

Interactive learning tutorials on quantum mechanics

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We discuss the development and evaluation of quantum interactive learning tutorials (QuILTs), which are suitable for undergraduate courses in quantum mechanics. QuILTs are based on the investigation of student difficulties in learning quantum physics. They exploit computer-based visualization tools and help students build links between the formal and conceptual aspects of quantum physics without compromising the technical content. They can be used both as supplements to lectures or as self-study tools. © 2008 American Association of Physics Teachers.

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I. INTRODUCTION

Quantum physics is a difficult and abstract subject,¹ and students struggle to master the basic concepts.^{2–13} In this paper I discuss the development and evaluation of quantum interactive learning tutorials (QuILTs) that help advanced undergraduate students learn quantum mechanics. QuILTs are designed to create an active learning environment in which students have an opportunity to confront their misconceptions, interpret the material learned, draw qualitative inferences from quantitative tools learned from the text, and build links between new material and prior knowledge. They are designed to be easy to implement regardless of the lecturer's teaching style and can be used in class as supplements to lectures or outside of the class as homework or self-study tools by students.

An important aspect of QuILTs is that they are research-based and target specific student difficulties and misconceptions.^{3–13} They often employ computer-based visualization tools^{14–18,20} to help students build their intuition about quantum concepts and keep students engaged in the learning process by asking them to predict what should happen in a particular situation, and then providing appropriate feedback. They attempt to bridge the gap between the abstract quantitative formalism of quantum mechanics and the qualitative understanding necessary to explain and predict diverse physical phenomena.

II. DETAILS OF THE QUILTS

The QuILTs use a learning cycle approach¹⁹ in which students engage in the topic via examples that focus their attention, explore the topic through facilitated questions and observation, explain what they have learned with the instructor facilitating further discussion to help refine their understanding, and extend what they have learned by applying the same concepts in different contexts. The guidance provided by the tutorials is decreased gradually.

In addition to the main tutorial, QuILTs often have a “warm-up” component and tutorial “homework.” Students work on the warm-up component of a QuILT prior to the main tutorials in class. These warm-ups typically review the prior knowledge necessary for optimizing the benefits of the main tutorial. The tutorial homework associated with a QuILT can be given as part of homework to reinforce concepts after students have worked on the main tutorial. The tutorial homework helps students apply the topic of a par-

ticular tutorial to many situations so that they learn about its diverse applications and generalize the concept appropriately.

We have designed a pretest and post-test to accompany each tutorial. The pretest assesses students' knowledge before they have worked on the corresponding tutorial, but typically after the lecture on the relevant concepts. The QuILT along with the pretest often makes students' difficulties with the relevant concepts clear, not only to the instructors but also to the students. The pretest can also have motivational benefits and help students better focus on the concepts covered in the tutorial that follows it. Pretest and post-test performances are also useful for refining and modifying the QuILT.

An integral component of the tutorials is the adaptation of visualization tools for helping students develop intuition. A visualization tool can be made more pedagogically effective if it is embedded in a learning environment such as QuILT. A simulation, preceded by a prediction and followed by questions, can help students reflect on what they visualized. Such reflection can be useful for understanding and remembering concepts.

We have adapted simulations from a number of sources.^{14,16–18,20} Some of the tutorials, for example, the one on the double-slit experiment,¹⁶ are also appropriate for courses on atomic physics. The double-slit tutorial uses simulations to teach students about the wave nature of particles, the importance of the phase of the probability amplitude for the occurrence of the interference pattern, and the connection between having information about which slit a “particle” went through (“which-path” information) and the loss of an interference pattern.

For the tutorials based on simulations, students need to first make predictions about what and why they expect a certain outcome in a particular situation before exploring the relevant concepts with the simulations. For example, students learn about the stationary states of a single particle in various types of potential wells. Students can change the model parameters and learn how they affect the stationary states and the probability of finding the electron at a particular position. They can also consider various linear combinations of stationary states to learn how the probability of finding the electron at a particular position is affected. Students can calculate and compare the expectation values of various operators in different states for a given potential. They can also better appreciate why classical physics can be a good approximation under certain conditions. Students can also develop intuition about the differences between bound states

Table I. Pre/post-test performance of undergraduates in advanced quantum mechanics at the University of Pittsburgh on the latest version of the three tutorials discussed.

Tutorial	Number of students	% Pretest Score	% Post-test Score
Time development of wave function	9	53	85
Uncertainty principle	12	42	83
Mach–Zehnder interferometer	12	48	83

and scattering states by using visual simulations. Guided visualization tools can also help students understand the new phenomena that occur when a system containing one particle is extended to many particles.²⁰

Similar to the development of tutorials for introductory and modern physics,^{21,22} the development of each QuILT goes through a cyclical iterative process. Preliminary tutorials are developed based on common difficulties in learning a particular topic,^{3–8,13} and how that topic fits within the overall structure of quantum mechanics. The preliminary tutorials are implemented in one-on-one interviews with student volunteers, and modifications are made. These modifications are essential for making the tutorials effective. After the one-on-one implementation with at least half a dozen students, the tutorials are tested and evaluated in classroom settings and further refined.

Working through QuILTs in groups is an effective way of learning because formulating and articulating thoughts can provide students with an opportunity to solidify concepts and benefit from one another’s strengths. It can also provide an opportunity to monitor their own learning because mutual discussions can help students rectify their knowledge deficiencies. Students typically finish a QuILT at home if they cannot finish it in class, and take the post-test associated with it individually in the following class for which no help is provided.

III. CASE STUDIES

Next we briefly discuss several case studies related to the development and evaluation of three QuILTs on the time development of wave function, the uncertainty principle, and the Mach–Zehnder interferometer. The development of each tutorial starts with an analysis of the difficulties students have in learning related concepts. After the preliminary development of the tutorials and the pretest and post-tests associated with them, we conduct one-on-one 1.5 hour interviews with six to seven student volunteers for each tutorial using a think-aloud protocol.²³ In this protocol, students are asked to work on a tutorial while talking aloud so that we can follow their thought processes. Hints are provided as appropriate. These individual interviews provide an opportunity to probe students’ thinking as they work through a tutorial and to gauge the extent to which students are able to benefit from them. After each of these interviews, the tutorials are modified based on the feedback obtained. They are then administered in the classroom and are modified further. Table I shows the performance on the pre/post-test of advanced undergraduate students in a quantum mechanics course on the last version of the tutorials. The pretest was given after traditional instruction on relevant concepts but before the tutorial. In Sec. III A we summarize each tutorial

and discuss student performance. Note that the pretest and post-test for a QuILT were not identical, but often had some identical questions.

A. Time-development QuILT

One difficulty with understanding the time development of wave functions stems from the fact that many students believe that the only possible wave functions for a system are stationary states.^{8,24} Because the Hamiltonian of a system governs its time development, we may expand a nonstationary state wave function $\Psi(x,0)$ at the initial time $t=0$ in terms of the stationary states and then include the appropriate phase factors $e^{-iE_n t/\hbar}$ with each term to find the wave function $\Psi(x,t)$ at time t . Students often append an overall time-dependent phase factor even if the wave function is in a linear superposition of the stationary states.⁸ To elicit this misconception, the pretest begins by asking students about the time dependence of a non-stationary state wave function for an electron in a one-dimensional infinite square well. If the students choose an overall phase factor similar to that for a stationary state, they are asked for the probability density, that is, the absolute square of the wave function. If the students incorrectly assume that the wave function is time-independent even for a nonstationary state, arguing that the overall phase factors cancel out, the tutorial asks them to watch the simulations of the time evolution of the probability densities.

Simulations for this QuILT are adapted from the open source physics simulations developed by Belloni and Christian.^{14,15} These simulations are highly effective in challenging students’ beliefs. Students are often taken aback when they find that the probability density oscillates back and forth for a nonstationary state.

When students observe that the probability density does not depend on time for the stationary-state wave function and depends on time for the nonstationary-state wave function, they are challenged to resolve the discrepancy between their initial prediction and observation. In our model of instruction, this time is a good time to provide students guidance and feedback to help them build a robust knowledge structure. Students then work through the rest of the tutorial to obtain appropriate support and help solidify their understanding of the basic concepts related to time development. Students respond to time development questions with stationary and nonstationary state wave functions in systems with different potential energies (for example, the harmonic oscillator and a free particle), and receive timely feedback. For each case they compare their calculations and predictions for the time-dependence of the probability density with the simulations. Figure 1 shows snapshots of the probability density of a nonstationary state wave function for a one-dimensional harmonic oscillator well. Students watch the probability density evolve in time. Within an interactive environment, they learn that the Hamiltonian governs the time development of the system, and that the eigenstates of the Hamiltonian are special with regards to the time evolution of the system. They learn that not all possible wave functions are stationary, and they learn the difference between the time-independent and time-dependent Schrödinger equation.

Table I shows that average student performance improved from 53% to 85% after working on the QuILT. As discussed earlier, the most common difficulty on the pretest was treating the time evolution of nonstationary states as though those

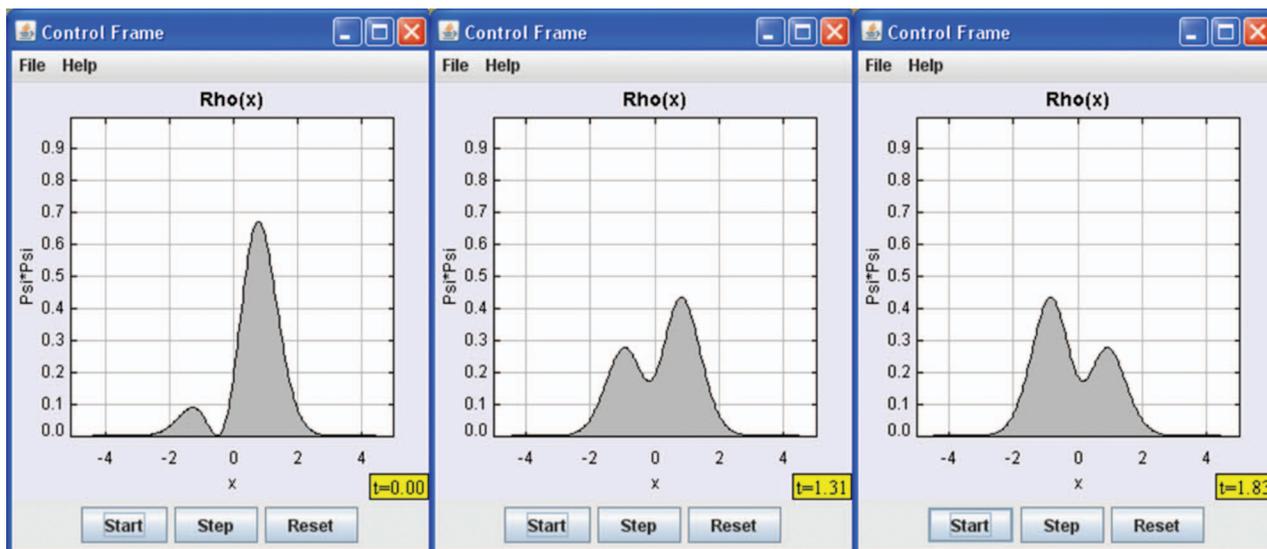


Fig. 1. Snapshots of the probability density for a nonstationary state wave function for the one-dimensional harmonic oscillator well (Ref. 14).

states were stationary states. In comparison, two students who were absent on the day the pretest and tutorial were given, but who were present for the post-test in the following class, obtained 30% and 0% on the post-test, respectively.

B. Uncertainty principle

The QuILT on the uncertainty principle contains three parts with increasing levels of sophistication. Depending on the level of students, the instructors may choose to use one or all three parts. The first part helps students understand that the uncertainty principle is due to the wave nature of particles. With the help of the de Broglie relation, the tutorial helps students understand that a sinusoidal wave has a well-defined wavelength and momentum, but does not have a well-defined position. In contrast, a wave pulse with a well-defined position does not have a well-defined wavelength or momentum.

Students gain further insight into the uncertainty principle in the second part of the QuILT by Fourier transforming the position-space wave function and noticing how the spread of the position-space wave function affects its spread in momentum space. Computations involving Fourier transforms are exploited in this part of the tutorial and students Fourier transform various position-space wave functions with different widths and check the corresponding changes in the momentum-space wave function. The third part of the tutorial helps students generalize the uncertainty principle for position and momentum operators to any two observables whose corresponding operators do not commute. This part of the QuILT also helps students bridge this new treatment with students' earlier encounter with the uncertainty principle for position and momentum in the context of the spread of a wave function in position and momentum space. The QuILT also helps students understand why a measurement of one observable immediately followed by the measurement of another incompatible observable does not guarantee a definite value for the second observable.

Table I shows that the average performance of 12 students who took the last version of the QuILT improved from 42% to 83% from pretest to post-test. In a question that was com-

mon for both the pretest and post-test, students were asked to make a sketch of the absolute value of the Fourier transform of a delta function. They were asked to explain their reasoning and label the axes appropriately. Only one student in the pretest drew a correct diagram. In the post-test, ten out of 12 students were able to draw correct diagrams with labeled axes and explain why the Fourier transform should be a constant extended over all space. Also, in the post-test, ten out of 12 students were able to draw the Fourier transform of a Gaussian position space wave function and discuss the relative changes in the spread of the position and the corresponding momentum space wave functions. These were concepts they had explored while working on the QuILT. Similar results were found in individual interviews conducted earlier with other students during the development of the QuILT.

One of the questions on both the pretest and the post-test of this tutorial was the following: Consider the following statements: "Uncertainty principle makes sense. When the particle is moving fast, the position measurement has uncertainty because you cannot determine the particle's position precisely...it is a blur...that's exactly what we learn in quantum mechanics...If the particle has a large speed, the position measurement cannot be very precise." Explain why you agree or disagree with the statement.

Out of the 12 students who took both pre/post-tests, seven students provided incorrect responses on the pretest. The following are examples of incorrect student responses on the pretest:

- (1) "I agree...when P is high, it is easy to determine, while x is difficult to determine. The opposite is also true, when P is small it is difficult to determine, while x is easy to determine."
- (2) "I agree because when a particle has a high velocity, it is difficult to measure the position accurately."
- (3) "I agree because I know the uncertainty principle to be true."
- (4) "Agree. When a particle is moving fast, we cannot determine its position exactly—it resembles a wave—at fast speed, its momentum can be better determined."

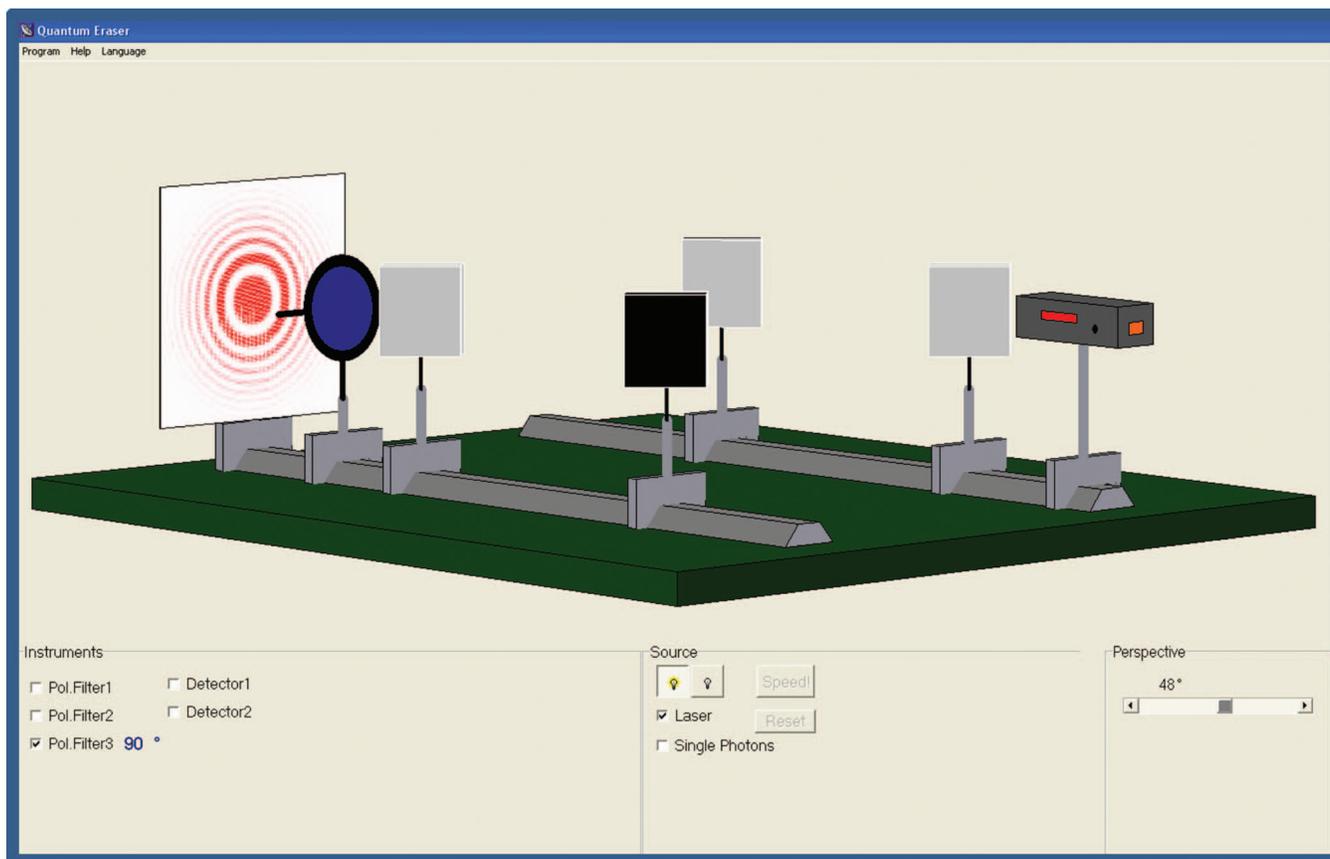


Fig. 2. A screen shot of the Mach-Zehnder interferometer simulation (Ref. 16).

In comparison, one student gave an incorrect response and one did not give clear reasoning on the post-test. The other ten students provided correct responses with good reasoning.

C. Mach-Zehnder interferometer

The goals of this QuILT are to understand the interference at a detector due to the superposition of light from the two paths of an interferometer. The tutorial adapts a simulation developed by Huber¹⁶ to help students understand the interference of a single photon with itself after it passes through the two paths of the Mach-Zehnder interferometer, the effect of placing detectors and polarizers in the path of the photon in the interferometer, and how information about the path along which a photon went (“which-path” information) destroys the interference pattern. A screen shot from the simulation is shown in Fig. 2.

The basic schematic for the Mach-Zehnder interferometer used in this QuILT is as follows (see Fig. 3) with changes made later in the tutorial, for example, changes in the position of the beam splitters, incorporation of polarizers, and detectors or a glass piece, to illustrate various concepts. All the angles of incidence are 45° with respect to the surface. For simplicity, we will assume that light can reflect from only one of the two surfaces of the identical half-silvered mirrors (beam splitters) BS_1 and BS_2 because of antireflection coatings. The detectors D_1 and D_2 are point detectors located symmetrically with respect to the other components of the interferometer as shown. The photons originate from a

monochromatic coherent point source. The light through both the U and L paths travels the same distance in a vacuum to reach each detector.

Students first learn about the basics of phase changes that take place as light reflects or passes through different beam splitters and mirrors in the interferometer by an analogy with a reflected or transmitted wave on a string with fixed or free boundary condition at one end. Then students use the simulation to learn that a single photon can interfere with itself and produce an interference pattern after it passes through both paths of the interferometer. Students explore and learn using simulations that which-path information is obtained by removing BS_2 or by placing detectors or polarizers in certain locations. Later in the tutorial, point detector D_1 is replaced with a screen.

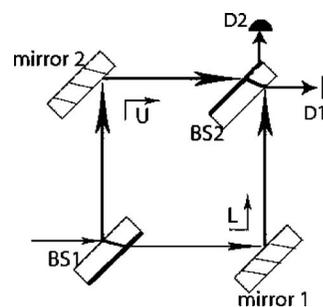


Fig. 3. A schematic diagram of the setup in the Mach-Zehnder interferometer simulation (Ref. 16).

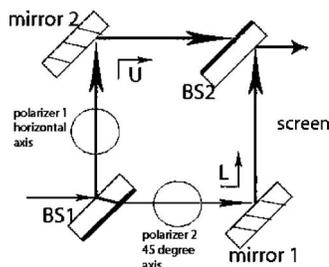


Fig. 4. A schematic diagram of a modified setup with two polarizers (after beam splitter 1 in the lower and upper paths) in the Mach-Zehnder interferometer simulation (Ref. 16).

Table I shows that the average performance of 12 students who took the last version of the Mach-Zehnder interferometer QuILT improved from 48% to 83% from pretest to post-test. All but one of the 12 students in the post-test obtained perfect scores on the following three questions [correct options (c), (b), and (b), respectively], which were similar but not necessarily identical to the kinds of questions they had explored using the simulation within the guided QuILT approach:

- (1) If you insert polarizers 1 and 2 (one with a horizontal and the other with a 45° transmission axis) as in Fig. 4, how does the interference pattern compare with the case when the two polarizers have orthogonal transmission axes?
 - (a) The interference pattern is identical to the case when polarizers 1 and 2 have orthogonal axes.
 - (b) The interference pattern vanishes when the transmission axes of polarizers 1 and 2 are horizontal and 45° .
 - (c) An interference pattern is observed, in contrast to the case when polarizers 1 and 2 were orthogonal to each other.
 - (d) No photons reach the screen when the transmission axes of polarizers 1 and 2 are horizontal and 45° .
- (2) If you insert polarizer 1 with a horizontal transmission axis and polarizer 2 (between the second beam splitter and the screen) with a 45° transmission axis (Fig. 5), how does the interference pattern compare with the case when only polarizer 1 was present?
 - (a) The interference pattern is identical to the case when only polarizer 1 was present.
 - (b) The intensity of the interference pattern changes but the interference pattern is maintained in the presence of polarizer 2.
 - (c) The interference pattern vanishes when polarizer 2 is

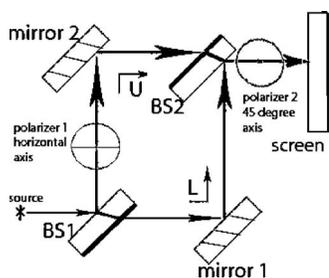


Fig. 5. A schematic diagram of a modified setup with two polarizers (one after beam splitter 1 but before mirror 2 and the other after beam splitter 2) in the Mach-Zehnder interferometer simulation (Ref. 16).

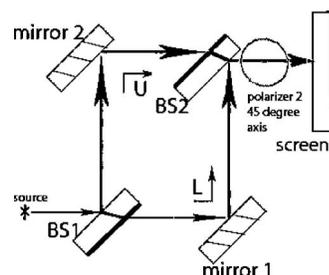


Fig. 6. A schematic diagram of a modified setup with a polarizer after the second beam-splitter in the Mach-Zehnder interferometer simulation (Ref. 16).

inserted but some photons reach the screen.

- (d) An interference pattern reappears that was absent when only polarizer 1 was present.
- (3) If you insert polarizer 2 with a 45° transmission axis between the second beam splitter and the screen (Fig. 6), how does the interference pattern compare with the case when polarizer 2 was not present?
 - (a) The interference pattern is unchanged regardless of the presence of polarizer 2 because all interference effects occur before beam splitter 2.
 - (b) The intensity of the interference pattern decreases but the interference pattern is maintained even in the presence of polarizer 2.
 - (c) The intensity of the interference pattern increases in the presence of polarizer 2.
 - (d) The interference pattern vanishes when polarizer 2 is inserted but some photons reach the screen.

D. Student Survey

A survey of 12 students whose pre/post-test data are presented in Table I was given to assess the effectiveness of QuILTs from the students' perspective. In the following we provide the questions and student responses:

- (1) Please rate the tutorials for their overall effectiveness where 1 means totally ineffective and 5 means very effective.

No student selected 1 or 2, one student selected 3, one selected 3.5, three selected 4, one selected 4.5, and six selected 5.
- (2) How often did you complete the tutorial at home that you could not complete during the class? (1) Never, (2) less than half the time, (3) often, (4) most of the time, or (5) always.

In response to this question, no student selected (1), one student selected (2), two students selected (3), six selected (4), and three selected (5).
- (3) How often were the hints/solutions provided for the tutorials useful? (1) Never, (2) less than half the time, (3) often, (4) most of the time, or (5) always.

No student selected (1) or (2), two students selected (3), six selected (4), and five selected (5).
- (4) Is it more helpful to do the tutorials in class or would you prefer to do them as homework? Please explain the advantages and disadvantages as you see it.

Ten students responded that doing them in class was more useful. The students who preferred doing them in class often noted that the tutorials focused on improving their conceptual understanding, which was best done via

group discussion and hence in class. They appreciated the fact that any questions they had could be discussed, and they benefited from the reasoning provided by their peers and the instructor. The few students who preferred doing them at home responded that more time and effort would go into them if they did them at home.

- (5) How frequently should the tutorials be administered in the class (for example, every other class, once a week, once every other week)? Explain your reasoning.

A majority of students liked having the tutorials once a week. This frequency was considered to be the best by some students who thought that the concepts learned in the tutorials made it easier for them to understand the textbook and homework problems later in the week and integrate the material learned. Others believed that once a week was the best because tutorials helped them focus on concepts that were missed in the lectures, book, and student/teacher conversations.

- (6) Do you prefer a multiple-choice or open-ended question format for the tutorial questions? Explain your reasoning.

Students liked the questions that were in multiple-choice format, but most of them also appreciated the open-ended questions. Some students noted that the multiple-choice questions helped focus their attention on important issues and common difficulties and misconceptions while the open-ended questions stimulated creative thought. Some students believed that multiple-choice format was better for the warm-up tutorial done at home and the open-ended questions were better for the main tutorial done in the class. Some students believed that a mix of the two types of questions was best because the multiple-choice format was a good way to get the fundamental concepts across and the open-ended questions gave them an opportunity to apply these concepts and deepen their understanding of the concepts.

IV. DISCUSSION

We have given an overview of the development of QuILTs and have discussed the preliminary evaluation of three tutorials using pre/post-tests in a classroom setting. During the development of the existing QuILTs, we have conducted more than 100 hours of interviews with individual students to assess the aspects of the QuILTs that work well and those that require refinement. QuILTs naturally lend themselves to dissemination via the web. They provide appropriate feedback to students and can be used as an online learning tool for undergraduates and beginning graduate students in addition to being suitable as supplements to lectures for a one or two-semester undergraduate quantum mechanics courses.

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¹⁷The original Spins program was written by Daniel Schroeder and Thomas Moore for the Macintosh and was ported to Java by David McIntyre of Oregon State University and used as part of the Paradigms project. Both of these versions are open source. See (www.physics.oregonstate.edu/~mcintyre/ph425/spins/).

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