  $A = -1$  ( $B = -1$ ). The Bell state yields for the correlation function

$$\begin{aligned} E(\hat{e}_A, \hat{e}_B) &= \frac{1}{2} \sin^2(\alpha - \beta)(+1) \cdot (+1) \\ &\quad + \frac{1}{2} \cos^2(\alpha - \beta)(-1) \cdot (+1) \\ &\quad + \frac{1}{2} \cos^2(\alpha - \beta)(+1) \cdot (-1) \\ &\quad + \frac{1}{2} \sin^2(\alpha - \beta)(-1) \cdot (-1) \\ &= \frac{1}{2} [\sin^2(\alpha - \beta) - \cos^2(\alpha - \beta)] \\ &= -\cos(2\alpha - 2\beta) \end{aligned} \quad (24-31)$$

Consequently, the Bell parameter is

$$S = E(\hat{e}_A, \hat{e}_B) + E(\hat{e}'_A, \hat{e}'_B) + E(\hat{e}'_A, \hat{e}_B) - E(\hat{e}_A, \hat{e}'_B) \quad (24-32)$$

$$= -\cos(2\alpha - 2\beta) - \cos(2\alpha' - 2\beta') - \cos(2\alpha' - 2\beta) + \cos(2\alpha - 2\beta') \quad (24-33)$$

For the suggested choice of angles, (p. XXIV-8).

$$S = -\cos\left(\pi - \frac{3\pi}{4}\right) - \cos\left(\frac{\pi}{2} - \frac{\pi}{4}\right) - \cos\left(\frac{\pi}{4} - \frac{3\pi}{4}\right) + \cos\left(\pi - \frac{\pi}{4}\right) \quad (24-34)$$

$$= -\cos\left(\frac{\pi}{4}\right) - \cos\left(\frac{\pi}{4}\right) - \cos\left(-\frac{\pi}{4}\right) + \cos\left(\frac{3\pi}{4}\right) \quad (24-35)$$

$$= -3\frac{1}{2}\sqrt{2} - \frac{1}{2}\sqrt{2} \quad (24-36)$$

$$= -2\sqrt{2} \quad (24-37)$$

$$\leq -2 \quad (24-38)$$

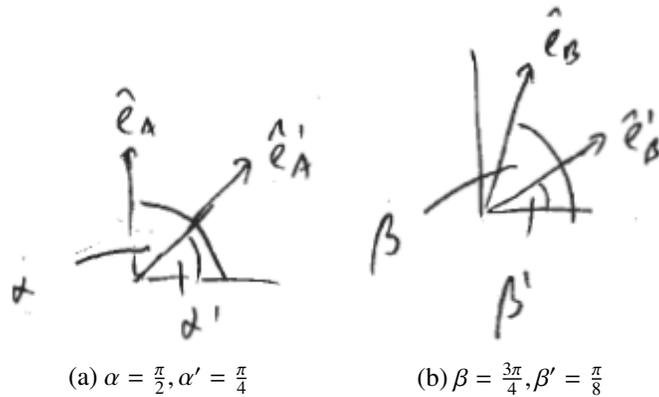


Figure VII: Polarizer angles.

So **QM** indeed violates Bell's inequality. The different predictions by hidden-variables theories ( $|S| \leq 2$  always) and **QM** ( $|S| > 2$  possible, one can however show that  $|S| \leq 2\sqrt{2}$  for **QM**) can be tested experimentally if one has a source of entangled photons. Such sources exist, e.g., a doubly excited atom that can decay to the ground state via two different pathways emits entangled photon pairs: In such a cascade the photons *A* and *B* always have

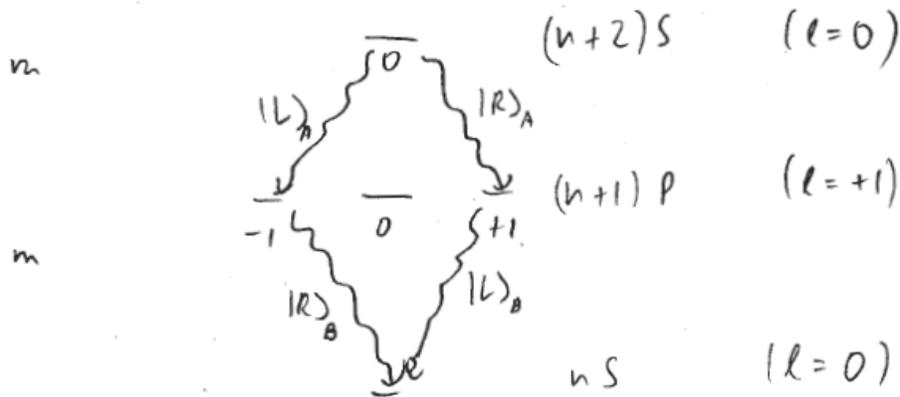


Figure VIII: A decay from a second excited state in an atom via two possible decay paths can produce entangled photons.

to have opposite polarizations, but each of them could be left or right circularly polarized. This corresponds to a Bell state such as the one we have considered. Experiments violate the hidden-variables prediction  $|S| \leq 2$ , with up to 20 standard deviations, but are in perfect agreement with **QM**.

**What does this mean?**

It means that for entangled states Alice's chosen measurement direction and outcome 'influence' Bob's measurement outcome beyond what could be possible if you assumed that once Bob has his photon, the properties of that photon are fixed. This is true even if Bob's measurement lies outside the light cone of Alice's measurement. Entanglement cannot be used for communication faster than the speed of light, but it can be used for quantum cryptography, i.e., cryptography that is protected by the laws of **QM**.