

# Harmonic Oscillator:

## Motion in a Magnetic Field

- \* The Schrödinger equation in a magnetic field
  - ⇒ The vector potential
- \* Quantized electron motion in a magnetic field
  - ⇒ Landau levels
- \* The Shubnikov-de Haas effect
  - ⇒ Landau-level degeneracy & depopulation

# The Schrödinger Equation in a Magnetic Field

An important example of harmonic motion is provided by electrons that move under the influence of the **LORENTZ FORCE** generated by an applied **MAGNETIC FIELD**

$$\mathbf{F} = -e\mathbf{v} \times \mathbf{B} \quad (16.1)$$

\* From **CLASSICAL** physics we know that this force causes the electron to undergo **CIRCULAR** motion in the plane **PERPENDICULAR** to the direction of the magnetic field

\* To develop a **QUANTUM-MECHANICAL** description of this problem we need to know how to include the magnetic field into the Schrödinger equation

⇒ In this regard we recall that according to **FARADAY'S LAW** a time-varying magnetic field gives rise to an associated **ELECTRIC FIELD**

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad (16.2)$$

# The Schrödinger Equation in a Magnetic Field

To simplify Equation 16.2 we define a **VECTOR POTENTIAL**  $\mathbf{A}$  associated with the magnetic field

$$\mathbf{B} \equiv \nabla \times \mathbf{A} \quad (16.3)$$

\* With this definition Equation 16.2 reduces to

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} = -\frac{\partial}{\partial t} \nabla \times \mathbf{A} \quad \therefore \quad \mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} \quad (16.4)$$

\* Now the **EQUATION OF MOTION** for the electron can be written as

$$\frac{\partial \mathbf{p}}{\partial t} = -e\mathbf{E} \quad \therefore \quad \hbar \frac{\partial \mathbf{k}}{\partial t} = e \frac{\partial \mathbf{A}}{\partial t} \quad \Rightarrow \quad \boxed{\hbar \mathbf{k}(B)} = \boxed{\hbar \mathbf{k}_0} + e\mathbf{A} \quad (16.5)$$

*1. MOMENTUM IN THE PRESENCE OF THE MAGNETIC FIELD*

*2. MOMENTUM PRIOR TO THE APPLICATION OF THE MAGNETIC FIELD*

# The Schrödinger Equation in a Magnetic Field

Inspection of Equation 16.5 suggests that in the presence of a magnetic field we **REPLACE** the momentum operator in the Schrödinger equation by

$$\mathbf{p} \rightarrow \mathbf{p} + e\mathbf{A} \quad (16.6)$$

\* To incorporate this result into the Schrödinger equation we recall that the first term on the LHS of its ( $B = 0$ ) time-independent form represents the **KINETIC ENERGY**

$$-\frac{\hbar^2}{2m} \nabla^2 \psi(x) + V\psi(x) = E\psi(x) \quad (16.7)$$

*NOTE THE THREE-DIMENSIONAL FORM*

\* Since kinetic energy is related to momentum as  $p^2/2m$  this in turn suggests that we define the **MOMENTUM OPERATOR** as

$$\frac{\hbar}{i} \nabla = \frac{\hbar}{i} \left[ \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right] \quad (16.8)$$

# The Schrödinger Equation in a Magnetic Field

With our definition of the momentum operator at **ZERO** magnetic field (Equation 16.8) we now use Equation 16.6 to obtain the momentum operator in the **PRESENCE** of a magnetic field

$$\left[ \frac{\hbar}{i} \nabla + e\mathbf{A} \right] \quad (16.9)$$

\* With this definition we may now **REWRITE** the Schrödinger equation in a form that may be used to describe the motion of electrons in a magnetic field

$$\frac{1}{2m} \left[ \frac{\hbar}{i} \nabla + e\mathbf{A} \right]^2 \psi(x) + V\psi(x) = E\psi(x) \quad (16.10)$$

⇒ The first term in the brackets on the LHS of this equation is known as the **CANONICAL** momentum and is the momentum in the absence of a magnetic field

⇒ The entire term in brackets is called the **MECHANICAL** or **KINEMATIC** momentum and corresponds to the **KINETIC ENERGY** of the electron

# Quantized Electron Motion in a Magnetic Field

We now apply the results of the preceding analysis to describe the motion of electrons in a magnetic field

\* We assume that this magnetic field is **CONSTANT** and points in the  $z$ -direction

$$\mathbf{B} = B\hat{\mathbf{z}} \quad (16.11)$$

\* **ONE** possible choice of vector potential that satisfies this equation is known as the **LANDAU GAUGE** and is given by (you can check this using Equation 16.3)

$$\mathbf{A} = Bx\hat{\mathbf{y}} \quad (16.12)$$

\* With this choice of gauge the Schrödinger equation now becomes

$$\left[ -\frac{\hbar^2}{2m} \nabla^2 - i \frac{e\hbar Bx}{m} \frac{\partial}{\partial y} + \frac{(eBx)^2}{2m} \right] \psi = E\psi \quad (16.13)$$

# Quantized Electron Motion in a Magnetic Field

Equation 16.12 reveals that the magnetic field produces **TWO** effects

$$\left[ -\frac{\hbar^2}{2m} \nabla^2 \underset{1}{-i \frac{e\hbar Bx}{m} \frac{\partial}{\partial y}} + \underset{2}{\frac{(eBx)^2}{2m}} \right] \psi = E \psi \quad (16.13)$$

\* The first effect is a derivative that **COUPLES** the motion in the  $x$ - and  $y$ -directions as we would **EXPECT** for a particle that undergoes **CIRCULAR** motion in the  $xy$ -plane

\* The second effect is that the magnetic field generates a **PARABOLIC MAGNETIC POTENTIAL** of the form that we have studied for the harmonic oscillator!

\* Now since the form of the vector potential we have chosen does **NOT** depend on  $y$  this suggests that we write the **WAVEFUNCTION** solutions for the electrons as

$$\psi(x, y) = u(x) e^{ik_y y} \quad (16.14)$$

**NOTE HOW THE  $y$ -COMPONENT CORRESPONDS TO A FREELY-MOVING PARTICLE**

# Quantized Electron Motion in a Magnetic Field

Substitution of our wavefunction into the Schrödinger equation

$$\left[ -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + \frac{1}{2} m \omega_c^2 \left[ x + \frac{\hbar k_y}{eB} \right]^2 \right] u(x) = E u(x) \quad (16.15)$$

\* This is just the Schrödinger equation for a one-dimensional **HARMONIC OSCILLATOR** with the magnetic-field dependent **CYCLOTRON FREQUENCY**

$$\omega_c = \frac{eB}{m} \quad (16.16)$$

\* An important difference with our previous analysis however is that the **CENTER** of the parabolic potential is **NOT** located at  $x = 0$  but rather at

$$x_k = -\frac{\hbar k_y}{eB} \quad (16.17)$$

# Quantized Electron Motion in a Magnetic Field

Solution of the Schrödinger equation yields a quantized set of energy levels known as **LANDAU LEVELS**

$$E_n = \left[ n + \frac{1}{2} \right] \hbar \omega_c \quad (16.18)$$

\* The wavefunctions are the usual **HERMITE POLYNOMIALS** and may be written as

$$\psi(x, y) = u(x) e^{ik_y y} = \sqrt{\frac{1}{2^n n! \sqrt{l_B}}} \exp\left[-\frac{(x-x_k)^2}{2l_B^2}\right] H_n\left[\frac{x-x_k}{l_B}\right] e^{ik_y y} \quad (16.19)$$

⇒ where we have defined the **MAGNETIC LENGTH**  $l_B$  as

$$l_B = \sqrt{\frac{\hbar}{eB}} \quad (16.20)$$

# The Shubnikov-de Haas Effect

- An important property of the quantized Landau levels is that they are highly **DEGENERATE**

$$E_n = \left[ n + \frac{1}{2} \right] \hbar \omega_c \quad (16.18)$$

\* By this we mean that each Landau level is able to hold a **LARGE** number of electrons

\* To obtain an expression for this degeneracy we begin by assuming that the circular motion of the electrons occurs in a plane with dimensions  $L_x \times L_y$

⇒ By assuming **PERIODIC** boundary conditions along the  $y$ -direction (ECE 352) we may write a quantization condition for the wavenumber  $k_y$

$$k_y = \frac{2\pi}{L_y} j, \quad j = 0, 1, 2, \dots \quad (16.21)$$

**WHEN A PARTICLE IS **CONFINED** IN A ONE-DIMENSIONAL BOX OF LENGTH  $L_y$  ITS ALLOWED WAVENUMBERS ARE **QUANTIZED** ACCORDING TO EQUATION 16.21**

# The Shubnikov-de Haas Effect

Since the **CENTER COORDINATE** of the harmonic oscillator must lie somewhere within the sample we may use Equation 16.21 to write the following condition

$$-L_x < x_k < 0 \quad \therefore \quad 0 < j < \frac{eBL_xL_y}{h} \quad (16.22)$$

\* According to Equation 16.22 **EACH** Landau level contains the same number of states at any given magnetic field

⇒ The number of states in each level **PER UNIT AREA** of the sample is just given by

$$\frac{j_{Max}}{L_xL_y} = \frac{eB}{h} \quad (16.23)$$

⇒ Since each state within the Landau level can hold **TWO** electrons with **OPPOSITE** spins the number of **ELECTRONS** that can be held in each Landau level is

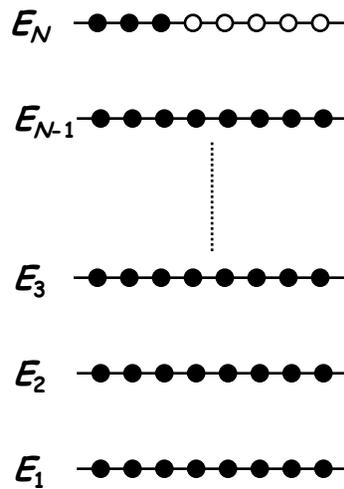
$$n = \frac{2eB}{h} \quad (16.24)$$

# The Shubnikov-de Haas Effect

Now let us consider what happens to a sample containing a **FIXED** number of electrons as we **VARY** the magnetic field

\* Starting at some **INITIAL** magnetic field a specific number  $N$  of Landau levels will be occupied by electrons

⇒  $(N - 1)$  of these levels will be filled completely while the  $N^{\text{th}}$  will typically be **PARTIALLY** filled with the remaining electrons that cannot be accommodated in the lower levels



• FILLING OF **LANDAU LEVELS** BY A **FIXED** NUMBER OF ELECTRONS AT AN ARBITRARY MAGNETIC FIELD

• EACH LANDAU LEVEL IS CAPABLE OF HOLDING THE **SAME** NUMBER OF ELECTRONS AND THESE LEVELS WILL BE FILLED IN A MANNER THAT **MINIMIZES** THE **TOTAL ENERGY** OF THE SYSTEM

• BECAUSE OF THIS AT ANY MAGNETIC FIELD THE LOWEST **(N-1)** LANDAU LEVELS WILL BE **COMPLETELY** FILLED BY ELECTRONS ACCOUNTING FOR  $(N-1)n$  ELECTRONS

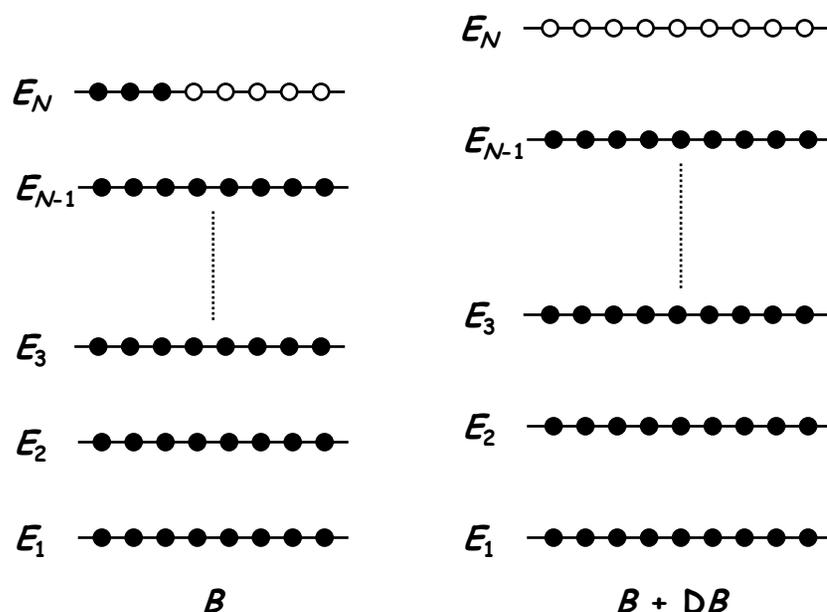
• THE REMAINING ELECTRONS WILL BE ACCOMMODATED IN THE **PARTIALLY-FILLED** UPPERMOST LEVEL

# The Shubnikov-de Haas Effect

If we now raise the magnetic field we increase the **ENERGY SPACING** of the Landau levels and also increase their **DEGENERACY**

\* Since more states are available in each level electrons **DROP** from higher levels to occupy empty states in the lower levels

\* Consequently we eventually reach a point where the  $N^{\text{th}}$  Landau level is **COMPLETELY** emptied of electrons and the number of occupied Landau levels is now just  $N-1$



• **EMPTYING OF THE UPPERMOST OCCUPIED LANDAU LEVEL** IN AN **INCREASING** MAGNETIC FIELD

• WITH INCREASING MAGNETIC FIELD THE **SPACING** OF THE LANDAU LEVELS **INCREASES** BUT THE **NUMBER OF ELECTRONS HELD BY EACH LEVEL** ALSO **INCREASES**

• AS ELECTRONS **DROP** TO FILL NEW STATES THAT BECOME AVAILABLE WITH INCREASING FIELD THE **UPPERMOST LANDAU LEVEL** EVENTUALLY **EMPTIES**

• THIS PROCESS IS REFERRED TO AS **MAGNETIC DEPOPULATION OF LEVELS**

# The Shubnikov-de Haas Effect

The **MAGNETIC DEPOPULATION** we have described **CONTINUES** with increasing magnetic field until all electrons occupy only the **LOWEST** Landau level at **VERY HIGH** magnetic fields

\* To obtain an expression for the magnetic field values at which the depopulations occur we consider a sample containing  $n_s$  electrons **PER UNIT AREA**

\* Since  $n$  is the number of electrons per unit area that occupy **EACH** Landau level we require those values of the magnetic field for which the following ratio is an **INTEGER**

$$\frac{n_s}{n} = \frac{n_s h}{2eB} \equiv N_L, \quad N_L = 1, 2, 3, \dots \quad (16.25)$$

⇒ This relation shows that as expected the number of occupied Landau levels **DECREASES** with increasing magnetic field

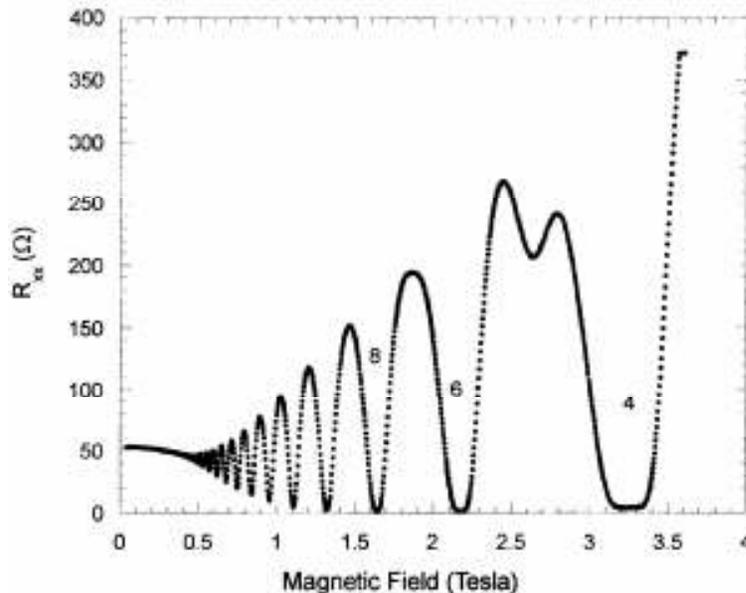
⇒ It also shows that an increasingly **LARGER** magnetic field increment is required to depopulate successively **LOWER** Landau levels

# The Shubnikov-de Haas Effect

The depopulation of Landau levels can actually be seen in the **MAGNETO-RESISTANCE** of semiconductors which **OSCILLATES** at low temperatures

\* The period of the oscillations **INCREASES** with magnetic field as expected from Equation 15.24 and the oscillations appear **PERIODIC** when plotted on an **INVERSE-FIELD** scale

⇒ The periodicity of these **SHUBNIKOV-DE HAAS** oscillations is often used in experiment as a means to determine the electron **CARRIER DENSITY**



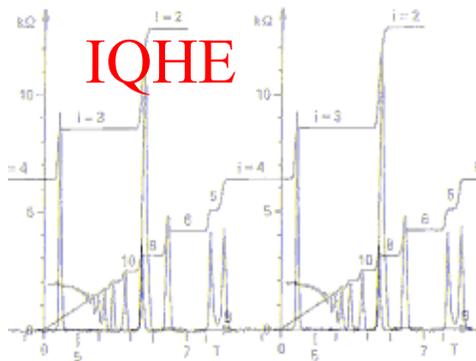
• **SHUBNIKOV-DE HAAS OSCILLATIONS** MEASURED IN A **GaAs/AlGaAs HETEROJUNCTION** AT 4 K

• THE NUMBERS ON THE FIGURE INDICATE THE NUMBER OF **OCCUPIED** LANDAU LEVELS AT SPECIFIC VALUES OF THE MAGNETIC FIELD

• THE **SPLITTING** OF THE PEAK IN THE REGION OF AROUND 2.5 T RESULTS AS THE MAGNETIC FIELD BEGINS TO LIFT THE **SPIN DEGENERACY** OF THE ELECTRONS

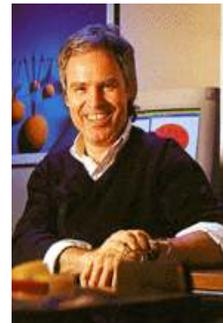
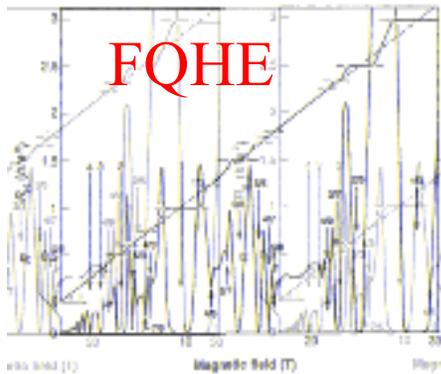
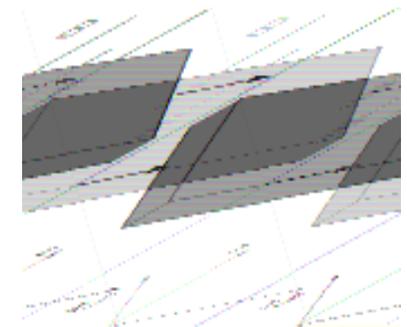
• **QUANTUM MECHANICS**, D. K. FERRY, IOPP (2001)

# quantum Hall history



**K. v. Klitzing**

discovery: 1980  
Nobel prize: 1985

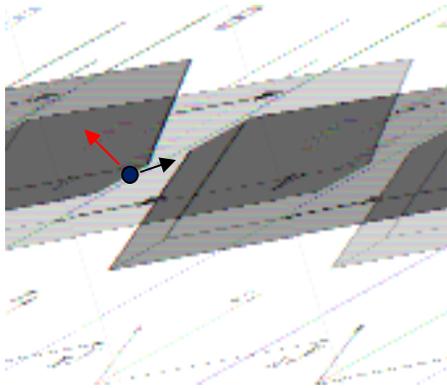


**D. Tsui H. Störmer R. Laughlin**

discovery: 1982  
Nobel prize: 1998

# classical Hall effect (1880 E.H. Hall)

Lorentz-force on electron:



~~$$m\vec{v} \Rightarrow e(\vec{E} + \vec{v} \times \vec{B})$$~~
~~$$m\vec{v} \Rightarrow e(\vec{E} + \vec{v} \times \vec{B})$$~~

stationary current:

~~$$j_x = j_y \frac{e\tau}{B} \frac{E_x}{B}$$~~
~~$$j_x = j_y \frac{e\tau}{B} \frac{E_x}{B}$$~~

Hall resistance:

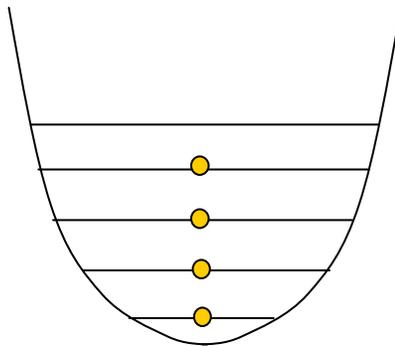
$$r_H = \frac{E_y}{j_x} = \frac{1}{\nu} \frac{2\pi\hbar}{e} = \frac{E_y}{j_x} = \frac{1}{\nu} \frac{2\pi\hbar}{e}$$

$$\frac{1}{\nu} = \frac{eB}{2\pi\hbar Q} = \frac{2\pi\hbar}{\Phi_{BQ}} \quad \frac{1}{\nu} = \frac{cB}{2\pi\hbar Q} = \frac{2\pi\hbar}{\Phi_{DQ}}$$

Dirac flux quantum

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# Landau levels



# 2D electrons in magnetic fields: Landau levels

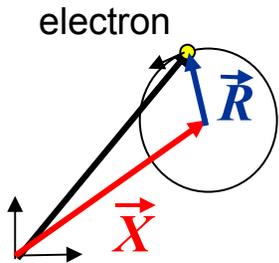
Hamiltonian

$$\hat{H} = \frac{1}{2m} \left[ (p_x - eAy)^2 + p_y^2 \right] + \frac{1}{2m} \left[ (p_x + eAy)^2 + p_y^2 \right]$$

coordinate transformation:

$$(x, p_x, y, p_y) \rightarrow (x, p_x, Y, p_y)$$

center of cyclotron motion      radial vector of cyclotron motion



$$X = y_0 - R_{y_0} \quad Y = y_0 - R_{y_0}$$

$$\frac{p_x + eAy}{eB} = R_x - R_y \quad \frac{p_x - eAy}{eB} = R_x - R_y$$

commutation relations

$$[X, Y] = -i \ell_m^2 \quad [Y, R_x] = i \ell_m^2 \quad [R_x, R_y] = -i \ell_m^2$$

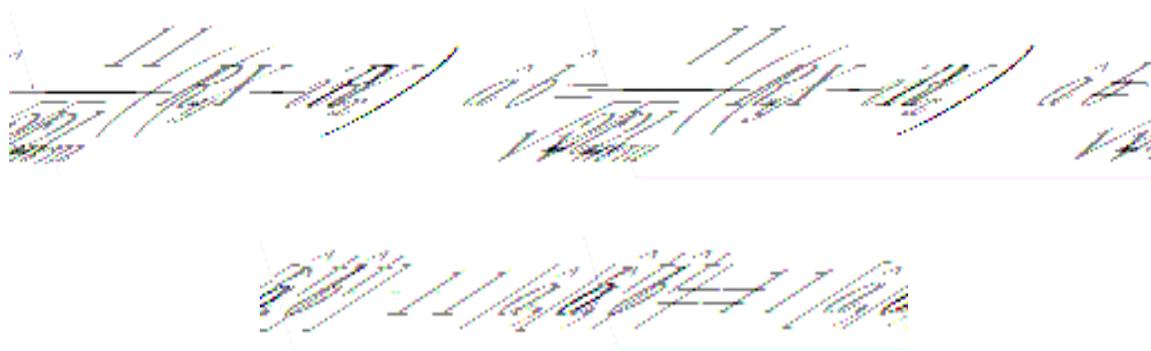
$$[X, R_x] = i \ell_m^2 \quad [Y, R_y] = i \ell_m^2 \quad [R_x, R_y] = -i \ell_m^2$$

$$[R_x, R_y] = -i \ell_m^2$$

# 2D electrons in magnetic fields: Landau levels

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mapping to oscillator:



$$H = \hbar \omega_c \left( \frac{R^2}{2 l_B^2} + \frac{1}{2} \right) = \hbar \omega_c \left( a^\dagger a + \frac{1}{2} \right)$$

**Landau levels**



# 2D electrons in magnetic fields: Landau levels

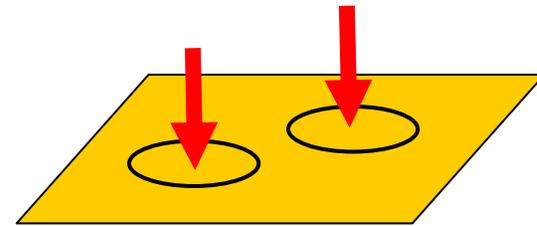
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degeneracy of Landau levels:

center of cyclotron motion  $(X, Y)$  arbitrary  $\rightarrow$  degeneracy

- 2D density of states (DOS)

$$\frac{2}{m} \frac{eB}{2\pi\hbar} \frac{1B}{2\pi\hbar} = \frac{eB}{2\pi\hbar} \frac{1B}{2\pi\hbar}$$



one state per area of cyclotron orbit

- filling factor

$$\frac{2}{m} \frac{\rho\Phi_D}{eBs} \frac{N}{2\pi\hbar} \frac{1}{N\Phi} = \frac{\rho\Phi_D}{eBs} = \frac{N}{2\pi\hbar}$$

# atoms / # flux quanta

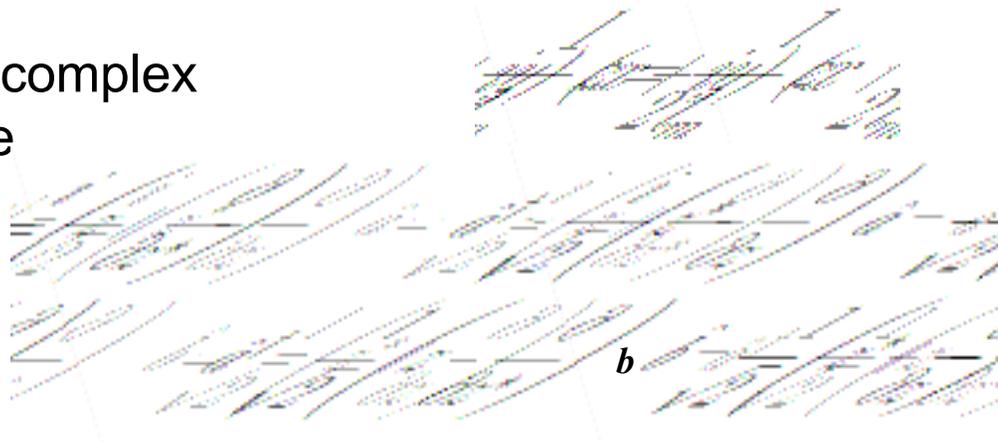
# 2D electrons in magnetic fields: Landau levels

wavefunction of lowest Landau level (LLL) in symmetric gauge

$$\left( \begin{array}{c} A_x A_y \frac{1}{2} B y \frac{1}{2} B x \\ A_x A_y \frac{1}{2} B y \frac{1}{2} B x \end{array} \right) \left( \begin{array}{c} A_x A_y \frac{1}{2} B y \frac{1}{2} B x \\ A_x A_y \frac{1}{2} B y \frac{1}{2} B x \end{array} \right)$$

symmetric  
gauge  
Landau gauge

introduce complex  
coordinate



LLL

$$\left( \hat{a} \frac{\partial}{\partial z^*} \right) \psi_0 \frac{i}{\sqrt{2}} \left( \hat{a} \frac{\partial}{\partial z^*} \right) \psi_0 \frac{i}{\sqrt{2}} \left( \hat{a} \frac{\partial}{\partial z^*} \right) \psi_0 \frac{i}{\sqrt{2}} \left( \hat{a} \frac{\partial}{\partial z^*} \right) \psi_0 \frac{i}{\sqrt{2}}$$

$$\psi_0 = M(z) e^{-|z|^2/2} \quad \psi_0 = M(z) e^{-|z|^2/2} \quad \text{analytic}$$

# 2D electrons in magnetic fields: Landau levels

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angular momentum of Landau levels:



eigenstates of  $n$ 'th Landau level:

$$\langle \hat{a}^\dagger \rangle^n \langle \hat{a} \rangle^{l+n} \sqrt{\frac{1}{n!(n+l)!}} \langle \hat{a}^\dagger \rangle^n \langle \hat{a} \rangle^{l+n} \sqrt{\frac{1}{n!(n+l)!}} \langle \hat{a}^\dagger \rangle^n \langle \hat{a} \rangle^{l+n} \sqrt{\frac{1}{n!(n+l)!}} \langle \hat{a}^\dagger \rangle^n \langle \hat{a} \rangle^{l+n} \sqrt{\frac{1}{n!(n+l)!}}$$

$$l = \hbar \omega_c (n + 1/2) \quad \hbar \omega_c (n + 1/2)$$

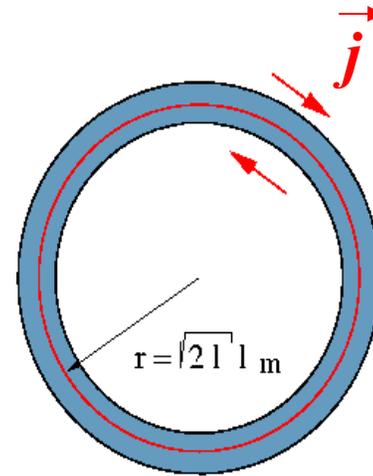
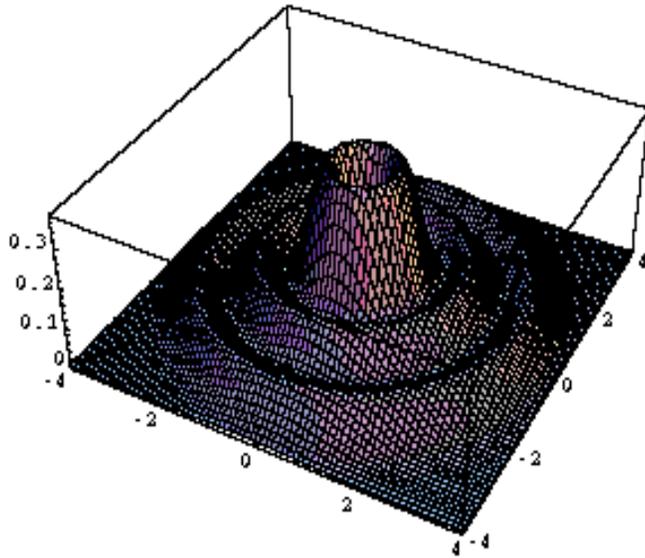
angular momentum states of LLL:

$$\frac{1}{\sqrt{2\pi}} \langle \hat{a}^\dagger \rangle^l \langle \hat{a} \rangle^l \frac{1}{\sqrt{2\pi}} \langle \hat{a}^\dagger \rangle^l \langle \hat{a} \rangle^l$$

# 2D electrons in magnetic fields: Landau levels

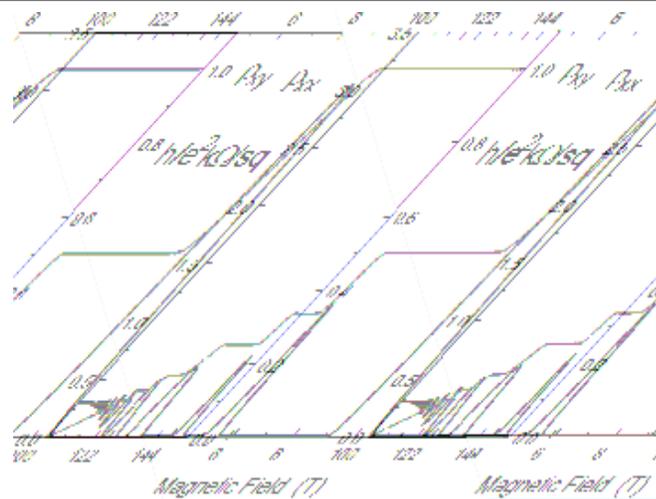
wavefunction:

$$\frac{1}{\sqrt{2\pi l_B}} e^{-z^2/4l_B^2} \frac{1}{\sqrt{2\pi l_B}} e^{-z^2/4l_B^2}$$



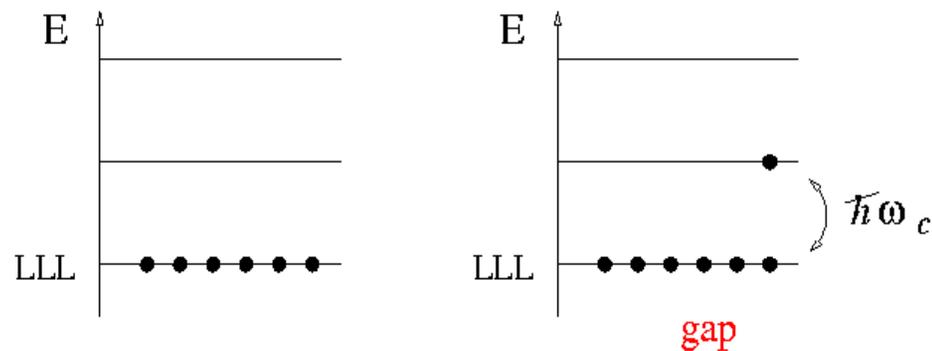
$$\frac{1}{\sqrt{2\pi l_B}} e^{-z^2/4l_B^2} \frac{1}{\sqrt{2\pi l_B}} e^{-z^2/4l_B^2}$$

# Integer Quantum Hall effect



# Integer Quantum Hall effect

spinless (for simplicity) and noninteracting electrons: *Pauli principle*



Slater determinant:

$$\begin{aligned}
 & \psi_1(1) \quad \psi_{N-1}(1) \quad \psi_1(1) \quad \psi_{N-1}(1) \quad \dots \\
 & \psi_2(1) \quad \dots \quad \psi_{N-1}(1) \quad \psi_1(1) \quad \dots \quad \psi_{N-1}(1) \quad \psi_1(1) \quad \dots \\
 & \vdots \\
 & \psi_1(N) \quad \psi_{N-1}(N) \quad \dots \quad \psi_{N-1}(N) \quad \psi_1(N) \quad \dots \\
 & \dots
 \end{aligned}$$

$$\frac{1}{\sqrt{N!}} \det \begin{pmatrix} \psi_1(1) & \dots & \psi_{N-1}(1) \\ \vdots & & \vdots \\ \psi_1(N) & \dots & \psi_{N-1}(N) \end{pmatrix}$$

$$\exp \left\{ - \sum_{r=1}^N \sum_{i < j} \frac{1}{|r-i|} \right\}$$

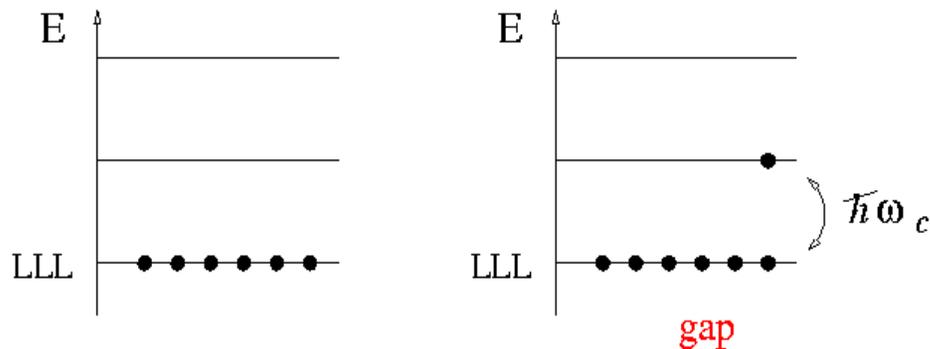
# Integer Quantum Hall effect

compressibility:

$$\begin{aligned} \frac{\partial^2 E}{\partial A^2} &= A \rho^2 \frac{\partial^2 E}{\partial N^2} & \frac{\partial^2 E}{\partial A^2} &= A \rho^2 \frac{\partial^2 E}{\partial N^2} = \\ &= A \rho^2 \frac{d\mu}{dN} & &= A \rho^2 \frac{d\mu}{dN} \end{aligned}$$

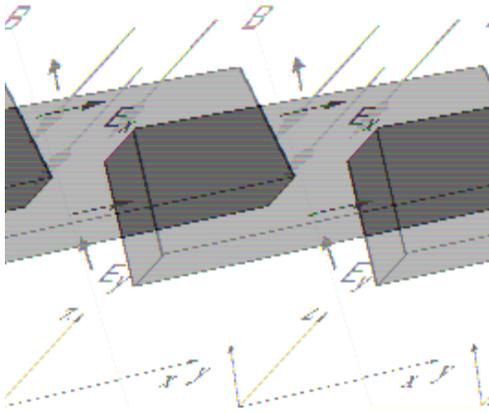
at integer fillings:

$$\frac{1}{\nu} \circ \circ \frac{1}{\nu} \circ \circ -$$



# Integer Quantum Hall effect

Hall current:



$$\frac{e v_x}{2m} (\hbar^2 y + \hbar^2) \quad \frac{e v_y}{2m} (\hbar^2 x + \hbar^2)$$

Heisenberg drift equations of cyclotron center

$$i\hbar \frac{d}{dt} X = \frac{i\hbar}{B} F_y X \quad i\hbar \frac{d}{dt} X = \frac{i\hbar}{B} F_x X$$

$$F_x [Y, \mathbb{H}] = \frac{d}{dt} Y \frac{i\hbar}{B} \quad F_y [Y, \mathbb{H}] = \frac{d}{dt} Y \frac{i\hbar}{B}$$

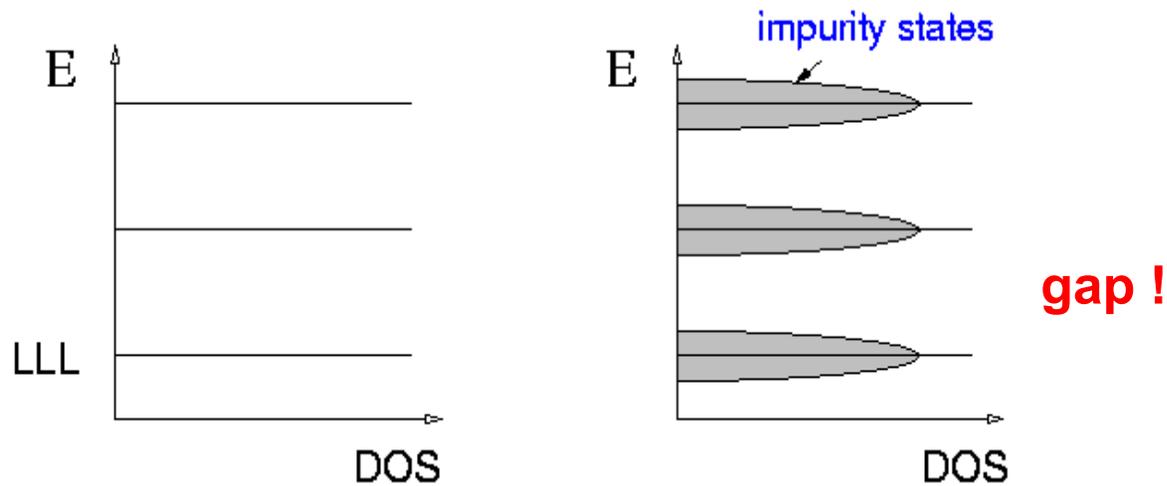
$$\frac{e v_x}{B} = \frac{e v_y}{B} = \frac{e v_z}{B}$$

$$\frac{e v_x}{B} = 0 \quad \frac{e v_y}{B} = 0 \quad \frac{e v_z}{B} = 0$$

*no plateaus*  
?!

# Integer Quantum Hall effect

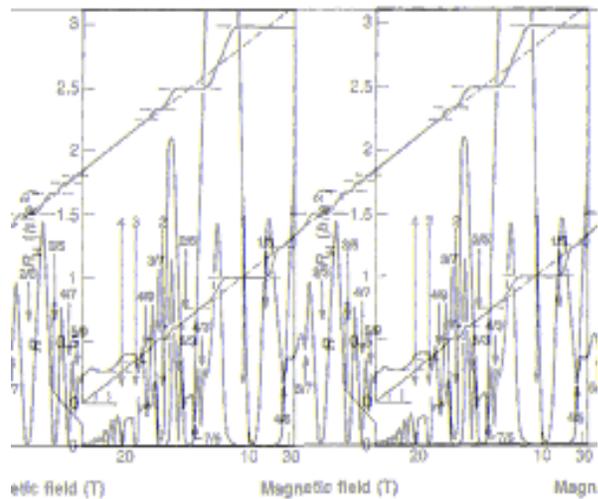
## Hall plateaus: impurities



**impurities** pin electrons to localized states  
⇒ electrons in impurity states do not contribute to current

**gap**  
⇒ impurity states fill first

# Fractional Quantum Hall effect



# Fractional Quantum Hall effect

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## Laughlin state:

- take e-e interaction into account
- generic wavefunction
- requirements
  - wave function antisymmetric
  - eigenstate of angular momentum
  - Coulomb repulsion  $\Rightarrow$  Jastrow-type of wave function







**Laughlin wave function**

# Fractional Quantum Hall effect

angular momentum of Laughlin wave function and filling factor

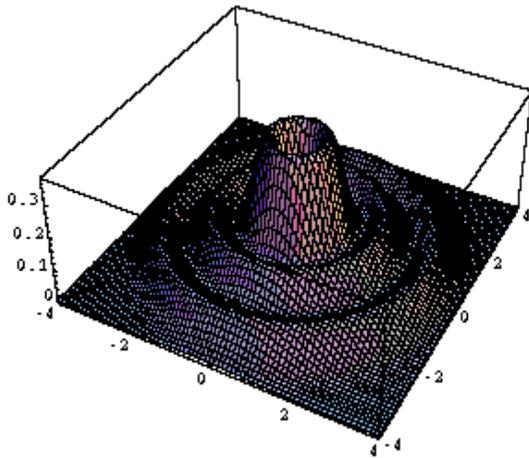
$$\psi \sim \frac{1}{\sqrt{N!}} \prod_{i < j}^N (z_i - z_j)^2 e^{-\sum_{i=1}^N |z_i|^2/4l_B^2}$$

$$L_z \psi = (L_{z1} + L_{z2} + \dots + L_{zN}) \psi = (l_B^2/2) (2 + 2 + \dots + 2) \psi = (N-1)l_B^2 \psi$$

maximum single-particle angular momentum

$$L_{zmax} = (N-1)l_B^2$$

$$\nu = \frac{N}{(N_{DS}-1)l_B^2} = \frac{N}{(N-1)l_B^2} = \frac{N}{N-1} \approx 1$$



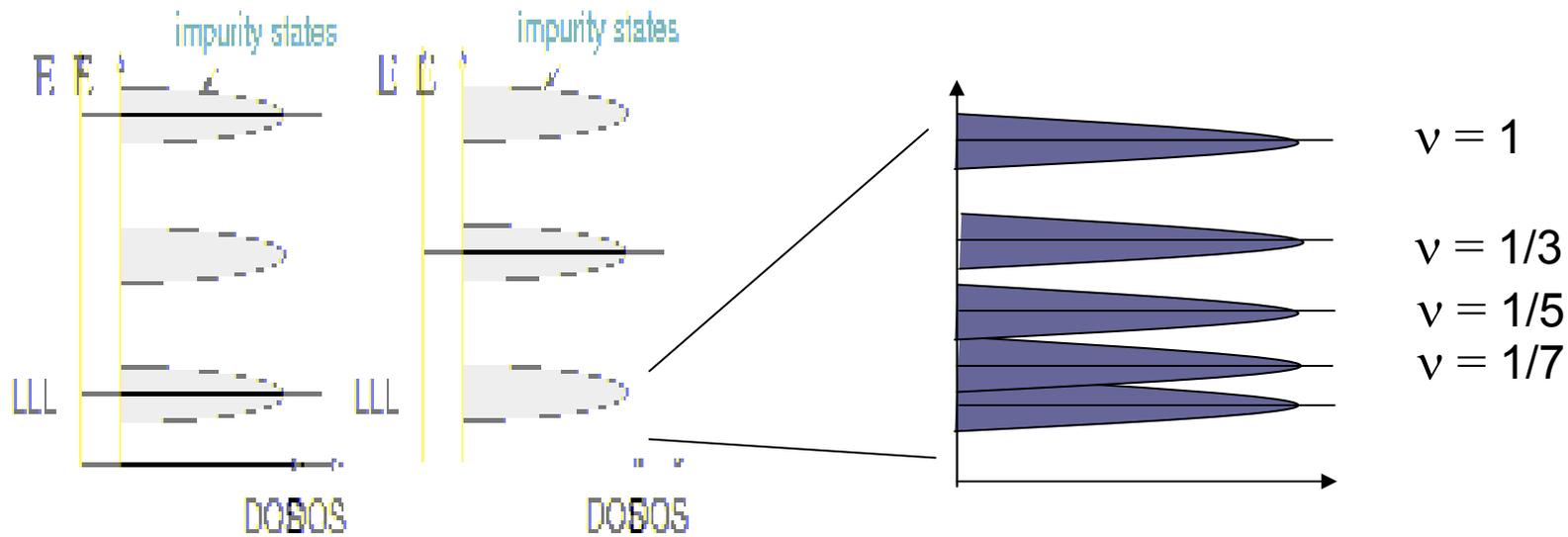
filling factor of Laughlin state

$$\nu = \frac{qN}{(N_{DS}-1)ml} = \frac{N}{(N_{DS}-1)ml} = \frac{N}{(N-1)ml} \approx \frac{1}{m}$$

# Fractional Quantum Hall effect

## fractional Hall plateaus:

fractional Hall states are gapped



# composite particle picture of FQHE

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composite particle = electron +  $m$  magnetic flux quanta



$m = 2p$  → composite fermion

$m = 2p + 1$  → composite boson

effective magnetic field

$$\cancel{B} \Phi_D \rightarrow B B_{\text{eff}} \Phi_D = B B_{\text{eff}}$$

composite particles are **anyons** (fractional statistics) exist only in 2D

# composite particle picture of FQHE

## some remarks about anyons:

- two-particle wave function  $\Psi(1, 2)$

- exchange particles

$$\frac{1}{\sqrt{2}}(\Psi(1, 2) \pm \Psi(2, 1))$$

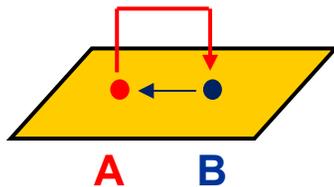
- exchange particles a second time

$$e^{i\alpha} \frac{1}{\sqrt{2}}(\Psi(1, 2) \pm \Psi(2, 1))$$

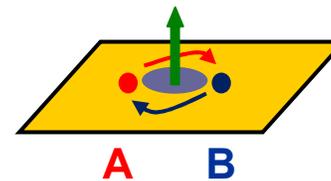
$\Rightarrow$  in 3D:  $\frac{1}{\sqrt{2}}(\psi(\mathbf{r}_1, \mathbf{r}_2) \pm \psi(\mathbf{r}_2, \mathbf{r}_1))$  **Boson**

$\frac{1}{\sqrt{2}}(\psi(\mathbf{r}_1, \mathbf{r}_2) - \psi(\mathbf{r}_2, \mathbf{r}_1))$  **Fermion**

3D: no projected area in (xy)



2D always projected area in (xy)



particles can pick up e.g. Aharonov-Bohm phase

# composite particle picture of FQHE

$\nu = 1/m$  FQE

(A) electron + ~~flux quanta~~ flux quanta

form composite boson  $\nu = \frac{1}{2p+1}$

**Bose condensation of composite bosons**

(B) electron + ~~flux quanta~~ flux quanta

form composite fermion  $\nu = \frac{1}{2p+1}$

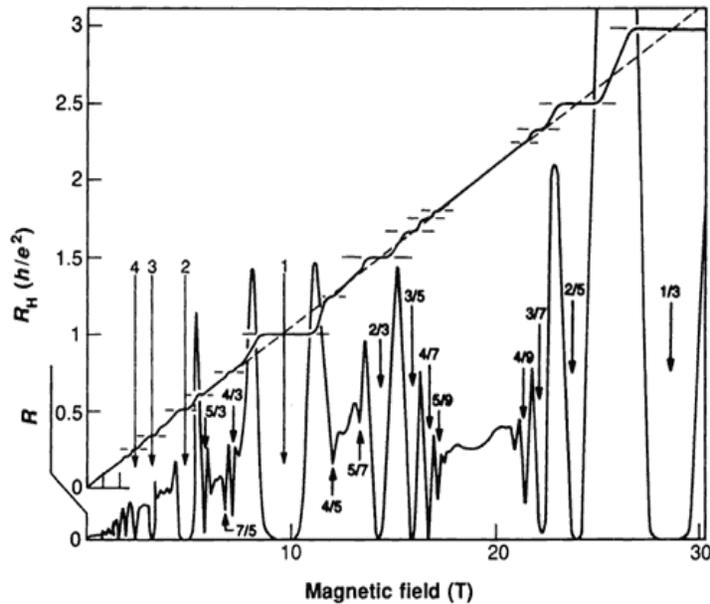
$$\frac{1}{m} \frac{\rho \Phi_D}{2p+1} = \frac{1}{m} \frac{\rho \Phi_D}{2p+1} \Rightarrow \frac{\rho \Phi_D}{B_{\text{eff}}} = m\nu (2p+1) \frac{\rho \Phi_D}{B_{\text{eff}}}$$

**IQHE for composite fermions**

# composite particle picture of FQHE

## Jain hierarchy:

- experiment: FQHE also for  $\frac{n}{n \pm 12p} = \frac{n}{n \pm 12p}$



composite fermion picture:

$$\frac{\nu_{eff}}{\nu_{eff} + 2p} = \frac{\nu_{eff}}{\nu_{eff} + 2p} = \frac{\nu_{eff}}{\nu_{eff} + 2p}$$

since

$$\nu_{eff} = \nu_{eff} - 2p \nu_{eff} \Rightarrow \nu_{eff} = \frac{\nu_{eff}}{1 + 2p}$$

$\Rightarrow$

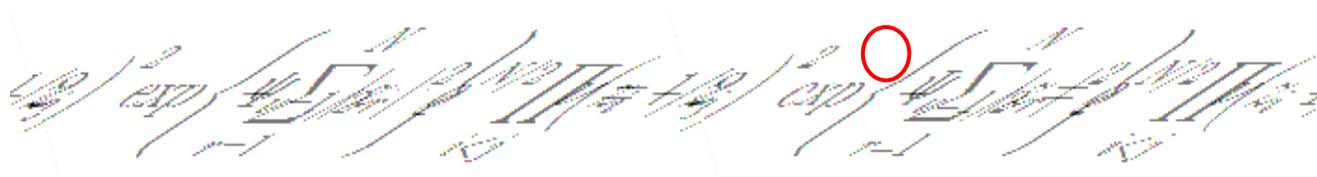
$$\frac{\nu_{eff}}{\nu_{eff} + 2p} = \frac{\nu_{eff}}{\nu_{eff} + 2p}$$

# FQHE for interacting bosons

exact diagonalization  $\Rightarrow$  FQH effect for

$$\frac{n}{\nu} \equiv \frac{n}{\pm 1} - \frac{n}{\pm 1} + \frac{n}{n}$$

**Laughlin state** for point interaction



**composite fermions:**

boson + single flux quantum



$$\frac{\nu_{\text{eff}}}{\nu} = \frac{\nu_{\text{eff}}}{\nu} - \frac{\nu_{\text{eff}}}{\nu} + \frac{\nu_{\text{eff}}}{\nu}$$

**IQHE for composite fermions**

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Thanks!

