

PX408: RELATIVISTIC QUANTUM MECHANICS

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Handout 2: Relativistic Wave Equations

We begin our search for suitable relativistic wave equations with the generalised Schrödinger equation,

$$H\psi = i\frac{\partial}{\partial t}\psi, \quad (1)$$

where H is the *Hamiltonian*, which (for our purposes, at least) represents the total energy of the system. To obtain the (non-relativistic) free-field Schrödinger equation, we simply take H to be the kinetic energy ($p^2/2m$) and quantise the momentum operator ($p \mapsto -i\nabla$).

$$-\frac{1}{2m}\nabla^2\psi = i\frac{\partial}{\partial t}\psi \quad (2)$$

The Klein-Gordon Equation

To obtain a relativistic wave equation we need an appropriate replacement of $E = p^2/2m$. We know the relativistic relation between energy and momentum to be

$$E^2 = p^2 + m^2, \quad (3)$$

so a possible choice would appear to be $E = \sqrt{p^2 + m^2}$. Unfortunately, we don't have a good way to handle the square root operator quantum mechanically, so let's instead quantise Eq. 3:

$$-\frac{\partial^2}{\partial t^2}\psi = -\nabla^2\psi + m^2\psi, \quad (4)$$

or

$$\begin{aligned} \left(\frac{\partial^2}{\partial t^2} - \nabla^2\right)\psi + m^2\psi &= 0 \\ (\partial_\mu\partial^\mu + m^2)\psi &= 0 \\ (\square + m^2)\psi &= 0 \end{aligned} \quad (5)$$

All of which are the same equation, just written differently. This is the Klein-Gordon equation. (The “*d'Alembertian operator*” $\square = \partial_\mu\partial^\mu$ is sometimes used in the literature – as are several other notations – but will not be used much in this module.)

It is an interesting historical aside to note that the Klein-Gordon equation was in fact discovered, but rejected, by Schrödinger before he obtained the non-relativistic equation that bears his name. The principal reason that he rejected it was that its solutions failed to describe the electronic energy levels in the hydrogen atoms (whereas the Schrödinger works quite well, since the electrons are reasonably non-relativistic, and spin-dependent effects are fairly small).

In spite of this problem, as well as others that we shall discuss momentarily, in the Klein-Gordon equation we have found a relativistically invariant, quantum mechanical equation, that at first sight appears to be a good candidate to describe electrodynamics. Let's now examine some of its issues.

The minimal solution for ψ is that it is a scalar field (*i.e.* its value at each point in space and time is given simply by a number). Such a quantity does not have any orientation, and must be

spin-zero. [If we know that electrons are spin-1/2 and photons are spin-1, we might already realise that this equation will struggle to describe electromagnetism.]

If we assume plane-wave solutions: *i.e.* $\psi \propto e^{-ip \cdot x}$ (where $p \cdot x = p_\mu x^\mu$), we find $E^2 = p^2 + m^2$ and hence $E = \pm \sqrt{p^2 + m^2}$. Thus, both positive and negative energy solutions are possible, yet negative energy solutions do not appear to be physically meaningful.

Q1 Show explicitly that plane wave solutions of the Klein-Gordon equation lead to negative energies.

We would also like to find the equivalent of the conserved current, which is familiar from the continuity equation (naturally satisfied from Maxwell's equations)

$$\frac{\partial \rho_{\text{em}}}{\partial t} = -\nabla \cdot \mathbf{j}_{\text{em}} \quad (6)$$

which we can write in terms of 4-vectors as $\partial_\mu j_{\text{em}}^\mu = 0$. It is natural to consider that $|\psi|^2$ may take the place of the density, ρ_{em} , since in our earlier discussion the squared magnitudes of wavefunctions formed probability densities. However, since ψ is a scalar, $|\psi|^2$ is just a constant number, and under Lorentz boosts does not transform as a time-like component, which clearly ρ_{em} must do.

To find the Klein-Gordon current, we start with the Klein-Gordon equation

$$(\partial_\mu \partial^\mu + m^2)\psi = 0 \quad (7)$$

and considering that ψ is a complex wavefunction, we can also write a similar equation for ψ^*

$$(\partial_\mu \partial^\mu + m^2)\psi^* = 0 \quad (8)$$

After multiplying the first by ψ^* and the second by ψ , subtracting and rearranging, we find

$$\partial_\mu i (\psi^* \partial^\mu \psi - \psi \partial^\mu \psi^*) = 0 \quad (9)$$

which is exactly of the desired form $\partial_\mu j^\mu = 0$, with $j^\mu = i (\psi^* \partial^\mu \psi - \psi \partial^\mu \psi^*)$.

Q2 What is $\partial_\mu (\psi^* \psi)$?

Q3 Explicitly derive Eq. 9 from Eqs. 7 and 8.

Q4 Applying a similar treatment to the Schrödinger equation (Eq. 2), find the Schrödinger current.

Our “density” is thus the time-like component of the above, which is $i \left(\psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t} \right)$. However, this is not a positive-definite quantity, and therefore cannot be interpreted as a probability density.

Q5 Using plane wave solutions for ψ , find the 4-current, and show that the time-like component is not positive definite.

Let us summarise the problems with the Klein-Gordon equation

- The simplest solutions are scalars, which cannot account for spin
- It fails to describe the hydrogen atom
- There are both positive and negative solutions for the energy.
- The density is not positive-definite
- The equation is second order in time-like derivatives, and thus to solve uniquely for ψ we need two boundary conditions (typically given by the values of $\psi(t=0)$ and $\frac{\partial \psi}{\partial t}(t=0)$).

Three comments are in order. First, the problems related to negative energy, the density and the second order nature of the equation are not really surprising, since we started with a second order equation ($E^2 = p^2 + m^2$) to find the Klein-Gordon equation. Secondly, all three of these problems are closely related. Finally, we could get around the problem related to the density by pointing out that in electromagnetism the current of interest is the electric current, and the density of interest is the charge density, which is allowed to be negative. If we simply multiply the j^μ of Eq. 9 by the electric charge, we recover something that is physically acceptable. Nevertheless, negative probability densities are not to be sniffed at, and we should move on to try to find a solution to the problem.

The Klein-Gordon Equation in an Electromagnetic Field

Everything above was in the free-field approximation. How about the more interesting case including electromagnetic fields?

We would start with a Hamiltonian

$$H = E = \sqrt{(\mathbf{p} - q\mathbf{A})^2 + m^2} + q\phi, \quad (10)$$

which, when squared gives

$$(E - q\phi)^2 = (\mathbf{p} - q\mathbf{A})^2 + m^2. \quad (11)$$

Quantising and rearranging then gives

$$(\partial_\mu + iqA_\mu)(\partial^\mu + iqA^\mu)\psi + m^2\psi = 0 \quad (12)$$

or

$$D_\mu D^\mu \psi + m^2\psi = 0. \quad (13)$$

This is the same Klein-Gordon equation as Eq. 7, but with electromagnetism introduced via the “minimal substitution”

$$\partial_\mu \mapsto D_\mu = \partial_\mu + iqA_\mu \quad (14)$$

Q6 Justify Eq. 10.

Q7 Quantise and rearrange Eq. 11 to obtain Eq. 13.

Q8 Obtain the conserved current for the Klein-Gordon equation in an electromagnetic field.

The Dirac Equation

The Dirac equation is absolutely central to Relativistic Quantum Mechanics, so we will look at two ways of deriving it.

In the first approach, we start by stating that we are searching for a first order relativistically invariant equation. We know it should depend on E , \mathbf{p} and m , so we can write, quite generally:

$$E = \boldsymbol{\alpha} \cdot \mathbf{p} + \beta m, \quad (15)$$

or, quantum mechanically

$$i\frac{\partial}{\partial t}\psi = (-i\boldsymbol{\alpha} \cdot \boldsymbol{\nabla} + \beta m)\psi. \quad (16)$$

We simply need to find the appropriate values of the constants $\boldsymbol{\alpha}$ and β .

To do so, we square the operators on both side, and assert that our solutions must still satisfy the relativistic mass-energy relation $E^2 = p^2 + m^2$:

$$-\frac{\partial^2}{\partial t^2}\psi = (-i\boldsymbol{\alpha}\cdot\nabla + \beta m)(-i\boldsymbol{\alpha}\cdot\nabla + \beta m)\psi \quad (17)$$

$$= -\alpha^2\nabla^2\psi + \beta^2m^2\psi - \sum_{i,j;j>i}(\alpha_i\alpha_j + \alpha_j\alpha_i)\frac{\partial^2\psi}{\partial x^i\partial x^j} - im\sum_i(\alpha_i\beta + \beta\alpha_i)\frac{\partial\psi}{\partial x^i} \quad (18)$$

$$= -\nabla^2\psi + m^2\psi. \quad (19)$$

After some rearrangement, we find that we require the following relations

$$\begin{aligned} \{\alpha_i, \alpha_j\} &= 2\delta_{ij} \\ \{\alpha_i, \beta\} &= 0 \\ \beta^2 &= 1 \end{aligned} \quad (20)$$

Here, δ_{ij} is the Kronecker delta.

Q9 Explicitly derive the results of Eq. 20.

The elements of $\boldsymbol{\alpha}$ and β cannot be simple numbers, since multiplication of numbers is commutative, and so it is impossible to satisfy a relation like $\alpha_i\beta + \beta\alpha_i = 0$, unless $\alpha_i = \beta = 0$ (in which case the equation proposed by Dirac (Eq. 16) would not be very interesting).

We might think we have a possible solution with the Pauli matrices, which we know (as shown previously) have the properties $\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = I$, $[\sigma_j, \sigma_k] = 2i\epsilon_{jkl}\sigma_l$ and $\{\sigma_i, \sigma_j\} = 2\delta_{ij}I$. Here, ϵ_{jkl} is the Levi-Civita symbol. [In the above the identity matrix I has been explicitly included, though it will not be elsewhere.] In fact, if we put $\boldsymbol{\alpha} = \boldsymbol{\sigma}$, this will solve the requirements on $\boldsymbol{\alpha}$ in Eq. 20, but it turns out that with this choice it is impossible to find solutions for β . Indeed, it is impossible to find solutions for any set of 2×2 matrices. (This is easy to understand once we know something about the Clifford algebra of the Pauli matrices, which we will discuss briefly later.)

Q10 Prove that for the choice $\boldsymbol{\alpha} = \boldsymbol{\sigma}$, it is impossible to satisfy the conditions required of β in Eq. 20.

In fact, the minimal solution to Eq. 20 is for the elements of $\boldsymbol{\alpha}$ and β to be 4×4 matrices. With such dimensions, there is a continuum of solutions, but the usual representation is

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (21)$$

Note the so-called block form for these matrices: each element of the 2×2 matrix is itself a 2×2 matrix (using 1 & 0 rather casually, to mean the unit & null square matrix, respectively, of any dimension).

Q11 Show explicitly that this choice for $\boldsymbol{\alpha}$ and β satisfies the requirements of Eq. 20.

Q12 Clearly matrices are playing an important rôle in our discussions. Make sure that you are familiar with matrix operations. What is the determinant of a matrix and how is it calculated? What is the trace of a matrix and how is it calculated? What is a diagonal matrix? What is the transpose of a matrix and how is it determined? What are symmetric and antisymmetric matrices? What is a unitary matrix? What is a Hermitian matrix?

Q13 Show that any set of matrices $\boldsymbol{\alpha}', \beta'$ related to the above forms by the transformation

$$\alpha'_i = U\alpha_iU^{-1} \quad \beta' = U\beta U^{-1} \quad (22)$$

where U is any unitary 4×4 matrix, would also give satisfactory solutions to the requirements of Eq. 20.

Q14 If α and β are 4×4 matrices, what must ψ be?

Q15 Write out Eq. 16 explicitly, including all elements of the 4×4 matrices.

We can put Eq. 16 into a manifestly Lorentz invariant form, making use of the *gamma matrices* $\gamma^\mu = (\gamma^0, \boldsymbol{\gamma})$ where $\gamma^0 = \beta$, $\boldsymbol{\gamma} = \beta\boldsymbol{\alpha}$.

We then find

$$(i\gamma^\mu\partial_\mu - m)\psi = 0 \quad (23)$$

Where ψ is a 4 component vector (but not a 4-vector) sometimes referred to as a *Dirac spinor*. γ^μ is a 4 component vector (but not a 4-vector) whose elements are 4×4 matrices. The invariant product $\gamma^\mu a_\mu = g_{\mu\nu}\gamma^\mu a^\nu$ appears quite frequently in quantum field theory, and is often abbreviated to \not{a} . This notation is called the ‘‘Feynman slash’’ – we will briefly discuss its significance later. Thus we can write Eq. 23 in even more compact form:

$$(i\not{\partial} - m)\psi = 0 \quad (24)$$

Q16 What are the commutation and anticommutation relations of the gamma matrices? Relate $\{\gamma^\mu, \gamma^\nu\}$ to $g^{\mu\nu}$.

Q17 What are the traces of the gamma matrices? What are their Hermitian conjugates (*i.e.* their adjoint matrices)?

Q18 If α and β transform as in Q13, how do the gamma matrices transform? Show that the transformed gamma matrices satisfy the same anticommutation relations. What are the traces of the transformed gamma matrices? (Note that traces will be discussed in more detail below.)

Q19 For a generic covariant four vector a_μ , write out explicitly the elements of \not{a} in the standard representation.

Q20 Using the standard representation, write out Eq. 23 explicitly, including all elements of the 4×4 matrices.

The Dirac Equation – Second Derivation

In the first derivation the relation to spin (*ie.* the need for the solutions to be four-component spinors, *etc.*) arose due to anticommutation relations. In fact, this connection between spin and statistics is a rather crucial point in quantum field theory. However, it is also useful to consider another viewpoint to see how the Dirac equation is related to spin. This derivation follows Feynman (QED, p.38).

We start with the relativistic mass-energy relation in an electromagnetic field.

$$(E - q\phi)^2 = (\boldsymbol{\sigma} \cdot (\mathbf{p} - q\mathbf{A}))^2 + m^2, \quad (25)$$

or quantum mechanically

$$\left(i\frac{\partial}{\partial t} - q\phi\right)^2 \psi = (\boldsymbol{\sigma} \cdot (-i\nabla - q\mathbf{A}))^2 \psi + m^2\psi. \quad (26)$$

Q21 Show that Eq. 25 is the correct relativistic mass-energy relation in an electromagnetic field.

We can rearrange this to give

$$\left(i\frac{\partial}{\partial t} - q\phi - \boldsymbol{\sigma}\cdot(-i\nabla - q\mathbf{A})\right)\left(i\frac{\partial}{\partial t} - q\phi + \boldsymbol{\sigma}\cdot(-i\nabla - q\mathbf{A})\right)\psi = m^2\psi \quad (27)$$

Now, substituting $D_0 = \frac{\partial}{\partial t} + iq\phi$ and $\mathbf{D} = -\nabla + iq\mathbf{A}$, *ie.* $D_\mu = \partial_\mu + iqA_\mu$, we find

$$(iD_0 - i\boldsymbol{\sigma}\cdot\mathbf{D})(iD_0 + i\boldsymbol{\sigma}\cdot\mathbf{D})\psi = m^2\psi \quad (28)$$

A convenient trick to solve this is to introduce χ , which satisfies

$$(iD_0 + i\boldsymbol{\sigma}\cdot\mathbf{D})\psi = m\chi \quad (29)$$

and therefore

$$(iD_0 - i\boldsymbol{\sigma}\cdot\mathbf{D})\chi = m\psi \quad (30)$$

Adding and subtracting these expressions gives

$$iD_0(\psi + \chi) + i\boldsymbol{\sigma}\cdot\mathbf{D}(\psi - \chi) = m(\psi + \chi) \quad (31)$$

$$iD_0(\psi - \chi) + i\boldsymbol{\sigma}\cdot\mathbf{D}(\psi + \chi) = -m(\psi - \chi) \quad (32)$$

which we can write in matrix form (after multiplying the second relation by -1):

$$\begin{pmatrix} iD_0 & i\boldsymbol{\sigma}\cdot\mathbf{D} \\ -i\boldsymbol{\sigma}\cdot\mathbf{D} & -iD_0 \end{pmatrix} \begin{pmatrix} \psi + \chi \\ \psi - \chi \end{pmatrix} = m \begin{pmatrix} \psi + \chi \\ \psi - \chi \end{pmatrix} \quad (33)$$

The matrix that appears in this expression can be written as $i\gamma^\mu D_\mu$, where

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \boldsymbol{\gamma} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix} \quad (34)$$

As if by magic, we have arrived at the Dirac equation in the presence of an electromagnetic field:

$$(i\gamma^\mu D_\mu - m)\Psi = 0 \quad \text{or} \quad (35)$$

$$(i\gamma^\mu(\partial_\mu + iqA_\mu) - m)\Psi = 0 \quad (36)$$

where the solutions Ψ have four components since

$$\Psi = \begin{pmatrix} \psi + \chi \\ \psi - \chi \end{pmatrix} \quad (37)$$

and both ψ and χ are two-component vectors. Clearly, this reduces to the free-field Dirac equation (from which it differs simply by the minimal substitution) by setting $A_\mu = 0$.

Q22 Find solutions for χ , and consequently for Ψ , for plane-wave solutions with $\psi \propto \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and with $\psi \propto \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. (Use the free-field form of the equations.)

Q23 How many linearly independent components do the solutions to the Dirac equation have?

One unfortunate consequence of this derivation is that it appears that there is something special about the choice of gamma matrices (*ie.* their definition in terms of the Pauli matrices) that is used. This is not the choice, as should be clear from the first derivation that was considered, and as will be further emphasised below.

Weyl Equations

Let us make a brief, but useful, digression and consider the case of massless fermions.

In this case we have

$$E^2 = p^2 = (\boldsymbol{\sigma} \cdot \mathbf{p})^2, \quad (38)$$

or, after quantisation, rearrangement and the introduction of electromagnetic fields by the minimal substitution

$$(D_0 - \boldsymbol{\sigma} \cdot \mathbf{D})(D_0 + \boldsymbol{\sigma} \cdot \mathbf{D})\psi = 0. \quad (39)$$

In this case we have two possible solutions:

$$(D_0 - \boldsymbol{\sigma} \cdot \mathbf{D})\phi = 0, \quad (40)$$

$$(D_0 + \boldsymbol{\sigma} \cdot \mathbf{D})\chi = 0. \quad (41)$$

Equations 40 and 41 are the *Weyl equations*. Their solutions, ϕ and χ respectively, are two-component spinors that are called *Weyl spinors*.

Historical note: this theory was developed by Hermann Weyl in 1929, very shortly after the publication of Dirac's work. However, it did not receive great attention: aside from the fact that it describes massless fermions, none of which were known to exist at that time, it also predicts that the particles corresponding to ϕ and χ have fixed "handedness" and hence violate mirror symmetry – as we shall shortly see when we discuss helicity and chirality. Thus it was not until after the prediction of the neutrino (Pauli, 1931) and its discovery (Cowan and Reines, 1956), together with the prediction of parity violation (Lee and Yang, 1956) and its discovery (Wu, 1957) that the utility of the theory was realised. Nowadays, neutrinos are known to have (a very small) mass, thus we are again at a state in which no massless fermions are known to exist. Nonetheless, Weyl's theory remains useful, since the Dirac spinor can be constructed from the Weyl spinors:

$$\psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix}. \quad (42)$$

Indeed, we can think of the effect of mass as being the mixing together of the massless solutions.

Equivalence Transformations

The defining characteristic of the gamma matrix is their anticommutation:

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}. \quad (43)$$

We have, until now, used a convenient convention with which to write down explicit expressions for the gamma matrices, but any other set that satisfy Eq. 43 would be equally acceptable.

To show this explicitly, we consider transformation of the wavefunction ψ , which is a solution of the Dirac equation.

$$\psi \mapsto \psi' = S\psi, \quad (44)$$

where S can be represented as a 4×4 matrix. (S must have an inverse S^{-1} . In fact, without loss of generality, we can state that S should be a unitary matrix.)

Let us assume that ψ' also satisfies the Dirac equation, so

$$(i\gamma^\mu \partial_\mu - m)S\psi = 0, \quad (45)$$

or, using the fact that S commutes with m and with ∂_μ ,

$$i\gamma^\mu S\partial_\mu\psi = Sm\psi. \quad (46)$$

Now left-multiply by S^{-1} to find

$$iS^{-1}\gamma^\mu S\partial_\mu\psi = S^{-1}Sm\psi = m\psi, \quad (47)$$

or

$$(i\gamma'^\mu\partial_\mu - m)\psi = 0, \quad (48)$$

where $\gamma'^\mu = S^{-1}\gamma^\mu S$. Note that this is exactly the same as the Dirac equation of Eq. 23, except for the *equivalence transformation* $\gamma^\mu \mapsto \gamma'^\mu = S^{-1}\gamma^\mu S$. To put it another way, γ'^μ would be an equally acceptable convention to use for the gamma matrices.

Q24 Show that γ'^μ satisfy the same anticommutation relations as γ^μ .

Gamma Matrix Algebra

The gamma matrices appear frequently in relativistic quantum mechanics and quantum field theory in various permutations. It is therefore very useful to become familiar with handling them.

Recall that their defining characteristic is their anticommutation relation (Eq. 43)

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}. \quad (49)$$

It is important to realise that in this equation $g^{\mu\nu}$ is not a matrix (in fact, there is an implied 4×4 unit matrix multiplying the right-hand side).

Since the operator $g^{\mu\nu}$ raises indices (*ie.* converts covariant four vectors into contravariant form), we can write

$$a^\mu = g^{\mu\nu}a_\nu = \frac{1}{2}\{\gamma^\mu, \gamma^\nu\}a_\nu. \quad (50)$$

We can find some useful identities. As examples

$$\gamma_\mu\gamma^\mu = \gamma^0\gamma^0 - (\gamma^1\gamma^1 + \gamma^2\gamma^2 + \gamma^3\gamma^3) = 4 \quad (51)$$

and

$$\gamma^\mu\gamma^\nu\gamma_\mu = (2g^{\mu\nu} - \gamma^\nu\gamma^\mu)\gamma_\mu = 2\gamma^\nu - 4\gamma^\nu = -2\gamma^\nu. \quad (52)$$

Recalling the Feynman slash notation

$$\gamma^\mu a_\mu = \not{a} \quad (53)$$

we can obtain further identities, such as

$$\gamma_\mu\not{a}\gamma^\mu = \gamma_\mu\gamma^\nu a_\nu\gamma^\mu \quad (54)$$

$$= \gamma_\mu(2g^{\mu\nu} - \gamma^\mu\gamma^\nu)a_\nu \quad (55)$$

$$= 2\not{a} - 4\not{a} = -2\not{a} \quad (56)$$

and

$$\not{a}\not{b} = \gamma^\mu a_\mu\gamma^\nu b_\nu \quad (57)$$

$$= (2g^{\mu\nu} - \gamma^\mu\gamma^\nu)a_\mu b_\nu \quad (58)$$

$$= 2a_\mu b^\mu - \not{a}\not{b} \quad (59)$$

$$\text{so } \{\not{a}, \not{b}\} = 2a_\mu b^\mu. \quad (60)$$

Notice for the latter that if $a_\mu b^\mu = 0$ then \not{a} and \not{b} anticommute. (In the above it is assumed that the elements of a and b are simple scalars, and therefore commute trivially. Indeed, the Feynman slash notation is generally only used for such quantities: we do **not** write $\gamma^\mu\gamma_\mu = \not{1}$, for example.)

Q29 Using the usual convention for the gamma matrices, find block form expressions for matrices in Eq. 66.

The product of more than four gamma matrices must contain a repeated pair, and therefore we will always be able to use the anticommutation relations to bring the pair together in the expression, after which it can be replaced by ± 1 as appropriate. For example

$$\gamma^0 \gamma^1 \gamma^2 \gamma^0 \gamma^3 = -\gamma^0 \gamma^1 \gamma^0 \gamma^2 \gamma^3 = \gamma^0 \gamma^0 \gamma^1 \gamma^2 \gamma^3 = \gamma^1 \gamma^2 \gamma^3. \quad (67)$$

The product of all four gamma matrices has a special importance, and is therefore given its own notation. The usual definition is

$$\gamma^5 = i\gamma^0 \gamma^1 \gamma^2 \gamma^3. \quad (68)$$

It is easy to show that $(\gamma^5)^2 = 1$, and that $\{\gamma^5, \gamma^\mu\} = 0$, *ie.* γ^5 has similar anticommutation relations to the other gamma matrices.

Q30 Show that $(\gamma^5)^2 = 1$.

Q31 Show that $\{\gamma^5, \gamma^\mu\} = 0$.

Q32 Show that $\{\gamma^5, \not{a}\} = 0$.

Q33 Show that $\text{Tr}(\gamma^5) = 0$.

Q34 Show that the product of three gamma matrices can be written as the product of the other one and γ^5 (up to factors of $\pm i$).

Q35 Show that the trace of the product of any odd number of gamma matrices is zero.

Q36 Using the usual convention, find an expression for γ^5 in block form.

Two comments are in order to conclude this discussion. Firstly, a more detailed discussion of the Clifford algebra than given here would allow the following interpretation of γ^μ . Rather than being a four-vector (as mentioned before, γ^μ does not transform as a four-vector), it can be understood as a mapping operator which transforms the four-vector a_μ into the Clifford algebra representation \not{a} .

Secondly, the set of matrices that define the Clifford algebra (Eq. 66 can alternatively be written

$$1 \quad \gamma^\mu \quad \sigma^{\mu\nu} \quad \gamma^\mu \gamma^5 \quad \gamma^5, \quad (69)$$

where $\sigma^{\mu\nu} = \frac{i}{2}(\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu)$ is an antisymmetric tensor. This representation is useful, since it relates the Clifford algebra to the possible forms of invariant quantities that can arise, being scalar, vector, tensor, axial-vector and pseudo-scalar respectively. (These phrases will be familiar to those who have studied particle physics, where they are used to distinguish between different transformation properties of fields under the parity operation. We will discuss this briefly later on.)

Q37 Show that the the products of three gamma matrices can be rewritten as $\gamma^\mu \gamma^5$.

Q38 Why is the choice of an antisymmetric tensor $\sigma^{\mu\nu}$ appropriate to represent the products of two gamma matrices? HINT: consider the value of a corresponding symmetric tensor.