

Double slit: mathematical model of interference pattern and photon scattering

To develop some more insight into interference, and the correlations between quantum system and (classical) apparatus that lie at the heart of the quantum measurement problem, we will postulate rules on how the interference pattern is formed, and how photon scattering changes the electron's wavefunction.

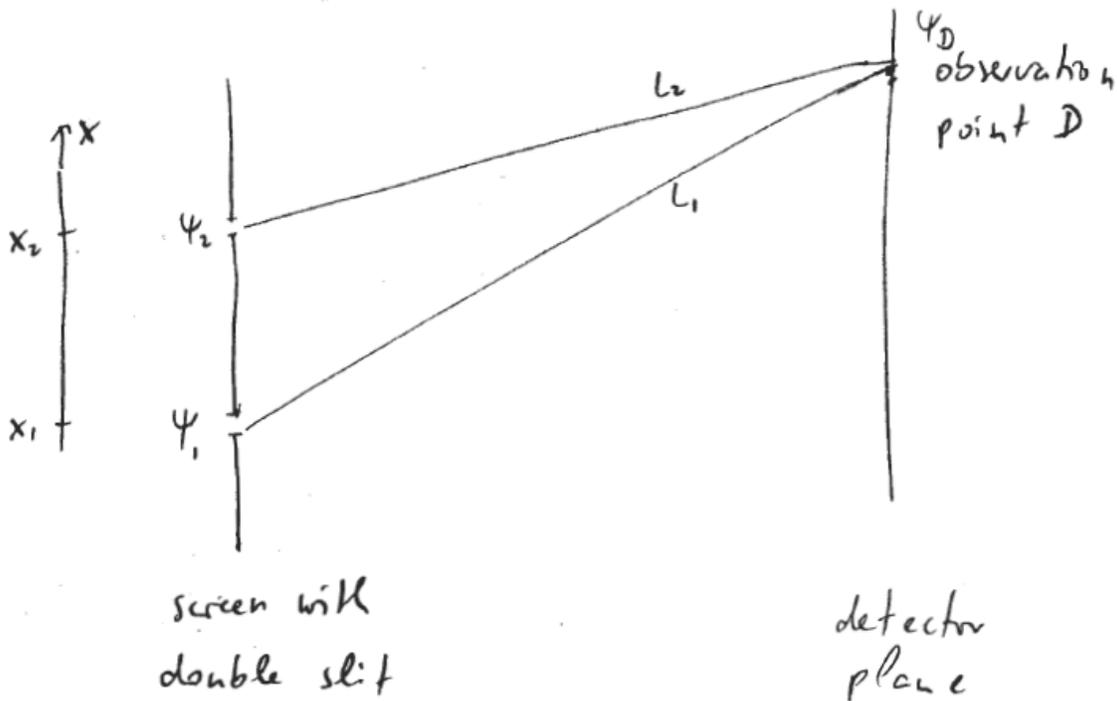


Figure I: Double-slit interference.

Rule 1. Wavefunction at slits 1, 2 is:

$$\psi_1 = Ae^{i\phi_1} \qquad \psi_2 = Ae^{i\phi_2} \qquad (7-1)$$

Complex numbers A real. For incident plane wave, and no observation:

$$\phi_1 = \phi_2 \qquad (7-2)$$

Rule 2. The wavefunction at the detection point D is given by

$$\psi_D = \psi_1 e^{i2\pi L_1/\lambda} + \psi_2 e^{i2\pi L_2/\lambda} \qquad (7-3)$$

$$\psi_D = \psi_1 e^{ikL_1} + \psi_2 e^{ikL_2} \qquad (7-4)$$

where λ is the de Broglie wavelength and $k = \frac{2\pi}{\lambda}$ the associated wave vector of the particle.

Rule 3. The intensity (and therefore particle arrival rate) at the detection point is proportional to:

$$\begin{aligned}
 |\psi_D|^2 &= |\psi_1 e^{ikL_1} + \psi_2 e^{ikL_2}|^2 \\
 &= |A|^2 |e^{i(kL_1+\phi_1)} + e^{i(kL_2+\phi_2)}|^2 \\
 &= |A|^2 \{e^{i(kL_1+\phi_1)} + e^{i(kL_2+\phi_2)}\} \cdot \{e^{-i(kL_1+\phi_1)} + e^{-i(kL_2+\phi_2)}\} \\
 &= |A|^2 \{1 + 1 + e^{i(kL_1+\phi_1-kL_2+\phi_2)} + e^{-i(kL_1+\phi_1-kL_2+\phi_2)}\} \\
 &= |A|^2 \{2 + 2 \cos(kL_1 + \phi_1 - kL_2 - \phi_2)\} \\
 &= 2|A|^2 \left\{ \underbrace{1}_{\text{background term}} + \underbrace{\cos(k(L_1 - L_2) + (\phi_1 - \phi_2))}_{\text{interference term}} \right\} \tag{7-5}
 \end{aligned}$$

Changing the relative phase $\phi_1 - \phi_2$ of the wavefunction at the two slits acts to shift the interference pattern in the detector plane. Changing $\phi_1 - \phi_2$ by π is equivalent to changing $L_1 - L_2$ by $\frac{\lambda}{2}$, i.e. to an angle change. If we average over random relative phases $\phi_1 - \phi_2$ of the wavefunctions at the two slits, the interference pattern and the interference term disappear.

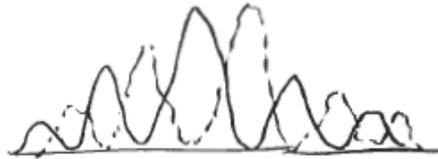


Figure II: Average over many interference patterns with random shifts.

Rule 4. If an electron at position \mathbf{x} scatters a photon with incident wavevector \mathbf{k}_{in} into a new direction characterized by a photon wavevector \mathbf{k}_{out} ($\mathbf{p}_{\text{in}} = \hbar\mathbf{k}_{\text{in}}$, $\mathbf{p}_{\text{out}} = \hbar\mathbf{k}_{\text{out}}$ incident, scattered photon momenta) its wavefunction at position \mathbf{x} acquires a phase shift $\phi_{\text{sc}} = (\mathbf{k}_{\text{in}} - \mathbf{k}_{\text{out}}) \cdot \mathbf{x} \implies \mathbf{QM}$ statement about momentum conservation

Wavefunctions at slit 1,2 before scattering:

$$\psi_1 = Ae^{i\phi_1} = A \qquad \psi_2 = Ae^{i\phi_2} = A \tag{7-6}$$

after scattering \mathbf{k}_{in} to \mathbf{k}_{out} :

$$\psi_1 = Ae^{i(\mathbf{k}_{\text{in}} - \mathbf{k}_{\text{out}}) \cdot \mathbf{x}_1} \qquad \psi_2 = Ae^{i(\mathbf{k}_{\text{in}} - \mathbf{k}_{\text{out}}) \cdot \mathbf{x}_2} \tag{7-7}$$

$$\rightarrow \text{at slit 1} \qquad \qquad \qquad \rightarrow \text{at slit 2} \tag{7-8}$$

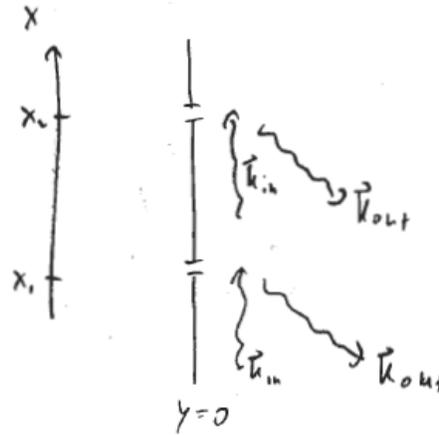


Figure III: Electron passing through double slit scattering a photon.

Interpretation. Momentum transfer onto electron.

A photon scattering event into a given direction does not destroy or “collapse” the wavefunction. It **merely shifts its phase**. Repeating the previous calculation, the intensity in the detector plane is now:

$$|\psi'_D|^2 = 2|A|^2 \{1 + \cos[(k(L_1 - L_2) + (\mathbf{k}_{in} - \mathbf{k}_{out})(\mathbf{x}_1 - \mathbf{x}_2))]\} \quad (7-9)$$

Note. Do not confuse

- $\mathbf{k} \rightarrow$ wavevector of the particle
- $\mathbf{k}_{in}, \mathbf{k}_{out} \rightarrow$ wavevector of incident, scattered photon.

For a given single scattering event characterized by outgoing photon wavevector \mathbf{k}_{out} there is an associated (and perfectly well defined) shift of the electron interference pattern in the detector plane. The momentum $\hbar\mathbf{k}_{out}$ (direction \mathbf{k}_{out}) of outgoing photons is perfectly correlated with the shift of the electron interference pattern in the detector plane.

Reconstructing the interference pattern from noise via correlations (conditional interference)

If we repeat the measurement many times, each one will have a different scattered photon direction \mathbf{k}_{out} , a different corresponding phase shift $(\mathbf{k}_{in} - \mathbf{k}_{out}) \cdot (\mathbf{x}_1 - \mathbf{x}_2)$, for the electron interference pattern, and the electron pattern on the screen will not show fringes.

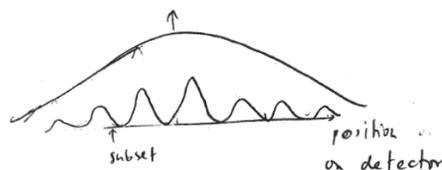


Figure IV: Averaging over interference patterns with random phases destroys interference.



Figure V: A subset of scattering events into a single angle, as recorded by a detector, shows interference.

However, if we record the photons and select only those events where a photon was recorded in a given, predetermined direction, this subset of electron arrival position will show perfect interference with a shift determined by the chosen detector direction (direction \mathbf{k}_{out}). The underlying interference is always there, it is the averaging over different phases of the interference pattern, or the correlated photon directions, that causes the interference pattern to wash out (disappear).

The above argument still holds if the photons fly off into vacuum, and the electrons are detected first. If we average over all photon directions, there is no electron interference pattern. However, if we post-select only those electron arrivals that correspond to a photon observation in a certain direction (even if that photon observation is performed after the electron arrival on the screen) an interference pattern with the predicted shift is observed.

Rule 5. A **measurement** is the interaction of our detector system (interaction here: electron scatters photon) **and** the averaging over many states of the detector (here: scattered photon directions). The averaging over the detector states is **crucial**. Before that averaging, we have a correlated (entangled) system, where the states of the quantum system (here: phase of the electron wavefunction, or position of the interference pattern) are correlated with the states of the detector:

”If the photon was scattered into direction 1, then the interference pattern is at position 1, and if photon was scattered into direction 2, then the interference pattern is at position 2, and if ...”

would describe the entangled state before averaging.

Note that if the wavelength of the light is too long to optically resolve the double slit ($\lambda_{\text{photon}} \gg x_2 - x_1 = d$), then even the maximum phase shift (obtained for $\mathbf{k}_{\text{out}} = -\mathbf{k}_{\text{in}}$; i.e. $(\mathbf{k}_{\text{in}} - \mathbf{k}_{\text{out}} = 2\mathbf{k}_{\text{in}})$ of magnitude $2k_{\text{in}}(x_2 - x_1) = \frac{4\pi}{\lambda_{\text{photon}}}(x_2 - x_1) \ll 1$, and the interference pattern continues to exist even when we average over all scattered-photon angles. If, on the other hand, the double slit is observed with a microscope with sufficient resolution (i.e., using sufficiently short photon wavelength), then the microscope's objective collects photons scattered into different angles, and the averaging over \mathbf{k}_{out} , and the corresponding electron phases $(\mathbf{k}_{\text{in}} - \mathbf{k}_{\text{out}}) \cdot (\mathbf{x}_2 - \mathbf{x}_1)$, washes out the electron interference pattern.

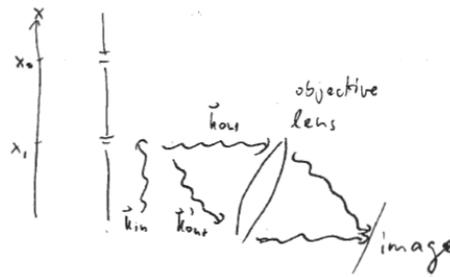


Figure VI: A microscope resolving which slit the electron passed through uses interference of the light scattered at different angles to image the electron. These angles correspond to different shifts of the electron interference pattern in the detector plane, i.e. the electron interference is washed out.

Atom model: Spectra and quantization of energy

Thomson model

In order to have an elastically bound electron that radiates monochromatic radiation, J.J. Thomson constructed a “plum-pudding” model of the atom where pointlike electrons are confined within a uniform positive charge distribution of radius $\sim 1\text{\AA}$.

Force on electron at x :

$$F = -\frac{|qQ|\frac{x^3}{R^3}}{4\pi\epsilon_0 x^2} = -\frac{|qQ|}{4\pi\epsilon_0 R^3}x \quad (7-10)$$

$$\ddot{x} = \frac{F}{m} = -\frac{|qQ|}{4\pi\epsilon_0 R^3 m}x \quad (7-11)$$

$$\ddot{x} = -\omega^2 x \quad \implies \quad \omega = \left[\frac{|qQ|}{4\pi\epsilon_0 R^3 m} \right]^{\frac{1}{2}} \quad (7-12)$$

For hydrogen:

- $Q = +q_e = 1.6 \cdot 10^{-19} \text{ C}$, $q = -q_e$
- $m = m_e = 9 \cdot 10^{-31} \text{ kg}$
- $R = 10^{-10} \text{ m} \implies \omega = 6 \cdot 10^{15} \frac{1}{\text{s}}$
- $\lambda = \frac{c}{\nu} = 2\pi \frac{c}{\omega} = 150 \text{ nm}$

The Thomson model yields a harmonically oscillating electron and the correct order of magnitude for the wavelength (shortest H wavelength Lyman α , $\lambda = 121 \text{ nm}$), but cannot explain the other spectral lines.

Rutherford scattering

Source. Marsden & Geiger 1906-, Rutherford 1911

Discovery of the nucleus through large-angle scattering. Marsden & Geiger mea-

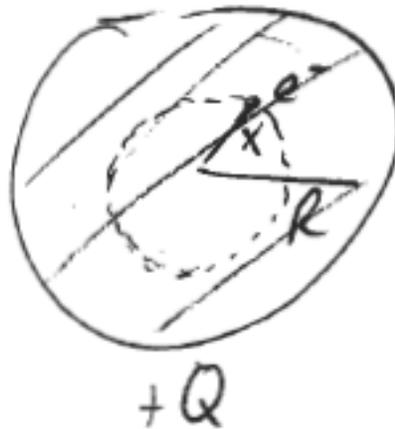


Figure VII: Thomson "plum pudding" model of the atom with a negatively charged electron imbedded into a uniformly distributed positive nuclear charge.

sured the angular distribution of scattered α -particles. According to the Thomson model, the positive charge is distributed evenly throughout the atom, so it should cause little deflection when the atoms are arranged in a solid. However, the observed distribution is dramatically different at large scattering angle. To observe large scattering angles, we need to scatter off massive particles so that momentum conservation allows large angles. Observations can be quantitatively explained by assuming that the mass of the nucleus is concentrated in a small volume. For quantitative description of observed scattering angle dependence we need the concept of cross section.

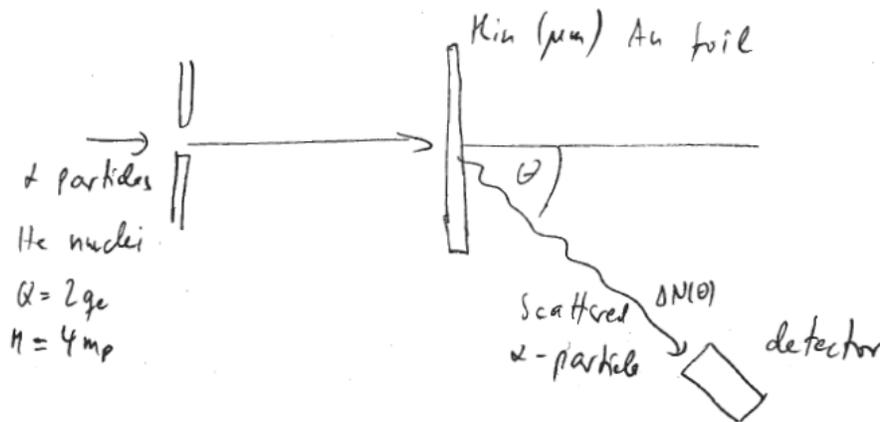


Figure VIII: Marsden and Geiger experiment on α particle scattering; explanation of results by Rutherford.

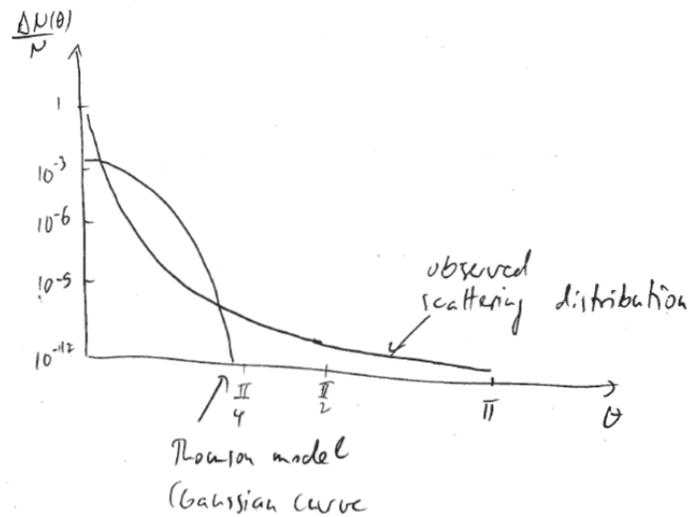


Figure IX: Observed angular dependence of α -scattering by gold foil, and prediction from the Thomson model.

Inadequacy of Thomson model

The electrons are $4 \cdot 1850 \sim 7000$ times lighter than an α -particle, conservation of momentum then results in a small deflection angle $\phi_e \sim \frac{m_e}{m_p} \sim 10^{-4}$ per α -electron collision. We expect a diffusion-type process for many scattering events (random walk in angle), this should result in a Gaussian distribution of scattering angles of width $\Delta\theta = \sqrt{M}\phi_e$, where M is the average number of α -particle scattering events. So we

expect for the fractional scattering into a particular angle θ :

$$\frac{\Delta N(\theta)}{N} = A e^{-\frac{\phi^2}{2(\Delta\theta)^2}} = A e^{-\frac{\phi^2}{2M\phi_e^2}} = A e^{-\beta\phi^2} \quad (7-13)$$

This would be the expected angular dependence of scattering events for Thomson model (only α -electron scattering causes deflection of α -particle). However, the observed angular distribution has a much longer tail at large scattering angles (see figure above).

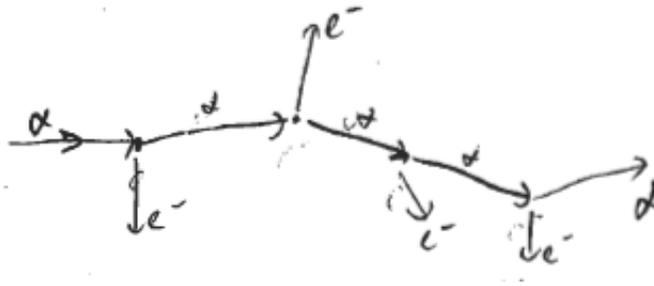


Figure X: Scattering of the heavy α particle off an electron causes only a small deflection angle ϕ_e , and for many scattering events a Gaussian distribution of scattering angles for the α particle.

Scattering problems and cross section

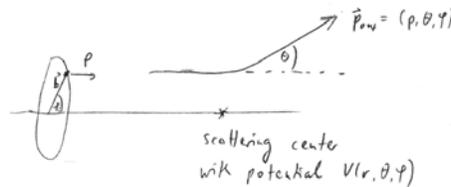


Figure XI: Scattering off a potential.

Problem is solved if **outgoing angles** θ , ϕ , can be calculated as function of **impact parameter** b and **incident angle** ϕ_i

- $\theta = \theta(b, \phi_i)$
- $\phi = \phi(b, \phi_i)$

For spherically symmetric potentials, $V = V(r)$, scattering is independent of ϕ : $\phi = \phi_i$ and problem solved if we know $\theta = \theta(b)$, i.e. if we can calculate the scattering angle θ as a function of impact parameter b .

For Coulomb scattering of a particle with charge $Z'q$ off a scattering center corresponding to a charge Zq , one can derive (see, e.g., Goldstein, "Classical Mechanics")

$$\cot \frac{\theta}{2} = \frac{8\pi\epsilon_0}{ZZ'q^2} bE \quad (7-14)$$

where E is the particle kinetic energy.

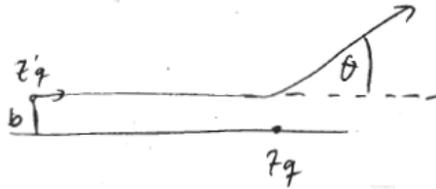


Figure XII: Coulomb scattering.

However, since the impact parameter b is not observable in a typical scattering experiment, we need a formalism to average over all impact parameters. We define the **total cross section**, σ_{tot} as the ratio of the total scattering rate R_1 to the incident intensity (for one scattering center)

$$\sigma_{\text{tot}} = \frac{R_1 \text{ [particles/s]}}{I \text{ [particles/(m}^2 \cdot \text{s)]}} \Rightarrow \text{[m}^2\text{]} \quad (7-15)$$

At density n , the number of atoms inside the volume $A \cdot l$ is $N = nAl$. Then in

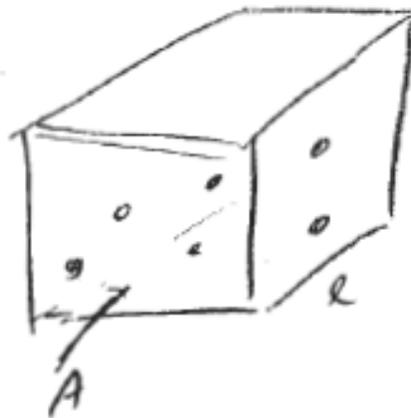


Figure XIII: Scatterers inside a volume.

the weak-scattering limit $N\sigma_{\text{tot}} \ll A$ the total scattering rate is proportional to the incident rate R_{in}

$$R_N = N\sigma_{\text{tot}}I = n\sigma_{\text{tot}}lIA = n\sigma_{\text{tot}}lR_{\text{in}} \quad (7-16)$$

and the fraction of scattered particles is

$$\frac{R_N}{R_{\text{in}}} = n\sigma_{\text{tot}}l = \frac{N\sigma_{\text{tot}}}{A} \quad (7-17)$$

The fraction of the particles removed from the beam is simply the total cross section $N\sigma_{\text{tot}}$ for N particles, divided by the beam area.

To describe the angular dependence of scattering, we slightly generalize the cross section concept. We define the **differential cross section** $\frac{d\sigma}{d\Omega}(\theta, \phi)$ as the ratio of the scattering rate per solid angle $\frac{dR}{d\Omega}$ to the incident intensity I (again for one scattering center).

$$\frac{d\sigma}{d\Omega} = \frac{dR_i/d\Omega \text{ [particles/(s·sterad)]}}{I \text{ [particles/(m}^2 \cdot \text{s)]}} \implies \left[\frac{\text{m}^2}{\text{sterad}} \right] \quad (7-18)$$

For spherically symmetric potentials, the differential cross section depends only on the angle θ . A detector of area A at distance R from the scatterer subtends an angle:

$$\frac{d\Omega}{4\pi} = \frac{A}{4\pi R^2} \quad (7-19)$$

or

$$d\Omega = \frac{A}{R^2} \quad (7-20)$$

Particles with impact parameters between b and $b + db$ scatter with angle between θ

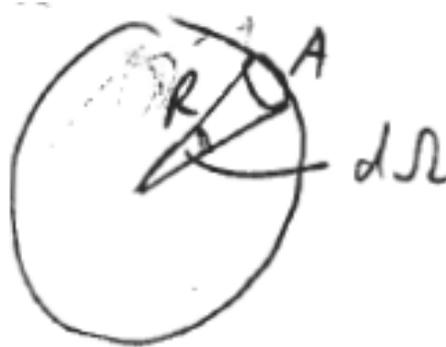


Figure XIV: Scattering into a solid angle $d\Omega$ and detector area A .

and $\theta + d\theta$ corresponding to a solid angle $d\Omega = 2\pi \sin\theta d\theta$. Thus the cross-sectional

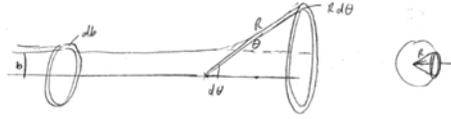


Figure XV: Differential scattering cross section

area $d\sigma = 2\pi b db$ corresponds to the solid angle $d\Omega = 2\pi \sin \theta d\theta$, and the differential cross section can be written as:

$$\frac{d\sigma}{d\Omega} = \frac{2\pi b db}{2\pi \sin \theta d\theta} \quad (7-21)$$

To eliminate the dependence on the impact parameter b (a quantity not directly observable in the experiment), we use the relation $\cot \frac{\theta}{2} = \frac{8\pi\epsilon_0}{ZZ'q^2} bE$ from Eq. 7-14 between θ and b , and differentiate:

$$\frac{8\pi\epsilon_0 E}{ZZ'q^2} db = d \left(\frac{\cos \theta/2}{\sin \theta/2} \right) \quad (7-22)$$

$$= \frac{-\frac{1}{2} \sin^2 \theta/2 - \frac{1}{2} \cos^2 \theta/2}{\sin^2 \theta/2} d\theta \quad (7-23)$$

$$= -\frac{d\theta}{2 \sin^2 \theta/2} \quad (7-24)$$

Substitution of b and db into the expression for $\frac{d\sigma}{d\Omega}$ yields

$$\frac{d\sigma}{d\Omega} = \left(\frac{ZZ'q^2}{8\pi\epsilon_0 E} \right)^2 \cot \frac{\theta}{2} \frac{-1}{2} \frac{1}{2 \sin^2 \theta/2 \sin \theta d\theta} \quad (7-25)$$

$$= \left(\frac{ZZ'q^2}{8\pi\epsilon_0 E} \right)^2 \frac{1}{4 \sin^4 \theta/2} \quad (7-26)$$

Differential cross section for scattering $Z'q$ off Zq :

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{Z'^2 Z^2 q^4}{256\pi^2 \epsilon_0^2 E^2} \frac{1}{\sin^4 \frac{\theta}{2}}} \quad (7-27)$$

Setting $Z' = 2$ for the He-4 particle we arrive at the Rutherford scattering formula for α -particles

$$\boxed{\frac{d\sigma}{d\Omega}(\theta) = \frac{Z^2 q^4}{64\pi^2 \epsilon_0^2 E^2} \frac{1}{\sin^4 \frac{\theta}{2}}} \quad (7-28)$$

To obtain scattering rate into detector at fixed θ , calculate solid angle $\Delta\Omega$ that detector occupies and use

$$R_N(\theta) = N \frac{d\sigma}{d\Omega}(\theta) \Delta\Omega I \quad (7-29)$$

- N : # of scatterers
- $\Delta\Omega$: detector solid angle
- I : incident intensity

The Rutherford scattering formula displays strong suppression for large scattering angles, but still predicts much more large-angle scattering than the Thomson model for scattering off the light electrons. For larger incident energy E there is less deflection and less scattering.

Note. The total cross section $\sigma_{\text{tot}} = \int d\Omega \frac{d\sigma}{d\Omega}$ diverges for Coulomb scattering because the potential has infinite range, so there remains some deflection even for very large impact parameter. The concept of cross sections remains very useful in the quantum theory of scattering. For very large scattering angles (and correspondingly small impact parameter) the experimental findings deviate from the Rutherford formula. As the α -particle enters the region of the nucleus ($R \sim 10^{-14}$ m), the Coulomb potential is modified by the internuclear forces. Measurements of this deviation at large scattering angle yield the result for the radius of the nucleus

$$R = 1.2 \cdot 10^{-15} \text{ m} \cdot \sqrt{A} \implies n_{\text{nucleus}} = 2 \cdot 10^{17} \frac{\text{kg}}{\text{m}^3} \quad (7-30)$$

where A is the number of nucleons (protons or neutrons). The density of the nucleus is constant and enormously large. (Steel has a density of 8×10^3 kg/m³.)