Quantum Mechanics Tutorials: Hidden Variables or Pure Probability?

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January 16, 2008

Abstract

Both classical mechanics and quantum mechanics can yield probabilistic results. However, in classical mechanics, probabilistic dynamics is generally attributed to high complexity of an underlying mechanism. In quantum mechanics no such underlying mechanism can be identified. Here is presented a thought experiment that would positively verify that such a mechanism *cannot* exist. A summary of necessary quantum mechanical formalism is included. The explanation is based on the properties of the Greenberger-Horne-Zeilinger quantum state.

1 Introduction

The proof in this paper is adapted from a talk by Sidney Coleman "Quantum Mechanics in Your Face[1]." Only the section on quantum mechanics and full derivations of all results were added by the author. Readers familiar with basic formalism of quantum mechanics should skip the first part. Readers not familiar with quantum mechanics on university level should study the section on quantum mechanical formalism before or after reading the proof.

The purpose of this tutorial is to present a demonstration of fundamental difference between classical and quantum concepts of uncertainty. Both classical and quantum physics deals with probabilistic results. Prominently, in classical mechanics, thermodynamics of an ideal gas involves random fluctuations of temperature and other parameters. In quantum mechanics, particle motion is not described by a trajectory, but by a wavefunction that provides *probabilities* of finding a particle at a

particular location. The difference is that in classical mechanics there is an underlying assumption that given unlimited computational power one would be able to account for the motion of every particle and make the fluctuations predictable. In quantum mechanics, however, we not only cannot imagine such a mechanism, but will demonstrate that such a mechanism *cannot* exist.

A more experienced reader will recognize this as an alternative to Bell's theorem on hidden variables. The great advantage of the proof presented here is its simplicity.

2 Quantum Mechanical Formalism

This section is intended to familiarize the reader with some basic concepts of quantum mechanics. A reader unfamiliar with quantum mechanics may feel that he/she is being presented with a collection of arbitrary concepts, and hence be inclined to assume that the following proof will depend on concepts that he/she does not understand. I ask the reader to look past certain omission in the explanation as a full explanation of every concept used would quickly grow into a graduate text on quantum mechanics. I also emphasize that the material presented here covers about all there is to say about the subject on an undergraduate level. A reader seeking a more complete basic introduction to quantum mechanics is encouraged to look through the famous text by Griffiths[2]. A reader with a strong mathematical/engineering background may find it more useful to refer to wonderful texts by L. Landau[4] or Shankar [3].

Quantum mechanical state contains all the information that is known about the particle. A state is generally written as a linear combination (sum) of *eigenstates*. An arbitrary state $|\psi\rangle$ would be written this way:

$$|\psi\rangle = C_a|a\rangle + C_b|b\rangle + ...$$

Here $|a\rangle$, $|b\rangle$ are *eigenstates* and C_a, C_b are constants (real or complex). An eigenstate is a state that has a well defined value of a particular physical parameter. If the particle state is equal to an eigenstate, the measurement of that parameter will always yield the same value (called the eigenvalue). If classical mechanics were defined in terms of states, *all* states would be eigenstates of *every* parameter. If a quantum mechanical state is not equal to an eigenstate, then the value of a given parameter is in a *superposition* of all eigenvalues that correspond to the eigenstates that make up the state. A measurement of that parameter will return any of the allowed eigenvalues

with probability equal to the square absolute value of the coefficient in front of that eigenstate.

A hermitian quantum operator (I'll refer to them as just operators) is a construct that corresponds to a physically observable quantity. Each operator has a set of corresponding eigenstates which can be used to construct particle states as mentioned above. The action of an operator on a state is best understood when the state is expressed in terms of eigenstates of that operator. For an operator $\bar{O}$:

$$\bar{O}|\psi\rangle = \bar{O}(C_a|a\rangle) + \bar{O}(C_b|b\rangle) + ... = a(C_a|a\rangle) + b(C_b|b\rangle) + ...$$

As we see, the action by an operator created a new state. In certain special cases an action by an operator corresponds to a measurement. Specifically, if the state is an eigenstate of the operator, then the operator will leave the state unchanged apart from multiplying it by its corresponding eigenvalue. The eigenvalue will be the result of the measurement.

Spin is a property of particles which has no direct analogy in classical mechanics. It is part of a vast collection of characteristics which only occur in quantum mechanics. Here I will discuss it briefly as it will be used in the proof which I am about to present. Spin of a particle determines the amount of *internal* angular momentum that it possesses. Every particle has a fixed spin value that can never be changed (like mass or charge). Spin values are discrete and have values such as 1/2, 1, 2/3 etc. In this paper we will only discuss particles with spin 1/2. Spin is a vector, hence the orientation of the reference frame will determine the value that will be measured. There are defined quantum operators which correspond to measurement of spin projections (components) on any of the 3 coordinate axes (x,y,z). Unlike classical angular momentum spin may only have a finite number of projections to any given axis. Specifically any measurement of spin projection of a spin-1/2 particle will return a value of either 1/2 or -1/2, regardless of choice of coordinate system. The spin of spin-1/2 particle may never be found at an angle to *any* axis of choice. This is not true for other spin values, however, the number of measurable projections is always finite. Components of spin are *non-commutative* which means that only one may be known exactly at a given time. A state that is an eigenstate of one component of spin cannot be an eigenstate of another. A famous analogy is the Heisenberg uncertainty principle, which postulates that position and momentum operators do not commute.

Also, the set of eigenstates of any of the spin operators is *complete* which means that eigenstates of one operator can be used to express any eigenstate of another operator.

In this tutorial I will use operations of spin projection operators on particle states. In order to explain how these results are obtained I'll use the matrix representation of the operators. The matrices representing the operators are linear algebra constructs that behave identically with regard to vectors representing the states as quantum operators behave with regard to particle states. It is important to remember, however, that it is just a representation.

Let's start by defining the matrices for spin-1/2 projection operators:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

These are the famous Pauli matrices. As mentioned before the only spin values that can be measured are 1/2 and -1/2 (From now on we will substitute 1 for 1/2 and -1 for -1/2 for mathematical simplicity. The given Pauli matrices also have eigenvalues 1 and -1). That means spin operators have only two eigenstates. Traditionally the eigenstates of σ_z operator are used to express all other spin states. Represented by vectors:

$$|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The up-arrow down-arrow notation is traditional and corresponds to the angular momentum being measured parallel or anti-parallel to the axis of choice.

From here it is very easy to explain how an operator acts on a state (it is equivalent to matrix multiplication):

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \sigma_z|\uparrow\rangle = |\uparrow\rangle$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow \sigma_z|\downarrow\rangle = -|\downarrow\rangle$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow \sigma_x|\uparrow\rangle = |\downarrow\rangle$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \sigma_x|\downarrow\rangle = |\uparrow\rangle$$

$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = i \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow \sigma_y|\uparrow\rangle = i|\downarrow\rangle$$

$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -i \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \sigma_y|\downarrow\rangle = -i|\uparrow\rangle$$

Multiple particle states may be written as such:

$$|\psi\rangle = C_1|\uparrow\downarrow\uparrow\rangle + C_2|\downarrow\uparrow\uparrow\rangle\rangle$$

This state describes 3 particles. Traditionally they are identified by their location in the list. An operator operating on this state would have to explicitly specify which particle it will operate on, hence a y-axis spin-1/2 projection operator operating on the second particle would be labeled σ_y^2. The operation would look like this:

$$\sigma_y^2|\psi\rangle = (C_1 \cdot (-i))|\uparrow\uparrow\uparrow\rangle + (C_2 \cdot i)|\downarrow\downarrow\uparrow\rangle$$

Note that the operation only affects the second particle.

3 Proof

Set up a thought experiment that has a particle emitter and three detectors. The detectors are positioned at equal distances from the emitter and are separated in space far enough that simultaneous events at the detectors are separated by a spacelike interval. That means that simultaneous events at the detectors cannot influence each other without requiring transmission of a signal faster than the speed of light.

Let the detector emit spin-1/2 particles in the state:

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\uparrow\uparrow\rangle - |\downarrow\downarrow\downarrow\rangle\right)$$

As we see, 3 particles are emitted each time. One particle is sent to each detector. The detector can measure two things: x-component of spin or y-component of spin. Let σ_x^1, σ_x^2, σ_x^3 correspond to x-component measurement of each particle (1,2,3) and equivalently for y. Every time a particle is scheduled to arrive at a detector the detector chooses randomly which component to measure. For algebraic simplicity lets assume that whenever a detector measures a value 1/2 it returns 1, and equivalently

-1 for -1/2. Imagine that at a particular time the first detector measures x component while the remaining two measure y component. In operator notation this looks like:

$$\sigma_x^1\sigma_y^2\sigma_y^3|\psi\rangle =$$

$$\frac{1}{\sqrt{2}}\left((\sigma_x^1|\uparrow\rangle)(\sigma_y^2|\uparrow\rangle)(\sigma_y^3|\uparrow\rangle) - (\sigma_x^1|\downarrow\rangle)(\sigma_y^2|\downarrow\rangle)(\sigma_y^3|\downarrow\rangle)\right) =$$

$$\frac{1}{\sqrt{2}}\left((1\cdot i\cdot i)|\downarrow\downarrow\downarrow\rangle - (1\cdot(-i)\cdot(-i))|\uparrow\uparrow\uparrow\rangle\right) =$$

$$\frac{1}{\sqrt{2}}\left(|\uparrow\uparrow\uparrow\rangle - |\downarrow\downarrow\downarrow\rangle\right)$$

See the section on quantum mechanical formalism for derivation of why this identity holds. We see that the state that we started with happens to be an eigenstate of the $\sigma_x^1\sigma_y^2\sigma_y^3$ operator. The state is unchanged by the operator, just multiplied by +1. Hence the eigenvalue is +1. This means that any time the detectors were set to measure X,Y,Y the product of their measured values would be 1. This might sound strange enough, but so far we have not discussed anything unusual. This is just a prediction of quantum mechanics that can be verified experimentally. Also note that since all detectors and particles are indistinguishable any time two y values and one x is measured the product of results will be 1(XYY=YXY=YYX).

Here we reach the main point. Let's imagine that a particle obeys the laws of quantum mechanics, but somewhere deep inside it there is information that specifies what the result of a particular measurement would be, if it was done. For example if we make an XYY measurement we may get -1-1+1 which is compatible with quantum mechanics since the product is +1. So long as the product is +1 quantum mechanics tells us that exact configuration of +1's and -1's cannot be predicted. What if we had taken a measurement XXX? We already know one rule the particles have to obey (XYY=1), so if we imagine that the result of every possible measurement was defined the moment XYY was performed, can we make a prediction of what XXX would have been? Figure 1 shows every possible result of an XXX measurement assuming a particular XYY.

Observe that any measurement XYY does not tell us with certainty what the specific measured values of X and Y would be had the other measurements were taken, but it does predict that a measurement of XXX would always yield +1.

Now let us look at the quantum mechanical prediction:

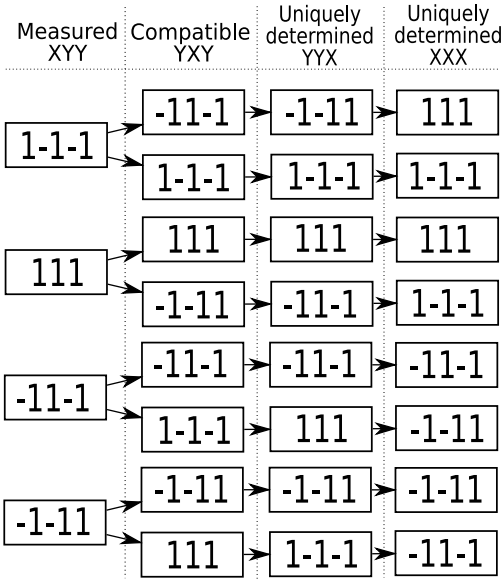

Figure 1: Every measurement of XYY does not uniquely determine the other possibilites, but just assuming that the product of any combination of two Y's and one X equals +1, we can find further patterns.

$$\sigma_x^1\sigma_x^2\sigma_x^3|\psi\rangle = \frac{1}{\sqrt{2}}\left((\sigma_x^1|\uparrow\rangle)(\sigma_x^2|\uparrow\rangle)(\sigma_x^3|\uparrow\rangle) - (\sigma_x^1|\downarrow\rangle)(\sigma_x^2|\downarrow\rangle)(\sigma_x^3|\downarrow\rangle)\right) =$$

$$\frac{1}{\sqrt{2}}\left((1\cdot 1\cdot 1|\downarrow\rangle|\downarrow\rangle|\downarrow\rangle - (1\cdot 1\cdot 1)|\uparrow\rangle|\uparrow\rangle|\uparrow\rangle\right) = -|\psi\rangle$$

Quantum mechanics predicts that the result of every XXX measurement will be -1!

That's it! The only thing we assumed was that verifiable predictions of quantum mechanics are correct and that the value of a parameter was defined regardless of whether it was measured or not. We see that if quantum mechanics predicts correct particle behavior then the second assumption must be wrong and parameter values *do not exist until they are measured*.

There has been a number of experiments aimed at verifying the predictions of quantum mechanics relevant to this result. They can be found in literature by searching for information about Bell's theorem and GHZ state.

References

[1] S. Coleman "Quantum Mechanics in Your Face" video available on video.google.com at the time of writing.

[2] D. Griffiths *Introduction to Quantum Mechanics* any edition.

[3] R. Shankar *Principles of Quantum Mechanics* 2nd ed.

[4] Landau,L., Lifshitz, L. *Quantum Mechanics Non-Relativistic Theory* Available in many languages and editions.