

Physics 195a
Course Notes
Preliminaries
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1 Introduction

This note sets conventions and gives some handy conversions for Ph 195.

2 Units

We adopt a system of units which tends to avoid carrying cumbersome constants in expressions. This has the benefit of permitting the physical meanings to be more readily apparent. The downside is the need to do conversions to “engineering” units. However, this is not a severe difficulty, as long as a couple of conversion constants are remembered. Indeed, many practicing physicists (and not only theorists) adopt these units in their everyday work.

Our system of units is such that Planck’s constant (over 2π), $\hbar = 1$, and the speed of light in a vacuum, $c = 1$. Note that this implies the conversion factors between SI (Système Internationale) units:

$$1 = \hbar = \frac{h}{2\pi} \approx \frac{6.62}{2\pi} \times 10^{-34} \text{ Js}, \quad (1)$$

$$1 = c \approx 3 \times 10^8 \text{ ms}^{-1}. \quad (2)$$

Thus, if we are given a quantity in Joules, we may convert it to inverse meters by multiplying by

$$\frac{2\pi}{(6.62 \times 10^{-34} \text{ Js})(3 \times 10^8) \text{ ms}^{-1}}. \quad (3)$$

We typically don’t find Joules to be an especially useful unit in quantum mechanics, so this isn’t a particularly common conversion, but the idea is the same for other conversions – you can multiply and divide by 1 as you please, to go from one unit to another.

We actually will use a variety of units, as dictated by the physical scales and desired intuition in any given situation. For example, atomic size scales are conveniently described in angstroms (\AA) and energy scales in electron volts (eV), while the nuclear size scales are more convenient to discuss in

femtometers (or “fermis”) (fm) and energy scales in millions of electron volts (MeV).

The above specification is not unique. The table below shows how Maxwell’s equations, and related electromagnetic formulae appear in one choice of the $c = 1$ (and $\hbar = 1$, as desired) units, the “Rationalized Heavyside-Lorentz” units, in comparison with the more familiar modified ($c = 1$) Gaussian units.

Example: Electromagnetism

Modified Gaussian	Rationalized Heavyside-Lorentz
$\nabla \cdot D = 4\pi\rho$	$\nabla \cdot D = \rho$
$\nabla \times E = -\partial_t B$	same
$\nabla \cdot B = 0$	same
$\nabla \times H = 4\pi J + \partial_t D$	$\nabla \times H = J + \partial_t D$
$D = \epsilon E; B = \mu H$	same
$E = -\nabla\Phi - \partial_t A$	same
$B = \nabla \times A$	same
$F = q(E + v \times B)$	same

Let us look at the implications of the first Maxwell equation for Coulomb’s law between two charges, q_1 and q_2 . If $\epsilon = 1$, then in the modified Gaussian

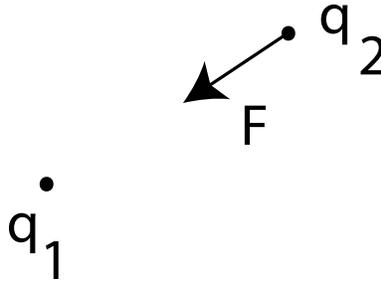


Figure 1: Force on charge q_2 due to charge q_1 .

units, $\nabla \cdot D = 4\pi\rho$. Integrating this over a sphere of radius $r_{12} = |x_1 - x_2|$ centered at q_1 gives an electric field at q_2 , due to q_1 of

$$E_1(r_{12}) = \frac{q_1}{r_{12}^2}, \tag{4}$$

or a force on q_2 of

$$F = \frac{q_1 q_2}{r_{12}^2}. \quad (5)$$

If we repeat the exercise in rationalized Heavyside-Lorentz units, we find

$$E_1(r_{12}) = \frac{q_1}{4\pi r_{12}^2}, \quad F = \frac{q_1 q_2}{4\pi r_{12}^2}. \quad (6)$$

Consider now the electrostatic potential energy of two electrons separated by an electron's Compton wavelength, $r = 1/m$, where m is the electron mass. We'll measure the strength of the electromagnetic force by comparing this energy with the electron's mass. This strength is a dimensionless constant α , known as the **fine structure constant**. Let $K = 1$ or 4π depending on whether we are using modified Gaussian or rationalized Heavyside-Lorentz units, respectively. Denote the electron charge by $-e$. Then:

$$\alpha = \frac{V}{m} = \frac{1}{m} \int_{\infty}^{1/m} F \cdot dx = \frac{1}{m} \int_{\infty}^{1/m} \frac{e^2}{K r^2} dr \quad (7)$$

$$= \frac{e^2}{K}. \quad (8)$$

Thus, the fine structure constant is expressed as e^2 in modified Gaussian units, and $e^2/4\pi$ in rationalized Heavyside-Lorentz units. Since this is a dimensionless constant, this may be confusing, but we can readily trace the origin to the manner in which we wrote the first Maxwell equation. Since the fine structure constant has a physical interpretation, it is numerically the same independent of the units, approximately $1/137$.

We will use the modified Gaussian system here, and $e^2 \approx 1/137$. But be aware that quantum electrodynamics work is typically done in the other system where we have $e^2/4\pi \approx 1/137$.

2.1 Handy conversions

There are two conversion constants that are especially useful in quantum mechanics computations:

$$1 = c \approx 3 \times 10^8 \text{ m/s}, \quad (9)$$

$$1 = \hbar c \approx 197 \text{ MeV}\cdot\text{fm}. \quad (10)$$

Of course, we also use different scales depending on the problem, for example:

$$1 = 10^5 \text{ fm}/\text{\AA}, \quad (11)$$

$$1 = 10^6 \text{ eV}/\text{MeV}. \quad (12)$$

When magnetic fields are involved, a handy formula to remember is:

$$p(\text{GeV}) = 0.3B(\text{T})\rho(\text{m})Q(\text{e}). \quad (13)$$

This relates the radius of curvature (ρ) which a particle of charge Q and momentum p has in a magnetic field B . The charge unit “e” is the electron charge. The 0.3 in the equation is more precisely the speed of light in nm/s.

2.2 Example: “Microscope Design”

- If an atom has a size $\sim 1 \text{ \AA} = 10^{-10} \text{ m}$, how high a momentum must a probe have, in order to be sensitive to internal structure?

Recall the deBroglie relation for the quantum mechanical wavelength (over 2π) of a particle of momentum p :

$$\lambda = \frac{1}{p}. \quad (14)$$

Thus, for a probe with a wavelength of the same scale as the atomic size, we must have:

$$\begin{aligned} p &= \frac{1}{\lambda} \sim \frac{1}{1 \text{ \AA}} = 10^{10} \text{ m}^{-1} \\ &= 10^{10} \text{ m}^{-1} (10^{-15} \text{ m}/\text{fm}) (200 \text{ MeV}\cdot\text{fm}) (10^3 \text{ keV}/\text{MeV}) \\ &= 2 \text{ keV} \end{aligned} \quad (15)$$

If the probe is a photon, it is in the X-ray regime.

- A nucleus has a size of order 1 fm. How high a momentum probe is required in this case?

$$p = \frac{1}{\lambda} \sim \frac{1}{1 \text{ fm}} 200 \text{ MeV}\cdot\text{fm} \sim 200 \text{ MeV} \quad (16)$$

2.3 Bohr Model of Atom

Even though the Bohr semi-classical model for the atom is incorrect (for one thing, it would seem that the orbiting electron should radiate its energy away), it is a handy model for estimating some general features. We consider here the Bohr model for the 1-electron atom, with nuclear charge Ze . Let m_e be the mass of the electron and m_A be the mass of the nucleus. The features are:

- The electron orbits in the Coulomb potential:

$$V(r) = -\frac{Ze^2}{r}, \quad (17)$$

where $r \equiv |\mathbf{x} - \mathbf{x}_A|$, \mathbf{x} is the electron position, and \mathbf{x}_A is the nucleus position.

Assuming the virial theorem is valid,

$$\langle T \rangle = \frac{1}{2} \left\langle r \frac{\partial V}{\partial r} \right\rangle = \left\langle \frac{Ze^2}{2r} \right\rangle = \frac{1}{2} \langle mv^2 \rangle, \quad (18)$$

where we are assuming that the motion is non-relativistic. We have expressed the kinetic energy, T , in terms of the reduced mass,

$$m = \frac{m_e m_A}{m_e + m_A} \approx m_e, \quad (19)$$

and the relative speed of the electron with respect to the nucleus,

$$v = |\mathbf{v}_e - \mathbf{v}_A|. \quad (20)$$

Alternatively, we could remember the force equation for circular motion:

$$F = \frac{Ze^2}{r^2} = \frac{mv^2}{r}, \quad (21)$$

which leads to the same result. We assume circular orbits, with constant r , and hence $\langle T \rangle = T$.

- The angular momentum is assumed to be quantized:

$$L = |\mathbf{r} \times \mathbf{p}| = n, \quad n = 1, 2, 3, \dots \quad (22)$$

This restricts the possible energy levels. Setting $mvr = n$, and

$$E = T + V = -\frac{1}{2} \frac{Ze^2}{r} = -\frac{1}{2} mv^2, \quad (23)$$

we find $v = Ze^2/n = Z\alpha/n$. The non-relativistic approximation is self-consistent if

$$Z \frac{\alpha}{n} \ll 1. \quad (24)$$

In particular, the ground state energy for hydrogen in this model is

$$\begin{aligned} E &= -\frac{1}{2} (0.511 \times 10^6 \text{ eV}) / (137)^2 \\ &= -13.6 \text{ eV}. \end{aligned} \quad (25)$$

This agrees nicely with experiment! Also, $v = \alpha = 1/137$ in the ground state, and $r = n^2/m\alpha$ leads to a ground state radius of

$$\begin{aligned} r &= \frac{1}{m\alpha} = \frac{137}{m} \\ &= \frac{137}{0.511 \text{ MeV}} 200 \text{ MeV-fm } 10^{-5} \text{ \AA/fm} \approx 0.5 \text{ Angstrom}. \end{aligned} \quad (26)$$

However, the orbital angular momentum in the ground state is $L = 1$ in this model, and that is ultimately not correct.

We notice that the quantities of interest can be described in terms of the ‘‘coupling strength’’ $\alpha = e^2$ and the reduced mass $m \approx m_e$. In other words, we have only one ‘‘scale’’ in the problem: m_e . In addition to this scale, much of atomic physics is then describable in terms of three dimensionless parameters:

1. α (strength of the interaction)
2. Z (number of protons in the nucleus)
3. m_e/m_A (corrections for finite mass of nucleus).

Note that, if m_N is the nucleon (proton, neutron) mass, $m_e/m_N \approx 0.511/940 \sim 1/2000$. This is a small number. Hence, the nucleus is essentially at rest in the atom’s rest frame, since $\mathbf{p}_A = -\mathbf{p}_e$ gives $m_A \mathbf{v}_A = -m_e \mathbf{v}_e$, and thus

$$|\mathbf{v}_A| = \frac{m_e}{m_A} |\mathbf{v}_e|. \quad (27)$$

3 Exercises

1. We have reviewed the Bohr atom briefly. This atom is held together by electromagnetism. We may consider the very similar problem of a gravitationally bound “Bohr atom”. Part of the point of this problem is to give you some practice making calculations with our simple $\hbar = c = 1$ units, so I hope you will attempt your calculations in this spirit. The gravitational potential between two objects of masses m_1 and m_2 is

$$V = -\frac{Gm_1m_2}{r},$$

where $r = |\mathbf{r}_1 - \mathbf{r}_2|$, and

$$G_N = 6.67 \times 10^{-11} \text{ m}^3\text{kg}^{-1}\text{s}^{-2}, \quad (28)$$

$$= 6.71 \times 10^{-39} \text{ GeV}^{-2}, \quad (29)$$

is Newton’s constant.

- (a) With the same quantization condition as for Bohr’s atom, find the formulas for the energy levels, relative velocities, and relative separations, for a “gravity atom”. Let n be the quantum level (*i.e.*, $n = 1, 2, 3, \dots$ is the orbital angular momentum).
- (b) Find n for the earth-sun system. Is quantum mechanics important here?
- (c) What would r be for the ground state of the earth-sun system (give answer in meters)?
- (d) Consider a gravitationally bound system of two neutrons, where we suppose that we have “turned off” other potentially important interactions (such as the strong interaction). What is the ground state energy (in GeV) and the ground state separation (in meters) for this system? Is gravity important compared with other forces for a real system of two neutrons in a real ground state?
- (e) Gravity may be important in a quantum mechanical system if the interaction potential is comparable with other forces. For example, the electromagnetic strength is characterized by $e^2 = \alpha = 1/137$, and the strong interaction (though not strictly Coulombic), is characterized by a larger number, $\alpha_s \approx 1$. Supposing we

have a system of two equal masses, determine the mass (in GeV), such that the corresponding gravitational interaction strength is equal to one. That is, the potential energy should be given simply by $1/r$ (*cf.*, $V = e^2/r$ for two electrons in electromagnetism). [This mass has a name; do you know what it is?]

2. Resonances I: An ensemble of neutron decays, observed from time $t = 0$, will exhibit the characteristic radioactive decay law:

$$N(t) = N(0)e^{-t/\tau}, \quad (30)$$

where $\tau = 886.7 \pm 1.9$ s (Review of Particle Properties, 2000) is the mean lifetime of the neutron (convince yourself that this should be the case). Also, if an ensemble of neutrons were each weighed very precisely, it would be found that the mass distribution has a (small) width. The full width at half maximum (FWHM) of this distribution is typically denoted by Γ . The mean lifetime is inversely related to this width: $\tau = 1/\Gamma$.

- (a) What is the width of the neutron, in electron volts?
 (b) Classical resonances: Let us probe a bit what this “width” means, and what it has to do with decay rate. We’ll start with a classical example: Consider a damped, driven harmonic oscillator:

$$\ddot{x} + \Gamma\dot{x} + \omega_0^2x = \cos\omega t$$

Determine the frequency response of this classical oscillator, *i.e.*, determine the square of the amplitude of oscillation as a function of ω . (Why the square?)

- (c) In the limit of a narrow resonance, what is the full width at half maximum of the distribution you found in part b.
 (d) Now suppose the driving force is turned off. How does the energy stored in the oscillator change with time? Find the “lifetime” of this oscillator. In this classical example, I mean the time it takes for the oscillator to reach e^{-1} of its original energy. Your answer should be very simply related to your answer for part c).
3. The Bohr model for the atom, while wrong, gave some remarkable agreement with experiment, and a means of estimating atomic scales

as long as we didn't push the model too hard. Let's play with another, wrong, model, in this problem, the "plum pudding" model of J.J. Thomson. In this model, the atomic electrons are embedded in a region of neutralizing positive charge. We assume that, within the radius of the atom, the positive charge is uniformly distributed. We consider an atom with atomic number Z , but which has been ionized such that only one electron remains. A simple calculation with Maxwell's divergence equation yields that the electric field inside the atom, due to the positive charge distribution, is linear in radius. The force on the electron can therefore be written as:

$$F_r = -eE_r = -\alpha kr,$$

where $-e$ is the electron charge, $\alpha = e^2$, and k depends on the radius R of the atom, and on Z .

- (a) Write down the Hamiltonian for the electron in this "atom", making sure you define any quantities not already defined above. Assume that any contribution from radii greater than R may be neglected.
- (b) Assuming circular orbits, use the Bohr quantization condition on angular momentum to derive the allowed energy spectrum.
- (c) For the hydrogen atom, the ground state energy is -13.6 eV. Note that we have to be a bit careful now in discussing the energy, since we need to know where our reference (zero) is. Also, our formula for the spectrum in part (b) must have a cut-off somewhere due to the finite atom size. However, we'll circumvent the complication here by considering the difference between the ground state and the first excited state, which for hydrogen is about 10 eV. Using this fact, determine the radius of the ground state orbit, expressing your answer in \AA to one significant digit.

[We really ought to check that your answer to part (c) is consistent with the model, *i.e.*, whether the radius obtained is less than R or not. But I'll leave this to your own amusement.]